NEW METHOD OF SOLVING THE EQUATIONS OF CASCADE THEORY

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A new method of solving the equations of electromagnetic cascade theory is described in detail. The method is based on replacing the integral operator that describes bremsstrahlung of the electrons and pair production by photons with a simple approximate differential operator. Many results obtained by applying the method to the solution of equations of one-dimensional cascade theory with and without allowance for ionization losses are presented.

W_E describe a new method of solving the equations of electromagnetic cascade theory. The method, whose main outlines were formulated by us earlier in[1], is based on replacing the integral operator describing electron bremsstrahlung and pair-production by photons by a simple approximate differential operator. In many cases this substitution greatly simplified the integro-differential equations of the cascade theory, reducing them to linear differential equations. An analysis of the solutions of the approximate differential equations shows that in many important cases these solutions are more accurate than the non-approximate solutions of the initial exact equations. By way of an example, we shall apply our method to the solution of the equations of one-dimensional cascade theory in approximations A and B[2]. We write the main equations in the form

$$\frac{\partial P(E,t)}{\partial t} = 2 \int_E^\infty \Gamma(E',t) W_p[E',E] dE'$$

$$+ \int_E^\infty P(E',t) W_e(E',E-E) dE'$$

$$- P(E,t) \int_0^E W_e(E,E') dE' + \beta \frac{\partial P(E,t)}{\partial E};$$

$$\frac{\partial \Gamma(E,t)}{\partial t} = \int_E^\infty P(E',t) W_e(E',E) dE'$$

$$- \Gamma(E,t) \int_0^E W_p(E,E') dE'. \quad (1)$$

Here $P(E,t)dE$ and $\Gamma(E,t)dE$ are respectively the number of shower electrons or photons with energy in the interval E, $E + dE$ at a depth t, and β is the magnitude of the ionization losses in one cascade length unit. The functions

$$W_e(E,E') dE' = \left[1.36 - 1.36 \frac{E'}{E} + \left(\frac{E'}{E}\right)^2 \right] \frac{dE'}{E'}$$

$$= \varphi_0(v) dv, \quad v = E'/E,$$

$$W_p((E',E) dE = \left[1.36 \left(\frac{E}{E'}\right)^2 - 1.36 \frac{E}{E'} + 1 \right] \frac{dE}{E'}$$

$$= \psi_0(u) du, \quad u = E/E'$$

describes the probabilities of bremsstrahlung and pair production in the total-screening approximation. We multiply (1) by $\exp(-\lambda t)$ and integrate with respect to t from zero to infinity. Eliminating from the obtained system of equations the function $\Gamma(E, \lambda)$, we obtain in the case of a shower produced by a primary electron with energy E_0 the following equation for the Laplace transform $P(E_0, E, \lambda)$:

$$L[P(E_0, E, \lambda)] - \beta \partial P(E_0, E, \lambda) / \partial E = \delta(E_0 - E). \quad (2)$$

The integral operator $L[P(E_0, E, \lambda)]$ is of the form

$$L[P(E_0,E,\lambda)] = - \int_E^{E_0} P(E_0, E', \lambda)[K(E,E'\lambda)$$

$$+ W_e(E', E'-E)] dE'$$

$$+ \left[\lambda + \int_0^E W_e(E,E') dE' \right] P(E_0,E,\lambda). \quad (3)$$

where

$$K(E,E',\lambda) = 2 \int_E^{E'} W_e(E',\varepsilon) W_p(\varepsilon,E) d\varepsilon / (\lambda + \sigma_0),$$

$$\sigma_0 = 0.773.$$

The expression for $L[P]$ can be simplified. To this end we multiply (3) by E^S and integrate with res-

pect to E from E_1 to E_0. Replacing further E by E' and E_1 by E, and changing the order of integration in some of the integrals, we obtain

$$\int_E^{E_0} E'^s L[P(E_0, E', \lambda)] dE' = \int_E^{E_0} P(E_0, E', \lambda) \chi(E, E', \lambda, s) dE',$$

(4)

where

$$\chi(E, E', \lambda, s) = -E'^s \psi(\lambda, s) - E'^s \int_{1-E/E'}^1 (1-v)^s \varphi_0(v) dv$$

$$+ E'^s \left[- B(s) \int_0^{E/E'} v^s \varphi_0(v) dv - 2 \int_{E/E'}^1 v^s \varphi_0(v) dv \int_0^{E/E'v} u^s \psi_0(u) du \right]$$

$$\times (\lambda + \sigma_0)^{-1}.$$

(5)

Here

$$\psi(\lambda, s) = \lambda + A(s) - B(s)C(s) / (\lambda + \sigma_0);$$

$A(s)$, $B(s)$, and $C(s)$ are known functions of the cascade theory[2,3]. After calculating the integrals in (5) we can represent the function $\chi(E, E', \lambda, s)$ in the form

$$\chi(E, E', \lambda, s) = -E^s \{ \psi(\lambda, s) / x^s + f(x, \lambda, s) \},$$

where $x = E/E'$, and the function $f(x, \lambda, s)$ varies slowly with variation of x for specified λ and s. If we take for the probabilities of the bremsstrahlung and pair production the simplified Bethe-Heitler cross sections for the case of total screening[2], then

$$f(x, \lambda, s) = B(s)C(s)(xs - s - 1) / (\lambda + \sigma_0)$$

$$- x \, _2F_1(1, s + 1; s + 2; x) / (s + 1),$$

where $_2F_1(1, s + 1; s + 2; x)$ is a hypogeometric function. In order for $\chi(E, E', \lambda, s)/E^s$ likewise to vary little with change of x, it is necessary to put $\psi(\lambda, s) = 0$. From this condition it follows that λ and s should be connected by the relations $\lambda = \lambda_1(s)$ or $\lambda = \lambda_2(s)$, where λ_1 and λ_2 are the roots of the equation $\psi(\lambda, s) = 0$. Putting $\lambda = \lambda_1(s)$ or $\lambda = \lambda_2(s)$, we obtain

$$\chi(E, E', \lambda, s) = -E^s f_i(x, s),$$

where the indices $i = 1$ and $i = 2$ correspond to the values $\lambda = \lambda_1(s)$ or $\lambda = \lambda_2(s)$. Substituting the obtained expressions $\chi(E, E', s)$ in (4), we have

$$\int_E^{E_0} E'^s L[P(E_0, E', \lambda_i(s))] dE' = E^s q_i(E_0, E, s) N_p^i(E_0, E, s),$$

(6)

where

$$q_i(E_0, E, s) = \int_E^{E_0} \frac{\partial N_p^i(E_0, E', s)}{\partial E'} f_i \left(\frac{E}{E'}, s \right) \frac{dE'}{N_p^i(E_0, E, s)},$$

$$N_p^i(E_0, E, s) = \int_E^{E_0} P(E_0, E', \lambda_i(s)) dE'.$$

(7)

In the first approximation we shall assume that

$$q_i(E_0, E, s) = q_i(s).$$

(8)

Then, differentiating (6) with respect to E and dividing the resultant expression by E^s, we obtain

$$L[P(E_0, E, \lambda_i(s))] = -q_i(s) \frac{dN_p^i(E_0, E, s)}{dE}$$

$$- \frac{s q_i(s)}{E} N_p^i(E_0, E, s).$$

(9)

Substituting (9) in (2) and introducing new variables $\epsilon_i = Eq_i/\beta$ and $\epsilon_{0i} = E_0 q_i/\beta$, we obtain a second-order linear differential equation with respect to the function $N_p^i(\epsilon_0, \epsilon, s)$:

$$\varepsilon \frac{d^2 N_p^i(\varepsilon_0, \varepsilon, s)}{d\varepsilon^2} - \varepsilon \frac{dN_p^i(\varepsilon_0, \varepsilon, s)}{d\varepsilon}$$

$$- s N_p^i(\varepsilon_0, \varepsilon, s) = \varepsilon \frac{\delta(\varepsilon_0 - \varepsilon)}{q_i(s)}.$$

(10)

Equation (10) replaces Eq. (2) with good accuracy. Thus we obtain in place of the integro-differential equation (2) a second-order linear differential equation (10). This equation is obtained from (2) by making the sole assumption (8), the validity of which can be readily verified.

If the ionization losses are disregarded, then Eq. (10) with $\beta = 0$ can be rewritten in the form

$$\frac{dN_p^i(E_0, E, s)}{dE} + \frac{s}{E} N_p^i(E_0, E, s) = -\frac{\delta(E_0 - E)}{q_i(s)}.$$

(10')

Its solution is

$$N_p^i(E_0, E, s) = \begin{cases} (E_0/E)^s / q_i(s) & E < E_0 \\ 0 & E \geqslant E_0. \end{cases}$$

(11)

Substituting (11) in (7) and carrying out the necessary integrations, we obtain for the function $q_i(E_0, E, s)$ the following expression:

$$q_i(E_0, E, s) = -(E/E_0)^s f_i(E/E_0, s) - s \int_{E/E_0}^1 f_i(x, s) x^{s-1} dx.$$

(7')

Figure 1 shows the function $q_1(E_0, E, s)$, while Fig. 2 shows the function $q_2(E_0, E, s)$ for different values of E/E_0 and s. We see from the figures that, accurate to several per cent, these functions are
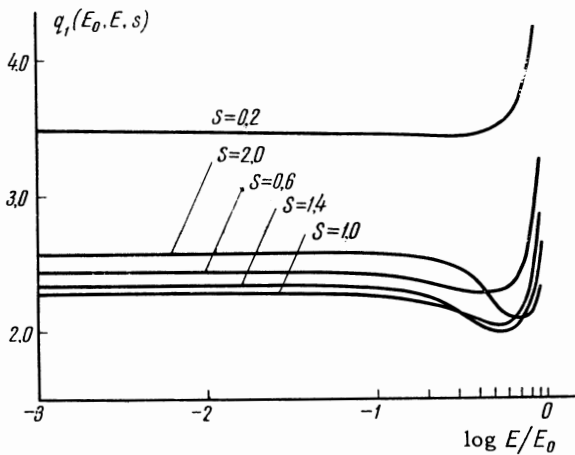
FIG. 1. Dependence of the function $q_1(E_0, E, s)$ on E for different values of s without account of ionization losses.

constant in a wide range of s when $0 \le E/E_0 \le 0.5$. The relative error of the functions $N_p^i(E_0, E, s)$, due to replacing $q_i(E_0, E, s)$ by $q_i(s)$, can be readily estimated with the aid of the following formula:

$$\frac{\Delta N_p{}^i}{N_p{}^i} = \frac{q_i(E_0, E, s) - q_i(s)}{q_i(E_0, E, s)}. \qquad (12)$$

If the error in the function N_p^i calculated in the zeroth approximation for $q_i = q_i(s)$ turns out to be too large, a method of successive-approximations in terms of q_i can be developed. Substituting the first-approximation expression for $q_i(E_0, E, s)$ in the right side of (6), differentiating (6) with respect to E, and dividing by E^s, we obtain the first-approximation expression for the operator L[P], etc. It can be shown that the expression for N_p in the n-th approximation is given by formula (11) in which $q_i(s)$ is replaced by the n-th approximation $q_i^n(E_0, E, s)$. The relative error of the function N_p in the n-th approximation is determined here by a formula similar to (12).
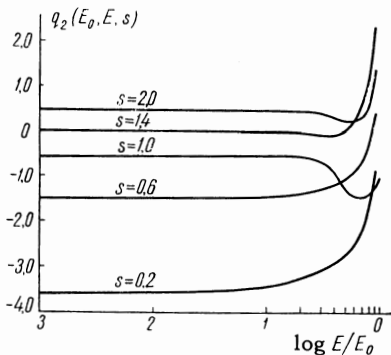

FIG. 2. Dependence of the function $q_2(E_0, E, s)$ on E for different values of s without account of the ionization losses.

Confining ourselves to the zeroth approximation $q = q_i(s)$, we determine the function $N_p(E_0, E, \lambda)$ in the form

$$N_p(E_0, E, \lambda) = \sum_{i=1}^{2} (E_0/E)^s / q_i(s), \qquad (13)$$

where s is determined from the relations $\lambda = \lambda_i(s)$. It can be shown that the functions $q_i(s)$ have an analytic representation

$$q_i(s) = -s\lambda_i'(s) / H_i(s), \qquad (7'')$$

where $\lambda_i'(s)$ and $H_i(s)$ are known cascade functions[2]. To obtain $N_p(E_0, E, t)$ it is necessary to take the inverse Laplace transform. In the calculation of the inversion integral we go from integration with respect to λ to integration with respect to s in accordance with the formula $\lambda = \lambda_i(s)$. As a result we obtain

$$N_p(E_0, E, t) = \frac{1}{2\pi i} \int_{\delta - i\infty}^{\delta + i\infty} \sum_{i=1}^{2} \frac{1}{q_i(s)} \left(\frac{E_0}{E} \right)^s e^{\lambda_i(s)t} d\lambda_i(s).$$

Taking into account formula (7''), we rewrite the expression for $N_p(E_0, E, t)$ in the form

$$N_p(E_0, E, t) = \frac{1}{2\pi i} \int_{\delta - i\infty}^{\delta + i\infty} \left[\frac{H_1(s)}{s} \left(\frac{E_0}{E} \right)^s e^{\lambda_1(s)t} \right.$$

$$\left. + \frac{H_2(s)}{s} \left(\frac{E_0}{E} \right)^s e^{\lambda_2(s)t} \right] ds. \qquad (14)$$

The expression (14) coincides exactly with formula (6.4) of[2] and formula (2.50) of[3], which are the exact solutions of the equations in the region $E \ll E_0$. Using (12) and (7'), we can show that the expression (14) represents the solution of the initial equations in the region $E < 0.3E_0$ with accuracy not worse than 5%. The expressions for $N_p(E_0, E, t)$ in the first, second, and succeeding approximations represent more accurately the solution in the region $0.3E_0 \le E \le E_0$.

From the presented method of obtaining the operator $L[P(E_0, E, s)]$ it follows that its expression in (2) does not depend on the form of the boundary conditions or on the form of the source functions in (1). The boundary conditions and the source functions determine the form of the right side of (2) and (10) or (10'). The solution of (10) or (10') with the right side specified in the form $\varphi(E, s)$ can be represented in the form

$$\{N_p(E_0, E, s)\}^{\varphi(E, s)} = \int_{E}^{E_0} \{N_p(E_0, E', s)\}^p \varphi(E', s) dE', \quad (15)$$

where $\{N_p(E_0, E, s)\}^p$ is the solution of the same equations with boundary conditions corresponding

to one primary electron with energy E_0 at t = 0. The integral in (14) can be calculated numerically with electronic computers[4] or approximately by the saddle-point method[2]. In Fig. 3 we show the function $\{N_p(E_0, E, t)\}^\Gamma$ in approximation A, calculated in a shower from a primary photon for several values of t and E/E_0 with a computer (solid line) and by the saddle-point method (dashed lines). We see from the figure that when t > 0.01 and $E/E_0 \lesssim 0.3$ the function $N(E_0, E, t)$ can be calculated by the saddle-point method with an error $\lesssim 10\%$.

We consider now the solution of (10) with allowance for the ionization losses. Eq. (10) with zero right-hand side is the Kummer differential equation for the confluent hypergeometric function (see[5]). Equation (10) itself and its solution for s = 1 coincide exactly with the well known equation and solution for the "equilibrium" Tamm-Belen'kiĭ spectrum[2]. For arbitrary values of the parameter s we can also write an analytic solution which is conveniently expressed in terms of two linearly independent particular solutions $M_{-s, 1/2}(\epsilon)$ and $W_{-s, 1/2}(\epsilon)$ of the Whittaker equation[5] in the form

$$N_p{}^i(\epsilon_0, \epsilon, s) = \frac{\Gamma(s+1)}{q_i(s)} \epsilon \epsilon_0 e^{-\epsilon_0}$$

$$\times [F_2(s+1, 2, \epsilon)F_1(s+1, 2, \epsilon_0)$$

$$- F_1(s+1, 2, \epsilon)F_2(s+1, 2, \epsilon_0)]; \quad (16)$$

here

$$F_1(s+1, 2, \epsilon) = e^{\epsilon/2}M_{-s, \frac{1}{2}}(\epsilon) / \epsilon,$$

$$F_2(s+1, 2, \epsilon) = e^{\epsilon/2}W_{-s, \frac{1}{2}}(\epsilon) / \epsilon.$$

The explicit expressions for the Whittaker functions $M_{-s, 1/2}(\epsilon)$ and $W_{-s, 1/2}(\epsilon)$ are given in[5]. Formula (16) defines the function $N_p^i(\epsilon_0, \epsilon, s)$ in a wide range of variation of E_0 and E and in the vicinity of Re s $\neq -1, -2, -3, \ldots$

At certain extremal values of the arguments we can obtain sufficiently simple asymptotic expressions for the function $N_p^i(\epsilon_0, \epsilon, s)$. Thus, for example, we can show that when $\epsilon_0 \gg 1$ and $\epsilon \ll 1$ we get

$$N_p(\epsilon_0, \epsilon, s) = -\frac{H_1(s)}{s\lambda_1'(s)} D(s) \left(\frac{E_0}{\beta}\right)^s G(s, \epsilon)$$

$$\times \left[1 - \frac{(1-s)s}{\epsilon_0} + \ldots\right]$$

$$- e^{-\epsilon_0} G(s, \epsilon_0)\epsilon\left[1 + \frac{1+s}{2}\epsilon + \ldots\right]\}, \quad (17)$$

where the functions D(s) and G(s, ϵ) are defined in accord with[2]; when $\epsilon_0 \gg 1$ and $\epsilon \sim \epsilon_0$ we have
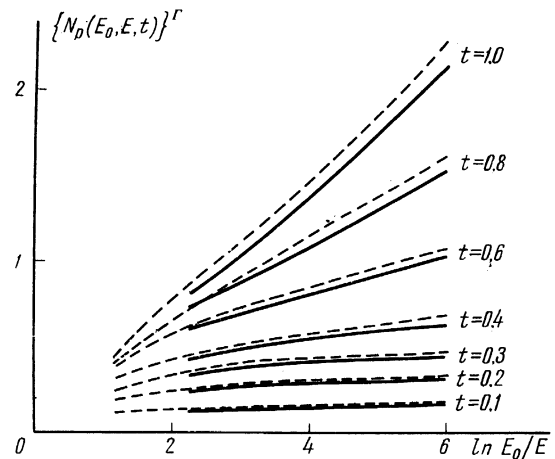

FIG. 3. Dependence of the total number of electrons $\{N_p(E_0, E, t)\}^\Gamma$ on the energy E in a shower from a primary photon with energy E_0 at different depths t. The solid lines are the results of calculations [4] with an electronic computer; dashed lines—results of calculations by the saddle-point method.

$$N_p(\epsilon_0, \epsilon, s) = \frac{1}{q_1(s)}\left\{\left(\frac{\epsilon_0}{\epsilon}\right)^s\left[1 - \frac{s(s+1)}{\epsilon} - \frac{s(1-s)}{\epsilon_0} + \ldots\right]\right.$$

$$\left. - e^{\epsilon-\epsilon_0}\left(\frac{\epsilon}{\epsilon}\right)^s\left[1 - \frac{(1-s)s}{\epsilon} - \frac{s(s+1)}{\epsilon} + \ldots\right]\right\}; \quad (18)$$

$$P(\epsilon_0, \epsilon, s) = \frac{1}{\beta}\left(1 - \frac{6s^2}{\epsilon_0{}^2}\right), \quad \epsilon_0 \gg 1 \quad (19)$$

and when $\epsilon_0 \sim 1$ and $\epsilon \ll 1$

$$N_p(\epsilon_0, \epsilon, s) = -\frac{\epsilon_0}{q_1(s)} e^{-\epsilon_0}\left\{G(s, \epsilon)F_1(s+1, 2, \epsilon_0)\right.$$

$$\left. - \frac{\epsilon}{\epsilon_0}\left(1 + \frac{s+1}{2}\epsilon + \ldots\right)G(s, \epsilon_0)\right\}. \quad (20)$$

The limitations on the region of variation of the variables E_0 and E are imposed by the following: a) physical consideration, which allow us to neglect the scattering of particles and the dependence of the photon absorption coefficient on the photon energy; b) the assumptions made in the derivation of the approximate expression (9) for $L[P(E_0, E, \lambda_i(s))]$. As in the approximation A, we have made a single assumption in the derivation of (9)—we have assumed that $q(E_0, E, s)$ does not depend on the energy E. The basis for such an assumption is the fact that the function $f_i(x, s)$, defined above, changes little in practically the entire interval of variation of x from zero to one. In that energy region where the approximation A is valid, we have $N_p^i(E_0, E, s) \sim (E_0/E)^S$ and consequently relation $q_i(E_0, E, s) = q_i(s)$ is satisfied.

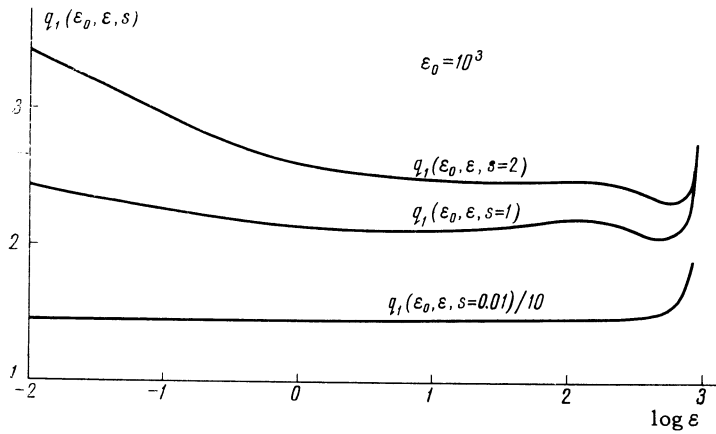To check on the validity of (8) in approximation B, let us calculate $q_i(E_0, E, s)$ in accord with (7),

FIG. 4. Dependence of the function $q_1(\epsilon_0,\ \epsilon,\ s)$, calculated with account of the ionization losses, on the energy ϵ for different values of s. The values of $q_1(\epsilon_0,\ \epsilon,\ s)$ for the parameter s = 0.01 are decreased by a factor 10.

using expression (16) for $N_p^1(E_0,\ E,\ s)$. The results of the calculation of $q_1(E_0,\ E,\ s)$ for $0 \ll E < E_0$ and $0 < s \leq 2$ are shown in Fig. 4. It is seen from the figure that when E changes in the region $0 \lesssim E \lesssim 0.5E_0$, the function remains constant accurate to several per cent. The number of particles with energy higher than E is given by expression (16) in the form

$$N_p{}^i(\varepsilon_0,\ \varepsilon,\ s) = E_0\beta^{-1}\varphi_p{}^i(\varepsilon_0,\ \varepsilon,\ s).$$

Since the function q changes somewhat with energy, it would be more accurate to put

$$N_p{}^i(\varepsilon_0,\ \varepsilon,\ s) = E_0\beta^{-1}\varphi_p{}^i(\varepsilon_0',\ \varepsilon',\ s),$$

where $\epsilon' = q(E_0,\ E,\ s)E/\beta$. Since $q(E_0,\ E,\ s)$ differs little from $q(s)$, the error arising when (16) is used has in first approximation the form

$$\Delta N_p{}^i/N_p{}^i = \varepsilon\frac{\partial\varphi_p{}^i(\varepsilon_0,\varepsilon,s)}{\partial\varepsilon}\Delta q/\varphi_p{}^i(\varepsilon_0,\varepsilon,s)q, \quad (21)$$

where $\Delta q = q(E_0,\ E,\ s) - q(s)$.

The results of the calculations of $\Delta N_p^1/N_p^1$ by means of formula (21) are given in the table. It is seen from it that when $0 \lesssim s \lesssim 2$ and in the region $0 \lesssim E \lesssim 0.5E_0$ formula (16) represents the solution of (2) with an error not larger than 20%. The next higher approximation for $N_p^1(\epsilon_0,\ \epsilon,\ s)$ can be obtained, just as in approximation A, by substituting in (6) $q = q_i(E_0,\ E,\ s)$ etc. Other cascade functions of the arguments E_0, E, s and also cascade functions of other boundary conditions can be easily expressed in terms of the integrals of (16). For example, in a shower due to a primary electron with energy E_0 we have

$$\{P(E_0,E,s)\}^p = -\frac{\{N_p(\varepsilon_0,\varepsilon,s)\}^p}{E} - \frac{\Gamma(s+1)}{\beta}\varepsilon_0\varepsilon e^{-\varepsilon_0}$$

$$\times \left[\frac{\partial F_2(s+1,2,\varepsilon)}{\partial\varepsilon}F_1(s+1,2,\varepsilon_0)\right.$$

$$\left. -\frac{\partial F_1(s+1,2,\varepsilon)}{\partial\varepsilon}F_2(s+1,2,\varepsilon_0)\right]. \quad (22)$$

Here

$$\frac{\partial F_1(s+1,2,\varepsilon)}{\partial\varepsilon} = \frac{s+1}{2}F_1(s+2,3,\varepsilon),$$

$$\frac{\partial F_2(s+1,2,\varepsilon)}{\partial\varepsilon} = \frac{s+1}{2}$$

$$\times [-F_2(s+1,2,\varepsilon) + sF_2(s+2,2,\varepsilon)].$$

When using the simplified expression for the cross section of the electron bremsstrahlung process in the form $W_e(E',\ E)dE' = dE'/E$, we obtain simple formulas for the photons:

$$\Gamma(E_0,E,s) = \frac{1}{E}\frac{N(E_0,E,\varepsilon)}{\lambda_1(s)+\sigma_0}, \quad (23)$$

$$N_\Gamma(E_0,E,s) = \frac{\Gamma(s+1)\varepsilon_0 e^{-\varepsilon_0}}{q_1(s)[\lambda_1(s)+\sigma_0]}\left[F_1(s\right.$$

$$+1,2,\varepsilon_0)\int_\varepsilon^{\varepsilon_0} F_2(s+1,2,\varepsilon')d\varepsilon'$$

$$\left. -F_2(s+1,2,\varepsilon_0)\int_\varepsilon^{\varepsilon_0} F_1(s+1,2,\varepsilon')d\varepsilon'\right] \quad (24)$$

To obtain the function $N_p(E_0,\ E,\ t)$ and also other functions of t with allowance for the ionization losses, it is necessary to take the inverse Laplace transform either approximately, analytically, or by numerical methods. Results of calculations of

Values of $|\Delta N_p^1/N_p^1|$ in per cent for $\epsilon_0 = 10^3$ and different ϵ and s.

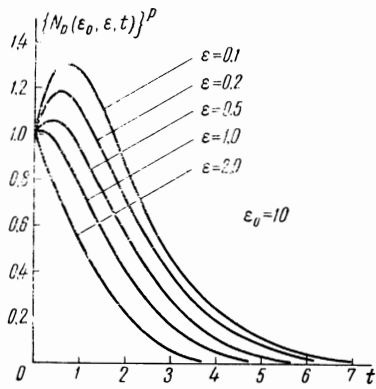| s	ε							
	10^{-3}	10^{-2}	10^{-1}	1	10	10^2	$5 \cdot 10^2$	$9 \cdot 10^2$
2.0	1	2	6	3	11	4	22	7
1.0	0.1	0.3	0.5	1	9	6	10	21
0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.02	—

FIG. 5. Dependence of the function $\{N_p(\epsilon_0, \epsilon, t)\}^P$ on the depth t in a shower from a primary electron with energy $\epsilon_0 = 10$.

$N_p(\epsilon_0, \epsilon, t)$ for several values of the arguments ϵ_0, ϵ, and t are shown in Fig. 5.

Thus, even in first approximation the proposed method of solution makes it possible to obtain, with acceptable accuracy, the Laplace transforms of the cascade functions of the arguments E_0, E, and s in the region $0 < s \lesssim 2$ and in a wide range of values of E_0 and E. If the inversion integral is calculated with sufficient accuracy, we obtain cascade functions for arbitrary values of the arguments E_0, E, and t. The limitations imposed on the region of variation of the arguments are determined by purely physical limitations of the region of appli-

cability of approximations A and B of the cascade theory.

In conclusion we note that the use of the approximate expression L[P] of the type (9) greatly simplifies the solution of many problems in the cascade theory: the construction of a theory with account of the dependence of the absorption coefficient of the photons on the energy, the solution of angular problems, and the solution of problems on the distribution of the particles in space.

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