THE PROPAGATION OF FOURTH SOUND IN HELIUM NEAR THE λ POINT

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Assuming that the velocity of the normal component of helium $V_n = 0$ we find equations for the hydrodynamics of a superfluid liquid, which take into account the relaxation of ρ_s near the λ point. We evaluate the velocity of the fourth sound and its absorption coefficient which is connected with the relaxation of ρ_s.

THE propagation of sound waves in superfluid helium has a number of interesting properties caused by the presence of two liquid components. The first sound waves corresponding to normal sound are connected with pressure (density) oscillations. In such a wave each volume element of the liquid oscillates as a whole; the normal and the superfluid masses move together. In the second sound wave the superfluid and normal masses of the fluid oscillate "in opposition" such that their center of mass in each volume element remains stationary. If, however, the situation is such that the normal part of the fluid is partly "pinned" and cannot freely take part in the oscillation, then both first and second sound will be strongly absorbed. Such a situation occurs when oscillations propagate in a porous medium saturated with liquid helium.[1, 2] In the limiting case where the normal part of the liquid is completely "stuck" to the walls of the channels, the wave corresponding to second sound, i.e., the temperature wave, is attenuated and the wave corresponding to first sound goes over into the weakly-damped fourth-sound wave in which both the pressure and the temperature oscillate.[1-7]

Such a situation occurs when the width of the channels along which the sound propagates is appreciably less than the penetration depth of the viscous wave. The penetration depth of the viscous wave is $\delta = (2\eta_n/\omega\rho_n)^{1/2}$, where η_n and ρ_n are, respectively, the viscosity and the density of the normal component of helium, ω the frequency of the oscillations. An estimate shows that near the λ point $\delta \approx 2.10^{-2}\,\omega^{-1/2}$ cm. For a sound frequency $\omega \leq 10^4$ sec^{-1} in channels with a width $d \leq 10^{-5}$ cm we can thus assume that the normal part of the liquid does not take part in the oscillations and that the velocity of the normal component of the liquid $V_n = 0$. Such a situation is realized in experiments on the propagation of fourth sound in porous substances saturated with superfluid helium.[4, 5, 7]

The purpose of the present work was a study of the propagation of fourth sound in helium near the λ point. When the λ point is approached, the density ρ_s of the superfluid component of the liquid tends to zero and the time taken by ρ_s to attain its equilibrium value increases steeply and this must lead to a damping of the fourth sound. Near the λ point the usual equations for the hydrodynamics of helium become unsuitable since they do not take into account the relaxation of ρ_s. In this temperature range the natural approach is the one used in the phenomenological theory of second order phase transitions. Such an approach was developed in a number of papers[8, 9] and we shall use them as our starting point.

1. EQUATIONS FOR THE HYDRODYNAMICS OF A SUPERFLUID LIQUID NEAR THE λ POINT

We describe the superfluid part of the liquid by a complex function $\psi(x, y, z, t) = \eta e^{i\varphi}$ defined in such a way that

$$\rho_s = m|\psi|^2, \qquad \mathbf{V}_s = \frac{\hbar}{m}\nabla\varphi. \tag{1.1}$$

The energy per unit volume of the liquid can, if we restrict ourselves to the first term in an expansion in the gradient of ψ, be written in the form

$$E = \frac{\hbar^2}{2m}|\nabla\psi|^2 + \varepsilon(\rho, S, |\psi|^2), \tag{1.2}$$

where ϵ is the internal energy per unit volume of the liquid. Minimizing the energy of the liquid with respect to ψ or ψ^*, we can obtain the equilibrium condition[8]

$$-\frac{\hbar^2}{2m}\Delta\psi + \left(\frac{\partial\varepsilon}{\partial\rho_s}\right)_{\rho,S}\psi = 0. \tag{1.3}$$

The imaginary part of (1.3) gives the condition

$$\operatorname{div}(\rho_s \mathbf{V}_s) = 0. \quad (1.4)$$

This condition follows also from the equation of continuity when $\mathbf{V}_n = 0$ and is, therefore, not an independent equilibrium condition. The actual equilibrium condition is thus the real part of (1.3). For small gradients of ρ_s it takes the simple form

$$\left(\frac{\partial \varepsilon'}{\partial \rho_s}\right)_{\rho S} + \frac{\mathbf{V}_s^2}{2} = 0. \quad (1.5)$$

The time-dependent equation for ψ can by analogy with quantum mechanics be written in the form

$$i\hbar\frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + \left[\left(\frac{\partial \varepsilon}{\partial \rho}\right)_{\rho_s S} + \left(\frac{\partial \varepsilon}{\partial \rho_s}\right)_{\rho, S}\right] m\psi + \hat{L}\psi, \quad (1.6)$$

where $\hat{L}$ is a non-Hermitean operator that describes the approach of ρ_s to its equilibrium value. Recognizing that the equilibrium condition for ρ_s is in fact the real part of (1.3) we can write $\hat{L}\psi$ in the form

$$-i\Lambda \operatorname{Re}\left\{-\frac{\hbar^2}{2m}\Delta\psi + \left(\frac{\partial \varepsilon}{\partial \rho_s}\right)_{\rho, S} m\psi\right\}. \quad (1.7)$$

Here Λ is a dimensionless coefficient which is inversely proportional to the relaxation time. Taking all this into account we can write the equation for ψ in the following form:

$$i\hbar\frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + \left[\left(\frac{\partial \varepsilon}{\partial \rho}\right)_{\rho_s S} + \left(\frac{\partial \varepsilon}{\partial \rho_s}\right)_{\rho, S}\right] m\psi - i\Lambda \operatorname{Re}\left[-\frac{\hbar^2}{2m}\Delta\psi + \left(\frac{\partial \varepsilon}{\partial \rho_s}\right)_{\rho, S} m\psi\right]. \quad (1.8)$$

To obtain the complete set of hydrodynamic equations we must add to (1.8) the conservation laws for mass and entropy:

$$\frac{\partial \rho}{\partial t} + \frac{i\hbar}{2}(\psi\Delta\psi^* - \psi^*\Delta\psi) = 0 \quad (1.9)$$

$$\frac{\partial S}{\partial t} + \operatorname{div}\left(\frac{\mathbf{q}}{T}\right) = \frac{R}{T}, \quad (1.10)$$

where the heat current $\mathbf{q}$ and the dissipation function of the liquid R must be chosen in such a way that the energy conservation law is satisfied. The momentum conservation law is not satisfied in our case as there is transfer of momentum from the superfluid part of the liquid to the normal part of the liquid which is "stuck" to the walls of the channel.

The energy conservation law requires that, if Eqs. (1.8) to (1.10) are taken into account, the relation

$$\partial E / \partial t + \operatorname{div}\mathbf{Q} = 0, \quad (1.11)$$

be identically satisfied; here $\mathbf{Q}$ is the energy current vector. Substituting (1.8) to (1.10) into (1.11) we get

$$\frac{\partial E}{\partial t} + \operatorname{div}\mathbf{Q} = R - \frac{2\Lambda}{\hbar} \times \left\{\operatorname{Re}\left[-\frac{\hbar^2}{2m}\Delta\psi + \left(\frac{\partial \varepsilon}{\partial \rho_s}\right)_{\rho, S} m\psi\right]\right\}^2 + \mathbf{q}\frac{\nabla T}{T}, \quad (1.12)$$

where $\mathbf{Q}$ is the energy current

$$Q_i = \frac{i\hbar}{2m}\frac{\partial \psi^*}{\partial x_i}\left\{-\frac{\hbar^2}{2m}\Delta\psi + \left[\left(\frac{\partial \varepsilon}{\partial \rho}\right)_{\rho_s S} + \left(\frac{\partial \varepsilon}{\partial \rho_s}\right)_{\rho, S}\right] m\psi\right\} + \text{c.c.} + q_i. \quad (1.13)$$

To satisfy the energy conservation law we must put the right-hand side of (1.12) equal to zero. Hence it follows that the dissipation function is

$$R = \frac{2\Lambda}{\hbar}\left\{\operatorname{Re}\left[-\frac{\hbar^2}{2m}\Delta\psi + \left(\frac{\partial \varepsilon}{\partial \rho_s}\right) m\psi\right]\right\}^2 - \mathbf{q}\frac{\nabla T}{T}. \quad (1.14)$$

Recognizing that R is positive we find

$$\mathbf{q} = -\varkappa \nabla T. \quad (1.15)$$

We now write down the equations which we have in this way finally obtained:

$$i\hbar\frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + \left[\left(\frac{\partial \varepsilon}{\partial \rho}\right)_{\rho_s, S} + \left(\frac{\partial \varepsilon}{\partial \rho_s}\right)_{\rho, S}\right] m\psi - i\Lambda \operatorname{Re}\left[-\frac{1}{2m}\Delta\psi + \left(\frac{\partial \varepsilon}{\partial \rho_s}\right)_{\rho, S} m\psi\right], \quad (1.16)$$

$$\frac{\partial \rho}{\partial t} + \frac{i\hbar}{2}(\psi\Delta\psi^* - \psi^*\Delta\psi) = 0, \quad (1.17)$$

$$\frac{\partial S}{\partial t} = \frac{1}{T}\operatorname{div}(\varkappa \nabla T) + \frac{1}{T}\frac{2\Lambda}{\hbar} \times \left\{\operatorname{Re}\left[-\frac{1}{2m}\Delta\psi + \left(\frac{\partial \varepsilon}{\partial \rho_s}\right)_{\rho, S} m\psi\right]\right\}^2 \quad (1.18)$$

For small gradients of ρ_s we get from Eqs. (1.16) to (1.18) the following set of equations:

$$\dot{\mathbf{V}}_s + \nabla\left\{\frac{\mathbf{V}_s^2}{2} + \left(\frac{\partial \varepsilon}{\partial \rho}\right)_{\rho_s, S} + \left(\frac{\partial \varepsilon}{\partial \rho_s}\right)_{\rho; S}\right\} = 0, \quad (1.19)$$

$$\frac{\partial \rho}{\partial t} + \operatorname{div}\rho_s\mathbf{V}_s = 0, \quad (1.20)$$

$$\frac{\partial S}{\partial t}=\frac{1}{T}\operatorname{div}(\varkappa\nabla T)+\frac{2\Lambda m}{\hbar}\left[\frac{\mathbf{V}_s^2}{2}+\left(\frac{\partial\varepsilon}{\partial\rho_s}\right)_{\rho,S}\right]^2\rho_s, \tag{1.21}$$

$$\frac{\partial\rho_s}{\partial t}+\operatorname{div}\rho_s\mathbf{V}_s=-\frac{2\Lambda m}{\hbar}\left[\frac{\mathbf{V}_s^2}{2}+\left(\frac{\partial\varepsilon}{\partial\rho_s}\right)_{\rho,S}\right]\rho_s. \tag{1.22}$$

In contradistinction to the usual equation of hydrodynamics, the quantity ρ_S is in these equations not assumed to be given but is an independent variable. Its approach to its equilibrium value is governed by Eq. (1.22). The coefficient Λ is determined from the absorption of first sound near the λ point. The expression $(\partial\epsilon/\partial\rho)_{\rho_S, S}+(\partial\epsilon/\partial\rho_S)_{\rho, S}$ is the chemical potential of the liquid written in terms of the variables (ρ, ρ_S, S).

The set of equations (1.19) to (1.22) could have been obtained from the equations derived by Pitaevskiĭ,[9] if we put in them $\mathbf{V}_n = 0$. However, in Pitaevskiĭ's paper the equation for $\mathbf{V}_S$ contains an extra term proportional to div $\rho_S(\mathbf{V}_S-\mathbf{V}_n)$ which arose from the incorrect use of the equilibrium condition (1.3) in the derivation of the equation for $\mathbf{V}_S$. We must also note that the equilibrium condition for ρ_S obtained by him when $\mathbf{V}_n \neq 0$ should contain an extra term $-(i\hbar/2\rho_S) \times \operatorname{div}\rho\mathbf{V}_n\psi$ arising when we take into account in the expression for the energy terms proportional to the spatial derivatives of $\mathbf{V}_n$.[1)]

2. FOURTH SOUND NEAR THE λ POINT

Using the hydrodynamical equations obtained in the preceding section we shall now consider the propagation of fourth sound. The characteristic length of the problem over which the wave function of the superfluid liquid changes appreciably is equal to[8] $l=\hbar[2m\alpha(T_\lambda-T)]^{-1/2}$, where $\alpha=\Delta c_p m(T_\lambda|\partial\rho_{sS}/\partial T|_\lambda)^{-1}$.

When $T_\lambda - T \sim 10^{-3}$ °K, $l\sim 10^{-6}$ cm. In order that the wavelength of the sound be of the order of l it is necessary to have a sound frequency $\omega\sim 10^{10}$ sec^{-1}. As the experiments are performed at much lower frequencies and the width of the channels in which the propagation of fourth sound is studied are $d\sim 5\times10^{-5}$ to 5×10^{-6} cm we can use the set of hydrodynamic equations valid for small gradients of ρ_S. Our considerations will be applicable for channels with a width d satisfying the following inequalities:

$$\hbar/\sqrt{2m\alpha(T_\lambda-T)}\ll d\ll(2\eta_n/\omega\rho_n)^{1/2}.$$

1)L. P. Pitaevskii has drawn the author's attention to this fact.

A numerical estimate gives $4\times10^{-8}(T_\lambda-T)^{-1/2}\ll d\ll 2\times10^{-2}\omega^{-1/2}$. All calculations will be valid for a temperature range for which $T_\lambda - T \ll T_\lambda$.

The propagation of sound in helium II can be described by the set of equations (1.19) to (1.22) which in our case can be linearized. After linearization the equations become of the form

$$\dot{\mathbf{V}}_s+\nabla\left\{\left(\frac{\partial\varepsilon}{\partial\rho}\right)_{\rho_s,\sigma}+\left(\frac{\partial\varepsilon}{\partial\rho_s}\right)_{\rho\sigma}\right\}=0, \tag{2.1}$$

$$\frac{\partial\rho}{\partial t}+\operatorname{div}\rho_s\mathbf{V}_s=0 \tag{2.2}$$

$$\frac{\partial(\rho\sigma)}{\partial t}=\frac{\varkappa}{T}\Delta T \tag{2.3}$$

$$\frac{\partial\rho_s}{\partial t}+\operatorname{div}\rho_s\mathbf{V}_s=-\frac{2\Lambda m}{\hbar}\left(\frac{\partial\varepsilon}{\partial\rho_s}\right)_{\rho,\sigma}\rho_s. \tag{2.4}$$

The equilibrium condition (1.5) for ρ_S takes in our case the form $\mu_2=(\partial\epsilon/\partial\rho_S)_{\rho,\sigma}=0$. We choose σ, ρ, and μ_2 as independent variables. Using (2.1) to (2.4) we can easily evaluate the non-equilibrium contribution to μ_2 caused by the propagation of the sound wave in the liquid,

$$\mu_2'=\frac{\hbar}{2\Lambda m\rho_s}\left[\left(\frac{\partial\rho_s}{\partial\rho}\right)_{\sigma,\mu_2}-\frac{\sigma}{\rho}\left(\frac{\partial\rho_s}{\partial\sigma}\right)_{\rho,\mu_2}-1\right]\frac{\operatorname{div}\rho_s\mathbf{V}_s}{1+i\omega\tau},$$
$$\tau=\left(\frac{\hbar}{2\Lambda m\rho_s}\right)\left(\frac{\partial\rho_s}{\partial\mu_2}\right)_{\sigma,\rho}, \tag{2.5}$$

where the meaning of τ is that of the relaxation time of the system.

The change in the chemical potential $\mu_1 = (\partial\epsilon/\partial\rho)_{\sigma,\rho_S}$ caused by the propagation of the sound wave is equal to

$$\mu_1'=\mu_{10}'+\left(\frac{\partial\mu_1}{\partial\mu_2}\right)_{\sigma,\rho}\mu_2', \tag{2.6}$$

where μ_{10}' is the change in the potential caused by the deviation of ρ and σ from their equilibrium values while $(\partial\mu_1/\partial\mu_2)_{\sigma,\rho}\,\mu_2'$ is the change in the chemical potential connected with the deviation of μ_2 from its equilibrium value. Substituting (2.5) and (2.6) into Eq. (2.1) we get

$$\dot{\mathbf{V}}_s+\nabla\{\mu_{10}'-\zeta\operatorname{div}\rho_s\mathbf{V}_s\}=0. \tag{2.7}$$

The coefficient of second viscosity ζ caused by the relation of ρ_S is equal to

$$\zeta=\frac{\hbar}{2\Lambda m\rho_s}\frac{1}{1+i\omega\tau}\left[\left(\frac{\partial\rho_s}{\partial\rho}\right)_{\sigma,\mu_2}-\frac{\sigma}{\rho}\left(\frac{\partial\rho_s}{\partial\sigma}\right)_{\rho,\mu_2}-1\right]^2 \tag{2.8}$$

In the low-frequency case of $\omega\tau\ll1$, which is of practical interest, we need not take into account the frequency dispersion of ζ. The dispersion of

ζ is important only for $\omega > 10^8 \text{ sec}^{-1}$ when $T_\lambda - T < 10^{-4}$ °K.

We eliminate $\mathbf{V}_s$ from (2.2) and (2.7) and use the thermodynamic identity

$$d\mu_1 = \rho^{-1}dP - \sigma dT. \tag{2.9}$$

We then obtain a set of two equations

$$\dot{\rho} + \rho_s\sigma\Delta\dot{T} - \frac{\rho_s}{\rho}\Delta\dot{P} - \zeta\rho_s\Delta\dot{\rho} = 0,$$

$$\sigma\dot{\rho} + \rho\dot{\sigma} - \frac{1}{T}\varkappa\Delta T = 0 \tag{2.10}$$

We can write the pressure P and the temperature T in the sound wave as the sum of constant equilibrium values and small additional terms varying as $\exp[i\omega(t - x/u)]$ (we choose the x axis in the direction of propagation of the wave; ω is the frequency and u the sound velocity). Substituting this form for P and T into (2.10) we get a set of algebraic equations.

The compatibility condition of this set of equations gives us an equation to determine u. If we neglect effects connected with thermal conductivity this equation is quadratic in u and we can write the square of the velocity of fourth sound in the form

$$u_4^2 = \frac{\rho_s}{\rho}u_1^2 + \frac{\rho_n}{\rho}u_2^2\left[1 - \frac{2\alpha_P}{\sigma}u_1^2\right] + i\omega\rho_s\zeta, \tag{2.11}$$

where u_1 is the first sound velocity, u_2 the second sound velocity, and α_P the expansion coefficient.

The imaginary part of the wave vector $k = \omega/u$ is equal to the sound absorption coefficient

$$\alpha = \operatorname{Im}\left(\frac{\omega}{u_4}\right) = \frac{\omega}{[u_4^4 + (\omega\rho_s\zeta)^2]^{1/2}} \times \left\{\frac{[u_4^4 + (\omega\rho_s\zeta)^2]^{1/2} - u_4^2}{2}\right\}^{1/2}. \tag{2.12}$$

In practice we have always $u_4^2 \gg \omega\rho_s\zeta$ so that the sound absorption coefficient has the form

$$\alpha = \omega^2\rho_s\zeta / 2u_4^3. \tag{2.13}$$

The second viscosity coefficient ζ has the following form in the variables P and T:

$$\zeta = \frac{\hbar}{2\Lambda m\rho}\left\{1 + \left[\frac{1}{\rho}\left(\frac{\partial\rho_s}{\partial T}\right)_P \frac{1}{\rho}\left(\frac{\partial\rho}{\partial T}\right)_P - \frac{c_P}{T}\left(\frac{\partial\rho_s}{\partial P}\right)_T + \frac{\sigma}{\rho}\left(\frac{\partial\rho_s}{\partial T}\right)_P \times \left(\frac{\partial\rho}{\partial P}\right)_T - \frac{\sigma}{\rho}\left(\frac{\partial\rho_s}{\partial P}\right)_T\left(\frac{\partial\rho}{\partial T}\right)_P\right] \times \left[\left(\frac{\partial\rho}{\partial P}\right)_T\frac{c_P}{T} - \frac{1}{\rho^2}\left(\frac{\partial\rho}{\partial T}\right)_P^2\right]^{-1}\right\}^2$$

It is clear from (2.13) that as the λ point is approached the fourth sound absorption coefficient increases steeply. A comparison of the experimental data on the absorption of fourth sound with the calculated value would give a possibility to determine the coefficient Λ appearing in the theory. Unfortunately, however, there are at present no such experimental data. Estimates using absorption of first sound data[10] lead to $\Lambda \approx 20$ to 35. Taking thermal conductivity into account leads in the fourth sound absorption coefficient to an additional term

$$\alpha_\varkappa = \frac{\omega^2}{2\rho u_4^3}\frac{\varkappa}{C}\left\{\frac{c_P}{c_v} - \left[1 + \frac{\rho_n}{\rho}\frac{u_2^2}{u_4^2}\left(1 - \frac{2\alpha_P}{\sigma}u_1^2\right)\right]^{-1}\right\},$$

where C is the specific heat per unit volume of the liquid. Because c_P and c_V differ little for liquid helium, α_κ is small compared with α.

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