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PLASMA INSTABILITY DUE TO PARTICLE TRAPPING IN A TOROIDAL GEOMETRY

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It is shown that toroidal devices with longitudinal magnetic fields are subject to an instability that is similar to the flute instability in a mirror device. This instability is due to particles which are trapped between mirrors, that is to say, regions of higher magnetic field.

INTRODUCTION

IT has been shown theoretically[1,2] and experimentally[3,4] that from the point of view of plasma confinement in mirror devices the most dangerous instability is flute instability which results from magnetic drift of charged particles in the inhomogeneous magnetic field. It appears from a hydrodynamic analysis[5,6] that in a toroidal geometry the flute instability can be stabilized quite easily by the introduction of "shear" associated with overlapping of the lines of force. The stabilization effect derives from the free motion of particles along the lines of force, making it easy to compensate the charges that arise by virtue of the magnetic drift; in this case perturbations in which the lines of force are not distorted and in which narrow localization occurs are found to be impossible.

However, the conclusion as to free interflow of the charges does not hold in a collisionless plasma. More precisely, this conclusion holds only for untrapped particles; it does not apply to particles that are trapped between mirror regions. It is precisely these latter particles which can be responsible for an instability which is similar to the flute instability in a mirror device; we have called this a "trapped-particle" instability.[7] Under these conditions the untrapped particles play the role of an environmental medium characterized by a high dielectric constant; they can only reduce the potential associated with the trapped particles but cannot eliminate it completely.

In the present work we shall analyze the trapped-particle instability in a toroidal system such as Tokomak. In Sec. 1 we consider briefly the equilibrium state and introduce a coordinate system convenient for the analysis. The motion of the charged particles is treated in Section 2 and in Sec. 3 we derive and investigate an integral equation for the potential; this equation plays the role of a dispersion relation for the determination of the characteristic oscillation frequency ω. Particle collisions are introduced in Sec. 4.

1. COORDINATE SYSTEM

In order to simplify the analysis we assume that the minor radius of the toroidal pinch a is much smaller than the major radius R_0. The quantity a/R_0 is then used as a small parameter. In order to investigate the oscillations it will be found convenient to introduce a special curvilinear coordinate system which becomes the usual cylindrical coordinate system when $R_0 \to \infty$. We denote the new coordinates by r, ϑ and ζ, also introducing the notation r′, φ' and z′ for the conventional cylindrical coordinates.

We assume that the toroidal pinch is obtained by bending a cylindrical pinch of circular cross-section. Then, as an approximation it can be assumed that the magnetic surfaces in the cross-section φ' = const are an ensemble of nested circles. The radius of these circles r is conveniently taken as one of the curvilinear coordinates so that the equation r = const specifies one of the magnetic surfaces. The second coordinate is taken to be the quantity ϑ, the azimuthal angle of the small circle (Fig. 1). The center of the circle r = const, which represents the cross-section of one of the magnetic

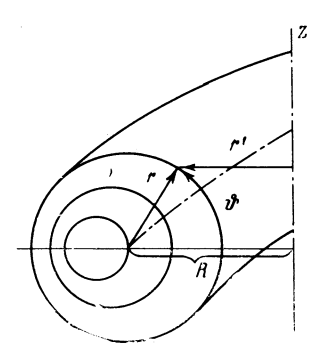

FIG. 1

surfaces is located at a distance $R = R_0 + \Delta(r)$ from the axis of symmetry where R_0 = const and the small quantity $\Delta(r)$ specifies the displacement of the magnetic surface due to the curvature. The third coordinate is taken to be the angle φ'; we can now write the following relations between the cylindrical coordinates r′, φ' and z′ and the coordinate system r, ϑ and ζ:

$$r' = R_0 - r\cos\vartheta, \quad z' = r\sin\vartheta, \quad \varphi' = \zeta, \qquad (1.1)$$

where, we have neglected $\Delta(r)$ because when $\epsilon = r/R \ll 1$, the quantity $\Delta(a)$ is small, being of order ϵ.

We note that the toroidal nature of the problem introduces the following modifications as compared with a straight pinch: in the first place, corrections of order ϵ which depend on the angle ϑ appear in the macroscopic quantities such as the pressure, magnetic field components and so on; second, and this is more important, there is a qualitative change in the motion of the charged particles. This change occurs because the toroidal system exhibits a variation of magnetic field along the line of force, that is to say, the system effectively acquires mirrors. As a result the particles can be divided into two groups: the trapped particles, which move between the mirrors, and the untrapped particles which are free to move along the entire system. It will be shown below that this effect is of order $\sqrt{\epsilon}$. For this reason, for small values of ϵ we retain the toroidal nature of the problem only when it has an important effect on the motion of the charged particles, that is to say, when it leads to effects of order $\sqrt{\epsilon}$.

When $\beta = 8\pi nT/H^2 \ll 1$ and $H_\vartheta^2/H^2 \ll 1$, the longitudinal magnetic field (in the sense corresponding to the straight pinch) can be written to order ϵ as follows:

$$H\zeta = H_0\left(1 + \frac{r}{R_0}\cos\vartheta\right), \qquad (1.2)$$

where H_0 is the value of the field at the magnetic axis. The azimuthal magnetic field can be written in the following form when $H_\vartheta \ll H_0$:

$$H_\vartheta = H_\vartheta^0(r) \qquad (1.3)$$

The quantity H_ϑ is related to H_0 by

$$H_\vartheta = H_0\frac{r}{R}\frac{1}{q} = H_0\varepsilon\frac{1}{q}, \quad q(r) = \frac{rH_0}{RH_\vartheta}, \qquad (1.4)$$

where q is the so-called reserve factor. In devices such as Tokomak the quantity q is of the order of several units in order to satisfy the Kruskal-Shafranov criteria. Thus, H_ϑ is small compared with H_0 and consequently corrections of order ϵ can be neglected in H_ϑ. The relations obtained above determine uniquely the geometry of the pinch and the corresponding coordinate system for any distribution of p and H_ϑ over the radius r.

2. DRIFT TRAJECTORIES

We shall now consider the motion of charged particles in a toroidal magnetic field. The results that are obtained will be used to integrate the kinetic equation and will also be useful in furnishing a physical interpretation of the trapped-particle instability itself.

The motion of a charged particle in a strong magnetic field can be described by the relation

$$\mathbf{r}(t) = \mathbf{r}_0(t) + \mathbf{n}\frac{v_\perp}{\Omega}\cos\alpha + \mathbf{b}\frac{v_\perp}{\Omega}\sin\alpha, \qquad (2.1)$$

where $\mathbf{r}_0(t)$ is the trajectory of the guiding center, **n** is the normal to a line of force, **b** is the binormal to the line of force, $v_\perp$ is the transverse velocity component and $\alpha = \alpha_0 - \Omega(t - t_0)$ is the azimuthal coordinate in velocity space.

The motion of the guiding center of the charged particle $\mathbf{r}_0(t)$ is described by the equation

$$\frac{d\mathbf{r}_0}{dt} = v_\parallel\mathbf{h} + \frac{c[\mathbf{h}\nabla\varphi_0]}{H} + \frac{Mc}{2eH^2}(v_\perp^2 + 2v_\parallel^2)[\mathbf{h}\nabla H]. \qquad (2.2)^*$$

Using the energy conservation relation

$$Mv^2/2 + e\varphi_0 = \text{const} \qquad (2.3)$$

and the conservation of the transverse magnetic moment

$$\mu = v_\perp^2/H = \text{const} \qquad (2.4)$$

we can now determine the motion completely.

As we have noted above, the most important effect that arises in the transition to a toroidal geometry is the appearance of trapped particles that oscillate between mirror regions. In order to treat these particles it is sufficient to retain the toroidal correction in the magnetic moment (2.4); the weak dependence of the factor $(1 + \epsilon\cos\vartheta)$ on ϵ can be neglected in the other terms. We limit ourselves to the case $\varphi_0 = 0$ and introduce the quantity ξ to denote the deviation from a line of force at a magnetic surface in accordance with the relation $\xi = \zeta - q\vartheta$; then, to order $\sqrt{\epsilon}$, we obtain the following system from (2.2):

$$\frac{d\vartheta}{dt} = \pm\frac{v}{R_0 q}\left[1 - \frac{\mu H_0}{v^2}\left(1 + \frac{r}{R_0}\cos\vartheta\right)\right]^{1/2}, \qquad (2.5)$$

$$\frac{dr}{dt} = \frac{1}{2\Omega R_0}(v^2 + v_\parallel^2)\sin\vartheta, \qquad (2.6)$$

*$[\mathbf{h}\nabla\varphi_0] \equiv \mathbf{h}\times\nabla\varphi_0$.

$$\frac{d\xi}{dt} = -\frac{1}{2\Omega R_0 r}(v^2 + v_\parallel^2)(q\cos\vartheta + q'r\vartheta\sin\vartheta), \quad (2.7)$$

where $\Omega = eH_0/Mc$ and $q' = dq/dr$.

The deviation Δr from the line of force is small and the quantity r in (2.5) can be regarded as a constant. As a result the equation for the longitudinal motion can be solved independently of (2.6) and (2.7). We now introduce the spherical coordinate system v, ψ_π and α in velocity space at the point $\vartheta = \pi$. Then $v_\perp^2/v^2 = \mu H_0(1-\epsilon)/v^2 = -\sin^2\psi_\pi$ and consequently (2.5) assumes the form

$$\frac{d\vartheta}{dt} = \pm\frac{v}{R_0 q}[\cos^2\psi_\pi - \varepsilon\sin\psi_\pi^2(1+\cos\vartheta)]^{1/2}, \quad (2.8)$$

where $\epsilon = r/R_0$. It is then evident that for small values of $\cos^2\psi_\pi$ the expression in the radical can vanish for certain values of ϑ, that is to say, the particles are reflected from the magnetic mirror. We now introduce the supplementary angle $\gamma_\pi = \pi/2 - \psi_\pi$. Since γ is small for the trapped particles, as an approximation we can write $\cos^2\psi_\pi \approx \psi_\pi^2$ and $\sin^2\psi_\pi \approx 1$ for these particles. Now, introducing the new variable $\kappa^2 = \gamma_\pi^2/2\epsilon$ we can write (2.8) for particles with small longitudinal velocity in the form

$$\frac{d\vartheta}{dt} = \pm\frac{v\sqrt{\varepsilon}}{R_0 q}\sqrt{2\varkappa^2 - 1 - \cos\vartheta}. \quad (2.9)$$

The turning point $\vartheta = \vartheta_0(\kappa)$ is determined from the condition $1 + \cos\vartheta_0 = 2\kappa^2$. A turning point exists when $\kappa < 1$. Hence, the value $\kappa = 1$ separates the untrapped particles from the trapped particles. The oscillation period of the trapped particles τ can be found by means of (2.9):

$$\tau = 4\frac{R_0 q}{v\sqrt{\varepsilon}}\int_{\vartheta_0}^{\pi}\frac{dv}{\sqrt{2\varkappa^2 - 1 - \cos\vartheta}} = 4\frac{R_0 q}{v\sqrt{\varepsilon}}\sqrt{2}K(\varkappa), \quad (2.10)$$

where K is a complete elliptic integral of the first kind.

In treating the untrapped particles ($\kappa > 1$) it will be convenient to introduce the quantity

$$\tau = 4\frac{R_0 q}{v\sqrt{\varepsilon}}\int_0^{\pi}\frac{d\vartheta}{\sqrt{2\varkappa^2 - 1 - \cos\vartheta}} = 4\frac{\sqrt{2}R_0 q}{v\sqrt{\varepsilon}}K\left(\frac{1}{\varkappa}\right), \quad (2.10a)$$

which has the meaning of the time required for two complete revolutions of the angle ϑ. The convenience of this definition of τ for untrapped particles will be made clear below when we go from integration over time to integration over the angle ϑ [cf. (3.11)], in which case the angle ϑ can be regarded as a continuous variable in the transition from trapped particles to untrapped particles.

We now introduce the notation $\omega_0 = 2\pi/\tau$. For large values of κ the particles move essentially freely along the magnetic field and $\omega_0 \approx v\sqrt{2\epsilon\kappa}/2R_0 q \approx v\gamma/2R_0 q$. When $\kappa \to 1$ the rotational frequency of the particles along ϑ goes to zero and the particles are trapped, in which case ω_0 assumes the significance of the angular frequency of the oscillations between mirrors.

It is evident that the existence of an inhomogeneity along the magnetic field causes a substantial modification of the particle motion. Specifically, in a straight field with slow waves $\omega/k_z \ll v_i$ the particles can achieve a Boltzmann equilibrium by virtue of their ability to move freely along the lines of force; in the present case, however, there will be a group of trapped particles with small longitudinal velocity $v_\parallel/v < \sqrt{2\epsilon}$ which do not move along the field on the average, that is to say, these particles do not achieve a Boltzmann distribution for very slow waves $\omega/k_z \ll v_i$. It is this lack of equilibrium which gives rise to the trapped particle instability. In order to investigate this instability it is necessary to treat the transverse motion in addition to the longitudinal motion. Assuming that $v_\parallel < v$ for the trapped particles, from (2.6) we have

$$\Delta r = \pm\int\frac{qv}{2\Omega\sqrt{\varepsilon}}\frac{\sin\vartheta d\vartheta}{\sqrt{2\varkappa^2 - 1 - \cos\vartheta}} = \pm\frac{vq}{\Omega\sqrt{\varepsilon}}\sqrt{2\varkappa^2 - 1 - \cos\vartheta}. \quad (2.11)$$

In moving along the magnetic field ($v_\parallel > 0$) the ions drift outward from the magnetic surface; ions moving in the opposite direction move inward. The quantity Δr is of opposite sign for the electrons. It follows from (2.11) that the ion deviation along the radius is of order $\Delta r \sim p_i q/\sqrt{\epsilon}$. We assume that this quantity is appreciably smaller than a, for otherwise a significant fraction of the ions can escape to the walls even in the absence of collisions or instability.

For the untrapped particles with $\kappa \gg 1$ we can assume $v_\parallel = \text{const}$; then, from (2.5) and (2.6) we have

$$\Delta r = -\frac{q}{2\Omega_0 v_\parallel}(v^2 + v_\parallel^2)\cos\vartheta. \quad (2.12)$$

It is then evident that the displacement of the untrapped particles is approximately $\sqrt{R/r}$ times smaller than that of the trapped particles.

In what follows the quantity $\langle\Delta\xi\rangle$ denotes the mean displacement of the charged particles per period along the pinch (i.e., along ζ). In accordance with (2.5) and (2.6) this displacement can be written in the form

$$\langle \Delta\xi \rangle \cong \frac{1}{\Omega}\frac{q}{r}\frac{\partial J_{\|}}{\partial r}; \qquad (2.13)$$

where

$$J_{\|} = \oint dlv_{\|} = 4qR_0\sqrt{\varepsilon}\, v \int_{\vartheta_0}^{\pi} \sqrt{2\varkappa^2 - 1 - \cos\vartheta}\, d\vartheta$$

$$= 8\sqrt{2\varepsilon}\, qR_0 v[E(\varkappa) - (1-\varkappa^2)K(\varkappa)], \qquad (2.14)$$

and E is a complete elliptic integral of the second kind. The quantity $J_{\|}$ represents the longitudinal invariant. We note that a relation such as (2.13) can be obtained in general form for any quasi-periodic motion.[9]

Carrying out the differentiation in (2.13) and taking account of (2.10) we find v_ζ, the mean drift velocity of the charged particles along ζ:

$$v_\zeta = \frac{\langle \Delta\xi \rangle}{\tau} = \frac{v^2 q \varepsilon}{\Omega r^2} G(\varkappa), \qquad (2.15)$$

where

$$G(\varkappa) = G_1(\varkappa) + \frac{2q'r}{q} G_2(\varkappa)$$

$$\equiv \left(\frac{E(\varkappa)}{K(\varkappa)} - \frac{1}{2}\right) + \frac{2q'r}{q}\left(\frac{E(\varkappa)}{K(\varkappa)} - 1 + \varkappa^2\right). \qquad (2.16)$$

The dependence of v_ζ on κ is shown in Fig. 2. When $\kappa > 1/\sqrt{2}$ the reflection points for the trapped particles lie in the region $\vartheta < \pi/2$, that is to say, in regions in which the magnetic field increases outward.

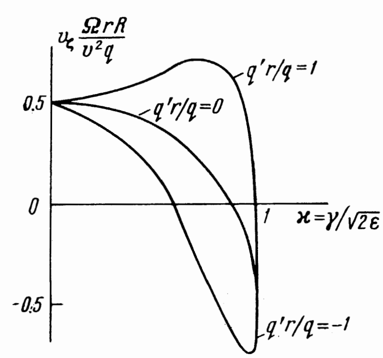

FIG. 2

For small values of κ the particle will be found close to the outer contour of the magnetic surface [between the points $\cos\vartheta_0 = -(1 - 2\kappa^2)$], i.e., in regions in which the magnetic field falls off in the outward direction. In this region, if $q' = 0$ the particle executes a drift which is unfavorable from the point of view of stability ($v_\zeta > 0$). But as the quantity κ increases the velocity is reduced and when $\kappa_1 = 0.9$ ($\cos\vartheta_0 = 0.66$) the velocity changes sign (Fig. 2) and the corresponding particles spend a large time fraction in the region $\vartheta < \pi/2$ in which the magnetic field increases in going outward from the magnetic surface. When $q' \neq 0$ the point κ_1 at which v_ζ vanishes is displaced into the region of large κ for $q'r/q > 0$ and into the region of small κ (i.e., the region of favorable magnetic drift is expanded) when $q'r/q < 0$.

3. DISPERSION RELATION FOR ELECTROSTATIC OSCILLATIONS

We can now proceed with our investigation of the instability of an inhomogeneous plasma in a toroidal geometry. We first consider the case of a collisionless plasma and assume that the oscillations are electrostatic. Under these conditions the kinetic equation for small oscillations can be written in the form

$$\frac{\partial}{\partial t} f' + \mathbf{v}\nabla f' + \frac{e}{M}\nabla\varphi_0 \frac{\partial}{\partial \mathbf{v}} f' + \frac{e}{Mc}[\mathbf{vH}]\frac{\partial}{\partial \mathbf{v}} f'$$

$$= -\frac{e}{M}\nabla\varphi' \frac{\partial}{\partial \mathbf{v}} f. \qquad (3.1)$$

Here, the equilibrium distribution function f satisfies the equation

$$\mathbf{v}\nabla f + \frac{e}{M}\nabla\varphi_0 \frac{\partial}{\partial \mathbf{v}} f + \frac{e}{Mc}[\mathbf{vH}]\frac{\partial}{\partial \mathbf{v}} f = 0, \qquad (3.2)$$

where φ_0 is the unperturbed potential of the electric field, φ' is the perturbation of the electric field, **H** is the unperturbed magnetic field, and f′ is the perturbation of the distribution function.

In the equilibrium state φ_0 is a function of the magnetic surface, i.e., a function of the variable r. In converting to a coordinate system that moves along ζ it is evident that $\nabla\varphi_0$ will vanish. In the analysis of the localized perturbations treated here we can assume that the translational velocity is independent of r so that the term with $\nabla\varphi_0$ in (3.1) and (3.2) can be neglected.

We assume further that the equilibrium distribution function f is approximately a Maxwellian f_0 in which the density n and temperature T depend on r. Then, to first-order accuracy in Ω^{-1}, from (3.2) we have

$$f = f_0 - \frac{1}{\Omega}[\mathbf{hv}]\nabla f_0. \qquad (3.3)$$

The solution of (3.1) is

$$f' = -\frac{e}{M}\int_{-\infty}^{t} \nabla\varphi' \frac{\partial f}{\partial \mathbf{v}} dt', \qquad (3.4)$$

where the integration is taken over the unperturbed trajectory (2.1). By virtue of the periodicity in ζ and ϑ, the functions φ and f′ can be written in the form

$$\varphi' = \exp(-i\omega t + im\vartheta - il\zeta)\,\varphi(r, \vartheta), \qquad (3.5)$$

where $\varphi(r, \vartheta)$ is a periodic function of ϑ on which we impose the requirement that it have the minimum number of nodes along ϑ. Substituting (3.3) and (3.5) in (3.4) and carrying out the usual calculations[10] it is a simple matter to obtain an expression for the perturbed particle density; then, using the quasi-neutrality condition, i.e., equilibrating the perturbed electron and ion densities, we can find the dispersion relation for the frequency of small oscillations ω:

$$\left(\frac{1}{T_e}+\frac{1}{T_i}\right)n^0\varphi = -i\sum_{j=e,i}\int_{-\infty}^{0}\int dk\exp\{-i\omega t + ik(r_j(t')-r)+im(\vartheta_j'-\vartheta)-il(\xi_j'-\zeta)\} \times J_0^2\left(\frac{k_\perp v_\perp}{\Omega}\right)\frac{1}{T_j}f_{0j}(\omega-\omega_j^*)\varphi dt'd\mathrm{v}. \quad (3.6)$$

Here, the summation on the right side is carried out over electrons and ions; $e_i = e$, $e_e = -e$, J_0 is the Bessel function, and

$$\omega_j^* = \frac{cT_jm}{e_jHr}\frac{1}{f_{0j}}\frac{d}{dr}f_{0j}$$

is the drift frequency for charges of the appropriate species. The relation in (3.6) is written for low-frequency oscillations $\omega \ll \Omega_i$. We also assume that $m \gg 1$ so that we have neglected $\partial\varphi/\partial\vartheta'$ compared with $m\varphi'$. The relation in (3.6) differs from the corresponding expression in cylindrical geometry in that the particle motion along the unperturbed trajectory is more complicated. The relation in (3.6) is a homogeneous integral equation and ω is an eigenvalue. The integration on the right side of (3.6) is carried out over the coordinates of the guiding center (since we have already carried out an averaging over the fast cyclotron gyration of the particles) and in order to write (3.6) in explicit form it is necessary to use the drift trajectories derived in Section 2.

From the point of view of macroscopic effects the most dangerous perturbations are the large scale perturbations; hence we assume that the localization of the perturbation over r is appreciably greater than Δr the amplitude of oscillation of the particles in the radial direction in the unperturbed drift; hence we can write $J_0 = 1$ in (3.6). On the other hand, we assume that φ is well localized as compared with a so that $\omega = \omega(r)$ is the local value for the eigenfrequency. We first consider the integral on the right side of (3.6) with respect to t′ for the untrapped particles. For these particles $\vartheta' - \vartheta \approx v_{||}t/Rq$, $\zeta' - \zeta \approx q(\vartheta' - \vartheta)$ and consequently the integration over dt′ leads to a factor of the form $i\{\omega - (m - lq)v_{||}/qR\}^{-1}$. In the oscillations being considered here $\omega \le \omega^* \sim m\rho_i v_i/a^2$ and this factor is appreciably smaller than ω^{-1} even when $m - lq \ll 1$ (but $m - lq \gg \rho_i m/a$); hence, the contribution in (3.6) due to the untrapped particles can be neglected compared with the term of order unity on the left side.

In computing the integral over t′ for the trapped particles we take $\zeta_j' - \zeta = q(\vartheta_j' - \vartheta) + \xi_j'$. Then, in the exponential factor that multiplies $\vartheta_j' - \vartheta$ we have the factor $(m - lq)$. The value of m has not yet been determined. It is evident that for any value of l we can take that value of m for which the difference $m - lq$ is less than 1/2. This choice means that we have taken φ to be a function which has the minimum number of nodes along ϑ. We shall first consider the simplest case $m - lq \ll 1$. Under these conditions the quantity $(m - lq)(\vartheta' - \vartheta)$ is a periodic function of t′ for the trapped particles and can be neglected since it is of the order of $(m - lq)$. In addition, neglecting the quantity $l(\xi' - \langle\xi\rangle)$ over one oscillation period, we can average over the period τ_j and write $\xi_j' \approx \langle\xi_j'\rangle = v_{\zeta j}t$ [where $v_{\zeta j}$ is defined by (2.15)]. Then (3.6) assumes the form

$$\left(\frac{1}{T_e}+\frac{1}{T_i}\right)n^0\varphi = \sum_j\frac{1}{T_j}\int d\mathrm{v}f_{0j}\frac{(\omega-\omega_j^*)}{\omega+lv_{\zeta j}}\int_{-\tau_j}^{0}\varphi'\frac{dt'}{\tau_j} \quad (3.7)$$

Here, instead of integrating over t′ we have carried out an integration over ϑ', writing [as follows from (2.9) and (2.10)]

$$\frac{dt'}{\tau} = \frac{1}{4\sqrt{2}K(\varkappa)}(r\varkappa^2 - 1 - \cos\vartheta')^{-1/2}d\vartheta'.$$

Furthermore, the quantity dv for the trapped particles with $\epsilon \ll 1$ is

$$d\mathrm{v} = 2\pi v^2 dv d\gamma_\vartheta, \quad (3.8)$$

where $\gamma_\vartheta = \pi/2 - \psi$ is the angle in velocity space at the point ϑ. By virtue of the conservation of the transverse adiabatic invariant we have

$$v_\perp^2/v^2 = \cos^2\gamma_\vartheta = \cos^2\gamma_\pi - \varepsilon(1+\cos\vartheta), \quad (3.9)$$

where γ_π is the angle in velocity space at the point $\vartheta = \pi$ that we have introduced earlier. According to (3.9) for small ϵ we have $\gamma_\vartheta^2 = \gamma_\pi^2 - \epsilon(1 + \cos\vartheta)$. Whence

$$d\gamma_\vartheta = \frac{\gamma_\pi d\gamma_\pi}{\gamma_\vartheta} = \frac{\sqrt{\varepsilon}\,d\varkappa^2}{\sqrt{2\varkappa^2 - 1 - \cos\vartheta}}. \quad (3.10)$$

Substituting the expressions that have been obtained for dt′ and $d\gamma_\vartheta$ in (3.7) we obtain an integral equation for the potential φ:

$$\left(\frac{1}{T_e}+\frac{1}{T_i}\right)\varphi = \sum_j\frac{1}{T_j}\sqrt{2\varepsilon}\int 2\pi v^2 dv\frac{f_{0j}}{n^0}$$

$$\times \int_{(1+\cos\vartheta)/2}^{1} \frac{d\varkappa^2}{K(\varkappa)\sqrt{2\varkappa^2-1-\cos\vartheta}}$$
$$\times \frac{(\omega-\omega_j^*)}{(\omega+lv_{\zeta j})} \int_{\vartheta_0}^{\pi} \frac{\varphi(\vartheta')\,d\vartheta'}{\sqrt{2\varkappa^2-1-\cos\vartheta}}. \qquad (3.11)$$

If the electron and ion temperatures are equal $T_e = T_i$ the electron and ion drift velocities $v_{\zeta e}$ and $v_{\zeta i}$ for a given energy $m_j v^2/2$ are equal in magnitude and opposite in sign. This result leads to a simplification of the equation:

$$\varphi = 2\pi\sqrt{2\varepsilon}\int_0^{\infty} v^2 dv \frac{f_0}{n^0}$$
$$\times \int_{(1+\cos\vartheta)/2}^{1} \frac{d\varkappa^2}{K(\varkappa)\sqrt{2\varkappa^2-1-\cos\vartheta}} \frac{(\omega^2-\omega^* lv_\zeta)}{(\omega^2-l^2v_\zeta^2)}$$
$$\times \int_{\vartheta_0}^{\pi} \frac{\varphi(\vartheta')\,d\vartheta'}{\sqrt{2\varkappa^2-1-\cos\vartheta}}. \qquad (3.12)$$

The integral equation (3.12) does not contain the particle mass; it is also independent of the sign of the charge.

4. TRAPPED-PARTICLE INSTABILITY

The existence of trapped particles leads to the possible appearance of an instability of the flute type. The trapped particles located in a given force tube between the magnetic mirrors are completely isolated from the other regions of the plasma and are consequently analogous to trapped particles in a conventional mirror machine. In general these particles execute an unfavorable magnetic drift in a magnetic field that diminishes toward the periphery; hence, as in the usual magnetic trap, a small perturbation of the flute type will lead to a charge separation which reinforces the initial perturbation. The only difference from the linear mirror machine lies in the fact that in the toroidal geometry the flutes of trapped particles are immersed in a plasma containing untrapped particles which, by virtue of their large longitudinal dielectric constant $\epsilon^0 = 1 + 8\pi ne^2/Tk_{\|}^2 \gg 1$, compensate the charge associated with the trapped particles to a large degree. However, since $\epsilon^0 \neq \infty$ total neutralization cannot be achieved and a highly retarded finite instability due to the trapped particles will develop in the plasma.

In order to demonstrate this result we consider the integral equation in (3.12). In place of ω we introduce the growth rate γ: $\omega = i\gamma$. In this case the denominator $\omega^2 - l^2v_\zeta^2 = -(\gamma^2 + l^2v_\zeta^2)$ becomes a monotonically increasing function of γ. We recall that v_ζ is a small quantity of order ϵ. Since the expression on the right side of (3.12) contains a small factor $\sqrt{\epsilon}$ it can be of order unity if the denominator $\gamma^2 + l^2v_\zeta^2$ is small enough. In this case we can neglect γ^2 in the numerator compared with $\omega^* l v_\zeta$. Thus, in order-of-magnitude terms, we have from (3.12)

$$1 \sim \sqrt{\varepsilon}\,\frac{\varepsilon\omega^*}{\gamma^2+\varepsilon^2\omega^{*2}},$$

where we have exploited the smallness of $lv_\zeta \sim \epsilon\omega^*$. It is then apparent that $\gamma^2 \sim \epsilon^{3/2}\omega^{*2}$ so that the quantity $\epsilon^2\omega^{*2}$ in the denominator of (3.12) can be neglected compared with γ^2 if ϵ is small enough.

In solving (3.12), in addition to using the Fourier representation it will be convenient to expand $\varphi(\vartheta)$ in a series in $\cos\vartheta$. We write

$$\varphi(\vartheta) = \sum_s e^{-is\vartheta}\varphi_s, \qquad (4.1)$$

where the $\varphi_s = \varphi_{-s}$ are real coefficients. Substituting this expression in (3.12) and neglecting small terms in the numerator and denominator, as indicated above, after integration over ϑ we find

$$\lambda(1+\delta_{0s})\varphi_s = \sum_{s'} F_{ss'}\varphi_{s'}, \qquad (4.2)$$

where

$$\lambda = \frac{2\pi}{3}\frac{\gamma^2}{\sqrt{2\varepsilon}\,\omega_M\omega_p^*}, \quad \omega_M = \frac{-2\,lq\,\varepsilon cT}{eHr^2} \cong \frac{-2mcT}{eHrR_0},$$
$$\omega_p^* = \frac{c}{eH}\frac{m}{rn^0}\frac{d}{dr}n^0T,$$

while the coefficients $F_{ss'}$ are given by the relations

$$F_{ss'} = F^1_{ss'} + \frac{2q'r}{q}F^2_{ss'}, \quad F^i_{ss'} = \int_0^1 \frac{G_i\Pi_s\Pi_{s'}}{K(\varkappa)}d\varkappa^2. \qquad (4.3)$$

The expression for G is given in Sec. 2, (2.16), while the function

$$\Pi_s = \int_{\vartheta_0}^{\pi} \frac{\cos s\vartheta\,d\vartheta}{\sqrt{2\varkappa^2-1-\cos\vartheta}} \qquad (4.4)$$

is expressed in terms of complete elliptic integrals of the first and second kind.

The eigenvalues λ are found from the condition that the determinant of the matrix corresponding to (4.2) must vanish. The values of the matrix elements $F^i_{ss'}$ for s, s′ ≤ 2 are given in the table.

It is evident that $F^i_{ss'}$ falls off rapidly with increasing s. Hence, in computing the largest value of λ (corresponding to the most unstable mode) it is valid to consider a system of finite order, taking $\varphi_s = 0$ for s larger than some value s_0. Limiting the system to an equation of second order, for the case q′ = 0, we find $\lambda = 0.74$ (if we solve the third-order system, then $\lambda = 0.76$). The second root is

	$F^1_{ss'}$				$F^2_{ss'}$		
s'	s = 0	1	2	s'	s = 0	1	2
0	0.681	—0.543	0.011	0	0.888	—0.145	—0.189
1	—0.543	0.380	—0.151	1	—0.145	0.128	0.029
2	0.011	—0.151	0.161	2	—0.189	0.029	0.091

close to zero and lies essentially at the limits of applicability of the analysis. Recalling the definition of λ and determining the numerical value we have

$$\gamma^2 \approx \sqrt{\varepsilon \omega_M \omega_p^*}\, / \, 2. \tag{4.5}$$

Using (4.2) we can find the expansion coefficients of $\varphi(\vartheta)$ in the Fourier series. When $q' = 0$ to accuracy of $\cos 2\vartheta$ we have

$$\varphi(\vartheta) = 1 - 1.4 \cos \vartheta + 0.4 \cos 2\vartheta. \tag{4.6}$$

It is evident that φ is localized in the region $\vartheta = \pi$. In other words the oscillations that develop at the outer contour of the torus and the oscillation amplitude tends to zero as $\vartheta \to 0,\ 2\pi$. Since $\varphi(\vartheta = 0) = 0$, m can be assumed close to lq even if lq (and consequently m) is not an integer. Thus, the solution obtained here extends to all values of r if $m \cong lq$.

We now consider the effect of a change in q' on the instability. As we have mentioned above, increasing q' leads to a strong instability and when $q' < 0$ there is a stabilization effect associated with the shortening of the segment of the line of force between the turning points in the outward motion.

Writing $\lambda = 0$ it is easy to find the critical value for the parameter $q'r/q$ below which the oscillations are stabilized. This value is found to be -1.5 so that the stabilization condition for the trapped-particle instability becomes

$$\frac{d \ln q}{d \ln r} < -\frac{3}{2}. \tag{4.7}$$

We now consider (3.12). We recall that in going to the simpler equation (4.2) we have neglected the quantity $l^2 v_\zeta^2$ of order ϵ^2 as compared with $\gamma^2 \sim \epsilon^{3/2}$. But since γ^2 is proportional to dp/dr while v_ζ is independent of the pressure, then even for small ϵ this procedure is not always valid; since γ^2/v_ζ^2 can only be $\sim \sqrt{\epsilon}$ the transition from (3.12) to (4.2) may not be valid even for modest pressure gradients. It is evident that the growth rate is reduced as dp/dr is reduced and that it vanishes for some critical value of dp/dr. In order to find this critical value of the gradient we write $\omega = 0$ in (3.12); then, again using the Fourier representation we find

$$\mu(1 + \delta_{0s})\varphi_s = \sum_{s'} P_{ss'}\varphi_{s'}. \tag{4.8}$$

Here

$$\mu = \frac{\pi}{\sqrt{2\varepsilon}}\left(\frac{d \ln n/T}{d \ln r}\right)^{-1}, \quad P_{ss'} = \int_0^1 \frac{\Pi_s \Pi_{s'}}{K(\varkappa) G(\varkappa)} d\varkappa^2, \tag{4.9}$$

where the integral over $d\kappa^2$ is taken in the sense of the principal value. It is evident that when $|rq'/q| \to \infty$ the matrix elements $P_{ss'}$ diminish. The largest value of μ corresponds to $rq'/q \sim 1$. We have also computed the roots μ for $rq'/q = 1/2$. For this value $P_{00} = 11.9$, $P_{01} = 2.02$, $P_{11} = 2.24$, whence $\mu_1 = 6.5$ and $\mu_2 = 1.7$. Taking the larger value and substituting in (4.9) we find the stability criterion

$$-\frac{(1-\eta)}{(1+\eta)} \frac{d \ln p}{d \ln r} < \frac{1}{3} \sqrt{\frac{r}{R_0}} \tag{4.10}$$

where $\eta = d \ln T / d \ln n$. In practice this condition can only be satisfied in separate narrow ranges of r.

5. DISSIPATIVE TRAPPED-PARTICLE INSTABILITY

The trapped-particle instability is very sensitive to collisions. As a result of collisions trapped particles can be scattered into the untrapped cone in velocity space, that is to say, the corresponding perturbation will be damped with some effective damping rate ν_{ef}. When $\epsilon \ll 1$ the angle γ in velocity space which separates the region of trapped particles from the untrapped particles is very small, being $\sim \sqrt{\epsilon}$. Correspondingly, the fraction of trapped particles is also small. Hence, in the collision integral in the Landau form for the trapped particles we need only keep the term with the distribution function so that as an approximation we can write it in the diffusion form

$$\mathrm{St}_j \approx \nu_j v_j^2 \Delta_\vartheta f', \tag{5.1}$$

where Δ_ϑ is the Laplacian in velocity space, ν_j is the collision frequency for particles of species j and $v_j^2 = 2T_j/M_j$. In this collision integral the largest term is the one which contains the second derivative with respect to γ and in order-of-magnitude terms $\mathrm{St} \sim f'\nu_j/\gamma^2 \sim f'\nu_j/\epsilon$. In other words $\nu_{ef} = \nu/\epsilon$.

In (3.1) for f′ if we add the collision integral and write it in the form $\nu_{ef}f'$, it is evident that in the denominator of the dispersion equation (3.7) the quantity ω will be replaced by $\omega + i\nu_{ef}$. In a tenuous plasma the untrapped particles can be described by a Boltzmann distribution; thus, taking account of collisions we can write the dispersion equation in the approximate form

$$2 = \sqrt{\varepsilon}\frac{\omega - \omega^*}{\omega + i\nu_i/\varepsilon - \omega_M} + \sqrt{\varepsilon}\frac{\omega + \omega^*}{\omega + i\nu_e/\varepsilon + \omega_M}. \quad (5.2)$$

Here, the terms on the right side which take account of the trapped particle contribution contain the factor $\sqrt{\epsilon}$ which is equal to the fraction of trapped particles while ω_M is the magnetic drift frequency; the factor $1/\epsilon$ that multiplies ν_e and ν_i takes account of the diffusional nature of the Coulomb collisions. For simplicity we take $T_e = T_i$. If $\nu_{ef} \gg \omega$, in (5.2) we can neglect the frequency ω compared with the frequency ω^*. Furthermore, we can neglect the magnetic drift completely, in which case we obtain the following expression for the frequency and growth rate:

$$\omega = \frac{\sqrt{\varepsilon}}{2}\omega^* + i\frac{\varepsilon^2}{4}\frac{(\omega^*)^2}{\nu_e} - i\frac{\nu_i}{\varepsilon}. \quad (5.3)$$

It is evident that there is an instability due to the trapped particles regardless of the sign of curvature of the lines of force (i.e., ω_M). This instability will be called the dissipative trapped-particle instability.

As ν diminishes the growth rate increases, reaching a maximum value $\gamma_{max} \sim \sqrt{\epsilon\omega^*}$ when $\nu \sim \epsilon^{3/2}\omega^*$; then it falls off. In a dense plasma, that is to say, if the collision frequency is high, the trapped-particle instability is stabilized, as is evident from (5.3). The stabilization condition is

$$\nu_i\nu_e > \varepsilon^3(\omega^*)^2/4. \quad (5.4)$$

It is evident that the long wave perturbations with small m are stabilized first. As $k_z = (m - lq)/qR$ is reduced, we must take account of the untrapped particles in the expression for the perturbed density and the analysis becomes the analysis of the drift-dissipative instability (cf.[11]).

CONCLUSION

We have considered a concrete example of a trapped-particle instability in a toroidal system with axial symmetry for the case $H_\vartheta \ll H_0$. However, the instability treated here is of a more general nature. In any toroidal system there must be regions with reduced magnetic field in which there are particles trapped between the effective magnetic mirrors. If these particles are subject to an unfavorable magnetic drift, that is to say, if they are located in a region in which the magnetic field falls off in going toward the periphery, these trapped trapped particles will give rise to an instability in a collisionless plasma. In a sufficiently dense plasma, in which the electron collision frequency exceeds the drift frequency ω^*, it is possible to have a dissipative trapped-particle instability (for which the sign of the magnetic drift is unimportant). As the collision frequency increases the growth rate for this instability falls off rapidly so that the trapped-particle instability will be unimportant in a sufficiently dense plasma.

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