NONLINEAR THEORY OF CURRENT INSTABILITY IN A NON-ISOTHERMAL PLASMA

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An analytical solution of the nonlinear problem of current stability in a non-isothermal plasma located in an external electric field is obtained within the framework of a theory in which pair collisions and particle scattering by ion-sound noise are taken into account. The shape of the quasistationary noise spectrum is found, and it is shown that if electron scattering by ion sound is taken into account an anomalously high resistance appears and restricts the electron current and prevents the appearance of runaway electrons. Equations are derived for the time variation of the mean electron and ion kinetic energies in the plasma. It follows from these equations that the presence of ion absorption leads to intense heating of the plasma ion component, at a rate proportional to the external electric field strength.

1. INTRODUCTION

It is well known[1,2] that a nonisothermal plasma situated in an external electric field becomes unstable if the mean directed velocity of the electrons exceeds the phase velocity of the ion-sound waves. By now an anomalously high resistance of a plasma to electric current was observed experimentally by a number of workers [3-5], who are inclined to believe that this result is the consequence of an instability that arises in the plasma.

Since such results cannot be explained within the framework of the linear theory, it becomes necessary to construct a nonlinear theory for a current-carrying plasma. The present paper is devoted to an analytic solution of the nonlinear problem of instabilities of ion sound in a plasma situated in an external electric field. We take account here of the interaction between the electrons and the ion-sound noise, as well as pair collisions[1)].

A similar problem, but with pair collisions neglected, was considered in[7,8]. Field and Fried[7] reported the results of a numerical calculation of the initial stage of the process, which is the most difficult for an analytic solution. However, when writing down the initial equations, they made a number of assumptions whose validity calls for additional justification. First, they assumed that in a direction perpendicular to the external field the distribution function was Maxwellian, and furthermore with constant temperature; second, they assumed that in the nonlinear theory the spectral density of the noise W(**k**) has a sharp maximum at a wave number **k** for which the increment of the linear theory is maximal; finally, they took absolutely no account of the damping of the waves by the ions, a very important factor and one particularly essential for the determination of the stationary noise spectrum.

A paper by Korablev and Rudakov[8] is devoted to an analytic investigation of the second quasistationary stage of the process. However, just as Field and Fried[7], Korablev and Rudakov[8] completely neglected pair collisions, and hence the collisional wave damping. Because of this, their results pertain, strictly speaking, only to the case when the ratio of the external field E to the so-called critical field E_c is infinitely large, and thus cannot explain the dependence of the number of runaway electrons on the magnitude of the electric field E.[2)] In our paper, to the contrary, we take into account also pair collisions, and consequently the finite ratio E/E_c. The latter, in turn, makes it possible to analyze more rigorously the conditions for the convergence of the employed method of successive approximations, and to indicate on this basis the value of the electric field below which the number of runaway electrons is negligibly small. We shall see that this value exceeds by many times the critical field E_c obtained within the framework of plasma-stability theory.

[1)]A similar problem is studied in a one-dimensional model in [6].

[2)]A definition of the critical field E_c is given below (see formula (44)).

2. FORMULATION OF THE PROBLEM AND FUNDAMENTAL EQUATIONS

Let us consider a spatially-homogeneous fully ionized non-isothermal plasma with $T_e \gg T_i$, situated in an external electric field $\mathbf{E}$. Let $f_e(\mathbf{v}, t)$ and $f_i(\mathbf{v}, t)$ be the distribution functions of the electrons and ions, respectively, normalized to unity. To avoid misunderstandings, we point out immediately that henceforth (with the exception of a few particular examples), we shall not assume that the functions f_e and f_i are Maxwellian, although for convenience we shall use the concepts of temperatures defined in the following manner:

$$T_e = mv_{Te}^2 = \frac{m}{3}[\overline{v_e^2} - (\overline{\mathbf{v}}_e)^2],$$

$$T_i = Mv_{Ti}^2 = \frac{M}{3}[\overline{v_i^2} - (\overline{\mathbf{v}}_i)^2], \tag{1}$$

where the bar with the index e or i denotes averaging over the corresponding distribution function, and m and M are the masses of the electron and the ion. However, for simplicity we shall assume that the ion distribution function $f_i(v, t)$ is isotropic.

We denote further by $W(\mathbf{k}, t)$ ($\mathbf{k}$ is the wave vector) the spectral density of the energy of the ion-sound waves in a plasma normalized to an average energy density $\mathscr{E}(t)$ such that

$$\mathscr{E}(t) = \int W(\mathbf{k}, t)\, d\mathbf{k}. \tag{2}$$

Then in the lowest order in the nonlinearity, and under the assumption that the intensity of the ion-sound noise W greatly exceeds their thermodynamic equilibrium value, the system of equations for the functions $f_e(\mathbf{v}, t)$ and $W(\mathbf{k}, t)$ can be written in the form[9]

$$\frac{\partial f_e}{\partial t} - \frac{e\mathbf{E}}{m}\frac{\partial f_e}{\partial \mathbf{v}} = \frac{\partial}{\partial v_i} D_{ij} \frac{\partial f_e}{\partial v_j} + \mathrm{St}(f_e),$$
$$i, j = 1, 2, 3 \quad (e > 0); \tag{3}$$

$$\partial W / \partial t = (\gamma_s - \gamma) W, \quad \gamma = \gamma_i + \gamma_{\text{coll}} \tag{4}$$

The first term in the right side of (3), which is proportional to

$$D_{ij} = \frac{\pi M}{m^2 n} \int d\mathbf{k} s^2 W(\mathbf{k}) k_i k_j \delta(ks - \mathbf{kv}), \tag{5}$$

takes into account the scattering of particles by the ion-sound noise, and the second takes into account the electron-ion and electron-electron collisions[3)]. In Eq. (4) the first term, which is proportional to

$$\gamma_s = \pi \frac{M}{m} ks^3 \frac{\partial}{\partial s} \int d\mathbf{v} f_e(\mathbf{v})\, \delta\left(s - \frac{\mathbf{kv}}{k}\right), \tag{6}$$

describes the excitation of ion sound by electrons, while the last two terms

$$\gamma_i = 2\pi^2 ks^4 f_i(s), \quad \gamma_{\text{coll}} = \frac{1}{2\sqrt{\pi}}\left(\frac{mT_e}{MT_i}\right)^{1/2} \frac{s_m^2}{s^2} \nu_T, \tag{7}$$

take into account the Landau damping by the ions and the damping of the sound by the ion-ion collisions[10]. In formulas (3)—(7) we use the following notation: n—electron density, $s = s(k) = \omega(k)/k$—phase velocity, and s_m—maximum phase velocity of the ion-sound waves (equal in the case of a Maxwellian distribution to $s_m = (T_e/M)^{1/2}$, and $\nu_T = \nu(v_{Te})$, where $\nu(v) = 4\pi e^4 nL/m^2v^3$ is the frequency of the electron-ion collisions (L—Coulomb logarithm).

We change over in (3)—(6) to the spherical coordinates $\mathbf{v} = \{v, \theta, \varphi\}$, $\mathbf{k} = \{k, \theta', \varphi'\}$, with z axis directed opposite to the vector $\mathbf{E}$. Then, by virtue of the axial symmetry of the problem, the functions $f_e(\mathbf{v})$ and $W(\mathbf{k})$ can be regarded as dependent only on the variables v, $\xi = \cos\theta$ and k, $x = \cos\theta'$, and Eq. (3) takes the form

$$\frac{\partial f_e}{\partial t} + \frac{eE}{m}\xi\frac{\partial f_e}{\partial v} + \frac{eE}{mv}(1-\xi^2)\frac{\partial f_e}{\partial \xi} = \mathrm{St}(f_e) + \frac{1}{v^2}\frac{\partial}{\partial v} v^2\left[D_{vv}\frac{\partial f_e}{\partial v} + \frac{\sqrt{1-\xi^2}}{v} D_{v\xi}\frac{\partial f_e}{\partial \xi}\right] + \frac{\partial}{\partial \xi}\left[\frac{\sqrt{1-\xi^2}}{v} D_{v\xi}\frac{\partial f_e}{\partial v} + \frac{1-\xi^2}{v^2} D_{\xi\xi}\frac{\partial f_e}{\partial \xi}\right] \tag{8}$$

where the coefficients are

$$D_{vv}(v, \xi) = \frac{2\pi M}{nm^2v^3}\int dk k^3 s^4 \int_{-1}^{1} \frac{d\eta W(k, x)}{\sqrt{1-\eta^2}},$$

$$D_{v\xi}(v, \xi) = \frac{2\pi M}{nm^2v^2}\int dk k^3 s^3 \left(1 - \frac{s^2}{v^2}\right)^{1/2} \int_{-1}^{1} \frac{\eta\, d\eta W(k, x)}{\sqrt{1-\eta^2}},$$

$$D_{\xi\xi}(v, \xi) = \frac{2\pi M}{nm^2v}\int dk k^3 s^2 \left(1 - \frac{s^2}{v^2}\right) \int_{-1}^{1} \frac{\eta^2 d\eta W(k, x)}{\sqrt{1-\eta^2}}, \tag{9}$$

[3)]We note that if the number of neutral atoms is sufficiently small, so that their contribution to the effective collision frequency is relatively small, then we can, without changing at all the computation scheme developed below, take account also of inelastic collisions which, while not playing a decisive role in the momentum conservation law, can make a rather appreciable contribution to the energy balance equation (see (41) and (42)).

with

$$x = \eta\left[\left(1 - \frac{s^2}{v^2}\right)(1 - \xi^2)\right]^{1/2} + \frac{s}{v}\xi.$$

On the other hand, formula (6) for the increment $\gamma_s(k, x)$ becomes

$$\gamma_s(k, x) = -2\pi^2\frac{M}{m}ks^3\Big\{sf_e(s, x)$$

$$-\frac{1}{\pi}\int_s^\infty dv\int_{-1}^{1}\frac{d\mu}{\sqrt{1-\mu^2}}\left[x - \mu s\frac{\sqrt{1-x^2}}{\sqrt{v^2-s^2}}\right]\frac{\partial f_e}{\partial \xi}\Big\}, \quad (10)$$

$$\xi = \mu\left[\left(1 - \frac{s^2}{v^2}\right)(1 - x^2)\right]^{1/2} + \frac{s}{v}x.$$

An analysis of Eqs. (8)—(10) shows that the entire evolution of the instability can be broken up into two stages. Immediately after the field is turned on, the average electron velocity begins to grow, and as soon as it exceeds a magnitude of the order of several times v_{Ti}, ion-sound waves are generated, with an intensity that increases first almost exponentially. With increasing noise intensity W, the role of scattering of electrons by waves begins to grow continuously, leading to a decrease in the increment γ_s and to a shift of its maximum value towards larger values of k; on the other hand, waves with small values of k begin to attenuate. The anisotropic part of the distribution function also changes quite rapidly during this stage. This is followed by a second quasistationary stage, during which the noise intensity W and the average electron velocity $\overline{v}_e$, which have already reached saturation, change relatively slowly, increasing only with increasing electron and ion temperatures.

The first stage, as already noted, is the most difficult for an analytic solution and apparently can be investigated in detail only by means of computers. We therefore confine ourselves here to a study of only the second quasistationary stage of the process. We note beforehand that inasmuch as $D_{vv} \sim s^2v^{-2}D_{\xi\xi}$ and $D_{v\xi} \sim sv^{-1}D_{\xi\xi}$, and the ratio $s/v_{Te} \lesssim m/m \ll 1$, the most rapid process for the bulk of the particles, i.e., in the velocity region $v \sim v_{Te}$, which proceeds with a characteristic time on the order of $\nu_{\rm eff}^{-1} \sim v_{Te}^2/D_{\xi\xi}(v_{Te})$, is the momentum relaxation process; on the other hand, the energy relaxation process is much slower (by approximately M/m times). Therefore, in analogy with the procedure used in plasma stability theory, it is natural to attempt to seek a quasistationary solution of Eq. (8) under the assumption that the isotropic part of the distribution function $f_e^{(0)}(v, t)$ is large compared with its anisotropic part $f_e^{(1)}(v, \xi, t)$ and neglect in the equation for the latter the derivative $\partial f_e^{(1)}/\partial t$ compared with the diffusion terms which are of the order of $\nu_{\rm eff}\partial f_e^{(1)}/\partial\xi$. In addition, we assume also that the characteristic time of variation of the spectral density W is much larger than the reciprocal increment γ_s^{-1} and accordingly, we can neglect in (4) the derivative $\partial W/\partial t$ compared with $\gamma_s W$. Under these assumptions, the solution of the system of nonlinear equations (3) and (4) can be obtained.

Thus, putting in (8) $f_e(v, \xi) = f_e^{(0)}(v) + f_e^{(1)}(v, \xi)$, recognizing that $f_e^{(0)}(v) \gg f_e^{(1)}(v, \xi)$ and does not depend on the variable ξ, and neglecting the derivative $\partial f_e^{(1)}/\partial t$, we get

$$\frac{\partial f_e^{(1)}}{\partial \xi} = -\frac{[u(v) + 2D_{v\xi}/v\nu(v)\sqrt{1-\xi^2}]}{[1 + 2D_{\xi\xi}/v^2\nu(v)]}\frac{\partial f_e^{(0)}}{\partial v},$$

$$u(v) = \frac{eE}{m\nu(v)}, \quad (11)$$

$$\frac{\partial f_e^{(0)}}{\partial t} = \mathrm{St}(f_e^{(0)}) + \frac{1}{2}\frac{1}{v^2}\frac{\partial}{\partial v}v^2\frac{\partial f_e^{(0)}}{\partial v}\int_{-1}^{1}d\xi$$

$$\times\left\{D_{vv} + \frac{1-\xi^2}{2}\nu\frac{[u^2(v) - 4D_{v\xi}^2/v^2\nu^2(1-\xi^2)]}{[1 + 2D_{\xi\xi}/v^2\nu]}\right\}. \quad (12)$$

In the derivation of these equations we took account of the fact that inasmuch as electron-electron collisions do not change the total momentum of the electron gas, and the frequency of the electron-ion collisions increases with decreasing velocity like v^{-3}, we can neglect, without noticeable error, the influence of the electron-electron collisions on the anisotropic part of the distribution function in the velocity region $v_{Ti} \ll v \lesssim v_{Te}$, and assume that

$$\mathrm{St}(f_e) = \mathrm{St}(f_e^{(0)}) + \frac{\partial}{\partial\xi}\frac{1-\xi^2}{2}\nu(v)\frac{\partial f_e^{(1)}}{\partial\xi}. \quad (13)$$

Thus, the quasistationary state will be described by the system (11) and (12), which should be solved simultaneously with the equation

$$[\gamma_s(k, x) - \gamma(k)]W(k, x) = 0. \quad (14)$$

Inasmuch as $s/v_{Te} \ll 1$, we can neglect the small quantities of order s/v in expressions (9) and (10) for the coefficients D_{ij} and the increment γ_s, and put

$$D_{v\xi}(v, \xi) = \frac{2\pi M}{nm^2v^2}\int dk k^3s^3\int_{-1}^{1}\frac{\eta d\eta}{\sqrt{1-\eta^2}}W(k, \eta\sqrt{1-\xi^2}), \quad (15)$$

$$D_{\xi\xi}(v, \xi) = \frac{2\pi M}{nm^2v}\int dk k^3s^2\int_{-1}^{1}\frac{\eta^2 d\eta}{\sqrt{1-\eta^2}}W(k, \eta\sqrt{1-\xi^2}),$$

$$\gamma_s(k,x) = -2\pi^2 \frac{M}{m} ks^3$$

$$\times \left\{ sf_e^{(0)}(0) - \frac{x}{\pi} \int_0^\infty dv \int_{-1}^{1} \frac{d\mu}{\sqrt{1-\mu^2}} \frac{\partial f_e^{(1)}}{\partial \xi} \bigg|_{\xi=\mu\sqrt{1-x^2}} \right\}. \quad (16)$$

After substitution of $\partial f_e^{(1)}/\partial \xi$ from (11) and integration over the velocities, (16) takes the form

$$\gamma_s(k,x) = -2\pi^2 \frac{M}{m} ks^3 f_e^{(0)}(0)\{s - Q(x)\},$$

$$u_T = \frac{3u(v_{Te})}{4\pi v_{Te}{}^3 f_e^{(0)}(0)} = \frac{3eE}{4\pi v_{Te}{}^3 f_e^{(0)}(0) m\nu_T}, \qquad \nu_T = \nu(v_{Te}); \quad (17)$$

$$Q(x) = \frac{x}{\pi} \int_{-1}^{1} \frac{d\mu}{\sqrt{1-\mu^2}} q(\mu\sqrt{1-x^2}),$$

$$q(\xi) = \frac{u_T + 2D_{v\xi}(v_{Te},\xi)/v_{Te}\nu_T\sqrt{1-\xi^2}}{1 + 2D_{\xi\xi}(v_{Te},\xi)/v_{Te}{}^2\nu_T}, \quad (18)$$

and consequently no longer depends explicitly on the concrete form of the distribution function. This dependence enters only via the velocity u_T which, if $f_e^{(0)}(v)$ is close to Maxwellian, is equal to $u_T = 3\sqrt{\pi/2}eE/mv_T$, i.e., it depends only on the electron temperature $T_e = mv_{Te}^2$.

3. QUASISTATIONARY SPECTRUM OF ION-SOUND NOISE

We consider first the equation (14)

$$W(k,x)\Gamma(k,x) = 0, \quad (19)$$

where $\Gamma(k, x)$ denotes the total increment

$$\Gamma(k,x) = \gamma_s^{(0)}(k)\left[\frac{Q(x)}{s} - 1\right] - \gamma(k),$$

$$\gamma_s^{(0)} = 2\pi^2 \frac{M}{m} ks^4 f_e^{(0)}(0), \quad (20)$$

and $Q(x)$ denotes the integral (18) which enters in formula (17). We assume that the noise intensity $W(k, x)$ differs from zero only inside a cone with aperture angle $\theta' = \theta_0$, i.e., only for $x > x_0 = \cos\theta_0$, and that it is identically equal to zero when $x \le x_0$, i.e., outside this cone. This assumption, obviously, does not contradict the initial equation (4) only if the increment $\Gamma(k, x)$ vanishes identically for $x \ge x_0$ and is negative when $x < x_0$. But since the functions $\gamma(k)$ and $\gamma_s^{(0)}(k)$ do not depend on x, this can take place only if the function $Q(x)$ is likewise independent of x in the region $x > x_0$. On the other hand, it is easy to show that the necessary and sufficient condition for the integral $Q(x)$ not to depend on x when $x > x_0$ and for it to equal a certain constant v_0 is the requirement that the following equality take place for all values of the variable ξ satisfying the condition $\xi^2 < 1 - x_0^2$:

$$q(\xi) = v_0/(1-\xi^2), \qquad \xi^2 < 1 - x_0^2. \quad (21)$$

However, since $W(k, x) = 0$ when $x \le x_0$, it follows from (15) that when $\xi^2 > 1 - x_0^2$ the coefficients $D_{v\xi}(v, \xi)$ and $D_{\xi\xi}(v, \xi)$ vanish identically, and consequently

$$q(\xi) = u_T, \quad \xi^2 > 1 - x_0^2. \quad (21')$$

Thus, formulas (21) and (21′) determine the function $q(\xi)$ for all $-1 \le \xi \le 1$, and consequently also the total increment $\Gamma(k, x)$ for all values of x. It remains to determine the constants v_0 and x_0, and to find the explicit form of the function $W(k, x)$. The constant v_0 is determined from the vanishing of the total increment $\Gamma(k, x)$ when $x > x_0$. Recognizing that, in accord with (21), we get $Q(x) = v_0$ when $x > x_0$, we obtain

$$\gamma_s^{(0)}(k)[v_0/s - 1] - \gamma(k) = 0. \quad (22)$$

However, since it follows from (15) and (18) that v_0 does not depend on the wave number k, the equality (22) can hold only for one value $k = k_0$ (and consequently one value $s = s_0 = s(k_0)$), namely one for which the increment $\Gamma(k, x)$, regarded here as a function of k (or s), has a maximum value, which according to (22) is equal to zero. From this it follows, in turn, that Eq. (19) can have nontrivial solutions only if

$$W(k,x) = \frac{\delta(k-k_0)}{k^2} W_0(x), \quad (23)$$

where k_0 is the root of the equation

$$\frac{\partial \Gamma(k,1)}{\partial k} = \frac{\partial}{\partial k}\left\{\gamma_s^{(0)}(k)\left[\frac{v_0}{s} - 1\right] - \gamma(k)\right\} = 0. \quad (24)$$

With this, according to (22), the constant v_0 will be equal to

$$v_0 = s_0[1 + \gamma(k_0)/\gamma_s^{(0)}(k_0)] > s_0, \qquad s_0 = s(k_0). \quad (25)$$

To find x_0 and the function $W_0(x)$ we use Eq. (21) which, taking (23) into account, can be represented in the form

$$y^2 - \frac{v_0}{u_T} = A \int_{x_0}^{y} \frac{W_0(x)x\,dx}{\sqrt{y^2 - x^2}} \left(\frac{x}{y^2} - \lambda\right), \qquad y^2 \geqslant x_0^2, \quad (26)$$

where

$$y^2 = 1 - \xi^2, \qquad \lambda = \frac{s_0}{v_0} < 1, \; A = \frac{4\pi}{\lambda} \frac{M}{m} \frac{k_0 s_0^3}{v_{Te} u_T \nu_T} \frac{1}{nT_e}.$$

We see therefore that since the right side of this equality vanishes when $y = x_0$, it is necessary to put $x_0^2 = v_0/u_T$ in order to make (26) consistent.

Substituting (21) and (21′) in (18), we obtain the expression for the total increment $\Gamma(k, x)$:

$$\Gamma(k,x)=\gamma_s^{(0)}(k)\left[\frac{v_0}{s}-1\right]-\gamma(k)$$

$$+\gamma_s^{(0)}(k)\frac{v_0}{s}\begin{cases}\dfrac{x}{|x|}-1 & \text{for } |x|\geqslant x_0\\ \dfrac{x}{|x|}-1+\dfrac{2}{\pi}\left[\dfrac{x}{x_0^2}\arccos\dfrac{\sqrt{1-x_0^2}}{\sqrt{1-x^2}}-\dfrac{x}{|x|}\operatorname{arctg}\dfrac{\sqrt{x_0^2-x^2}}{|x|\sqrt{1-x_0^2}}\right] & \text{for } |x|\leqslant x_0.\end{cases} \quad (27)^*$$

Taking into account the definitions of s_0, v_0, and x_0, we can readily verify that, in accordance with the assumptions made, the increment $\Gamma(k, x)$ vanishes when $k = k_0$ and $x \geq x_0$, and is negative for all other values of the variables k and x.

Solving the integral equation (26) (see the appendix), we get

$$W_0(x)=\mathscr{E}_0\begin{cases}\varphi(x) & \text{for } x\geqslant x_0\\ 0 & \text{for } x\leqslant x_0\end{cases}\quad\left(x_0^2=\frac{v_0}{u_T}\right), \quad (28)$$

where

$$\mathscr{E}_0=\frac{4}{3\pi A}=\frac{\lambda nT_e}{3\pi^2}\frac{m}{M}\frac{v_{Te}u_T\nu_T}{k_0s_0^3}=\frac{\lambda}{4\pi^3}\frac{m}{M}\frac{eEn}{k_0s_0^3}\frac{1}{f_e^{(0)}(0)},$$

$$\varphi(x)=\frac{1}{x(1-\lambda x)^2}\Big\{(x^2-x_0^2)^{1/2}[(x^2-x_0^2)+3x^2(1-\lambda x)] -\frac{3}{4}\lambda x_0^4\ln\frac{x+\sqrt{x^2-x_0^2}}{x_0}\Big\}. \quad (29)$$

Thus, the quasistationary noise spectrum is determined by formulas (23) and (28), from which it follows that it contains only waves with one value $k = k_0$, determined only by the parameters of the plasma and not depending explicitly on the electric field E. With this, the directions of wave propagation are bounded by a cone with aperture angle $\theta_0 = \cos^{-1}\sqrt{v_0/u_T}$.[4)] Since u_T is proportional to E, the angle θ_0 decreases with decreasing field E and vanishes when $u_T = v_0$. This value corresponds precisely to the stability limit, since, as follows from the expression for the increment $\Gamma(k, x)$, instability can take place only when the velocity u_T exceeds v_0. We note, finally, that since $\lambda < 1$ (see (25)), the function $W_0(x)$ is always bounded[5)].

The total noise energy density is

$$\mathscr{E}=2\pi\int_{-1}^{1}W_0(x)\,dx=2\pi\,\mathscr{E}_0 I(\lambda,x_0),$$

$$I(\lambda,x_0)=\int_{x_0}^{1}\varphi(x)\,dx. \quad (30)$$

From this it follows, in particular, that $\mathscr{E}$ depends very little on the collision frequency and increases in direct proportion to the field intensity E. In the case of greatest interest, when $x_0^2 \ll 1$, the last term in expression (29) for $\varphi(x)$ can be neglected, and we get for the integral

$$I(\lambda,x_0)=\frac{1}{\lambda^3(1-\lambda)}\Big\{1+(1-\lambda)\ln\frac{1}{1-\lambda}-\frac{9}{2}(1-\lambda) +5(1-\lambda)^2-\frac{3}{2}(1-\lambda)^3\Big\}+O(x_0^2). \quad (31)$$

To conclude this section, let us analyze Eqs. (24) and (25), which determine the parameters s_0, v_0, and λ. We confine ourselves here for simplicity to the case when the velocity distribution of the electrons and the ions are nearly Maxwellian, so that $s_m = \sqrt{T_e/M}$, and the collision frequency is not very large, so that the following inequality is satisfied[6)]:

$$\nu_T/\omega_0\ll(mT_i/MT_e)^{1/2},\qquad \omega_0^2=4\pi e^2n/m.$$

Substituting expressions (7) for $\gamma_i(k)$ and $\gamma_{coll}(k)$ in (24), and taking (25) into account, we find that when the collision frequency ν_T varies in the interval

$$\omega_0\left(\frac{mT_i}{MT_e}\right)^{1/2}\gg\nu_T\gg\omega_0\left(\frac{m}{M}\right)^{1/2}\left(\frac{T_i}{T_e}\right)^3\left[\ln 4\frac{M}{m}\left(\frac{T_e}{T_i}\right)^3\right]^{5/2} \quad (32)$$

the principal role is assumed by collision damping, and

$$s_0\approx\left(\frac{T_e}{M}\right)^{1/2}\left[\frac{\nu_T}{\omega_0}\left(\frac{MT_e}{mT_i}\right)^{1/2}\right]^{1/5}, \quad (33)$$

i.e., the phase velocity decreases slowly with decreasing collision frequency. The constant λ is in this case independent of the plasma parameters and is equal to

*arctg ≡ $\tan^{-1}$.

[4)]It is interesting to note that this angle is smaller than follows from the linear theory, according to which it should be equal to $\cos^{-1}(v_0/u_T)$.

[5)]If we neglect completely the damping by the ions and put $\nu_T = 0$ and $\gamma(k) = 0$, then $\lambda = 1$, $x_0 = 0$, and we obtain for $W_0(x)$ the same expression as in [8]. At the same time, however, we obtain for the total noise energy a divergent expression, which indicates that such a neglect is not valid.

[6)]In the case of large values of ν_T, when the inverse inequality is satisfied, we have for the velocity $s_0 \approx 2(T_e/M)^{1/2}/5^{1/2}$, and $\lambda \approx [1 + (MT_e/mT_i)^{1/2}\nu_T/\omega_0]^{-1} \ll 1$. It must be remembered, however, that the theory developed here is valid only so long as $\gamma_s \ll \omega$. We note also that if the function $f_e^{(0)}(v)$ differs from Maxwellian, then the quantity T_e in the formulas (32)–(37) obtained below must be replaced by Ms_m^2.

$$\lambda = 4/5. \quad (34)$$

With further decrease of the collision frequency, the main role is assumed by the Landau damping γ_i and when

$$\omega_0 \left(\frac{m}{M}\right)^{1/2}\left(\frac{T_i}{T_e}\right)^3\left[\ln 4\frac{M}{m}\left(\frac{T_e}{T_i}\right)^3\right]^{3/2} \gg \nu_T \geqslant 0 \quad (35)$$

the phase velocity s_0 is equal to

$$s_0 \approx \left(\frac{T_i}{M}\right)^{1/2}\left[\ln 4\frac{M}{m}\left(\frac{T_e}{T_i}\right)^3\right]^{1/2}. \quad (36)$$

The constant λ increases in this case somewhat and becomes a function of the temperature ratio[7]:

$$\lambda \approx 1 - \left[\ln 4\frac{M}{m}\left(\frac{T_e}{T_i}\right)^3\right]^{-1}. \quad (37)$$

4. DISTRIBUTION FUNCTION AND EQUATIONS FOR THE TEMPERATURES

We now proceed to study the electron distribution function. We consider first expression (11) for $\partial f_e^{(1)}/\partial\xi$. Taking (18), (21), (21′), and (28) into account, this expression can be rewritten in the form

$$\frac{\partial f_e^{(1)}}{\partial \xi} = -\frac{v_0}{1-\xi^2}\frac{\partial f_e^{(0)}}{\partial v}\times\begin{cases}\dfrac{1-\xi^2}{v_0}u(v) & \text{for } \xi^2 \geqslant 1 - x_0^2\\[2mm] 1 + \dfrac{u(v)/u_T - 1}{1+\psi(\sqrt{1-\xi^2})} & \text{for } \xi^2 \leqslant 1 - x_0^2\end{cases}, \quad (38)$$

where

$$\psi(y) = \frac{4}{3\pi}\frac{\lambda}{y^2}\int_{x_0}^{y}\frac{x\varphi(x)\,dx}{\sqrt{y^2-x^2}}, \quad (39)$$

and $\varphi(x)$ is defined by (29). The integral in (39) can be calculated, but the expression obtained for $\psi(y)$ thereby is quite cumbersome and will not be presented here. We note only that $\psi(y) \approx 32\lambda y/9\pi$ when $\lambda y \ll 1$ and $\psi(y) \approx 4/3(1-\lambda y)^{3/2}$ when $\lambda y \sim 1$.

Using (38), we can determine the mean electron velocity $\bar{v}_{ze} = (\bar{\mathbf{v}}_e \cdot \mathbf{E})/E$, and consequently the current density $j = en\bar{v}_{ze}$ flowing through the plasma. Assuming for simplicity that the ratio $x_0^2 = v_0/u_T \ll 1$, and neglecting quantities $\sim x_0^2$ compared with unity, we get

$$\bar{v}_{ze} = \varepsilon v_0, \qquad \varepsilon = \alpha\beta + 3/2(1-\alpha), \quad (40)$$

where $\beta = 4\pi v_e^3 f_e^{(0)}(0)$ (in the case of a Maxwellian distribution $\beta = 16/\pi$), and α stands for the integral

$$\alpha = \int_{x_0}^{1}\frac{y\,dy}{\sqrt{1-y^2}[1+\psi(y)]}$$

which, as shown by analysis, is a monotonic function of the parameter λ, varying from 1 at $\lambda = 0$ to ~ 0.3 at $\lambda = 1$.

It follows from (40) that inasmuch as the critical velocity v_0 does not depend on the field intensity E, and the parameter ϵ is determined only by the form of the function $f_e^{(0)}(v)$ (more accurately, by its moments), then the average electron velocity $\bar{v}_{ze}$, and consequently also the current density j, likewise does not depend on the field E. In other words, the resistance of the plasma in the quasistationary mode is anomalously large and increases in direct proportion to the intensity of the external electric field.

We now turn to Eq. (12) for the function $f_e^{(0)}(v, t)$. A rigorous analysis of this equation, which in general is nonlinear, is quite complicated, and we cannot dwell on it here[8]. We confine ourselves only to a derivation of equations for the time variation of the electron and ion temperatures, and to a brief analysis of this equation. We indicate at the same time the conditions under which the function $f_e^{(0)}(v, t)$ can be regarded as close to Maxwellian.

The sought equations can be obtained by multiplying (12) by $mv^2/3$ and integrating over the velocities. Recognizing, however, that the energy and momentum conservation laws are satisfied in the scattering of the electrons by the waves[9], it is simpler to start directly from the system (3) and (4). We multiply (3) by $mv^2/3$ and then by $mv/3$, after which we integrate over the velocities. Subtracting then the second equation multiplied by $2\bar{v}_{ze}$ from the first equation, and taking (4), (5), and (6) into account, we get

$$\frac{dT_e}{dt} = \frac{2}{3}\left[eE\bar{v}_{ze} - \gamma(k_0)\frac{\mathcal{E}}{n}\right] - \frac{2\sqrt{2}m}{3\sqrt{\pi}M}\nu_T[T_e - T_i] - \frac{\delta T_e}{\delta t}. \quad (41)$$

In the derivation of this equation we took account of the fact that $d\mathcal{E}/dt \ll \gamma\mathcal{E}$ and neglected the small corrections of the order of $\bar{v}_{ze}/u_T \ll 1$ compared with unity.

The physical meaning of the different terms in (41) is obvious: the first takes into account the ohmic heating of the electrons, the second the cooling due to the excitation of ion sound absorbed by the ions, and the third the exchange of energy between the electrons and the ions upon collision.

[7]In these calculations we assume for simplicity that the plasma is sufficiently strongly non-isothermal, so that $m/M \gg (T_e/T_i)^3\exp(-T_e/T_i)$, i.e., $T_e > 16T_i$.

[8]The nonlinearity of this equation is due to the dependence of its coefficients D_{ij} and on the moments and to the need for taking into account the electron-electron collisions.

Finally, the fourth term $\delta T_e/\delta t$ describes losses occurring as a result of inelastic collisions or some other processes (for example, bremsstrahlung).

The equation for the ion temperature T_i can be obtained simply from the energy conservation law, and takes the form

$$\frac{dT_i}{dt} = \frac{2}{3}\gamma(k_0)\frac{\mathscr{E}}{n} + \frac{2\sqrt{2}\,m}{3\sqrt{\pi}\,M}\nu_T[T_e - T_i] - \frac{\delta T_i}{\delta t}. \quad (42)$$

The term $\gamma(k_0)\,\mathscr{E}/n$ can be transformed to a simpler form which does not contain $\gamma(k_0)$. Indeed, taking in account relations (29) and (30) for $\mathscr{E}$, and also (25), we can easily verify that

$$\gamma(k_0)\,\mathscr{E}\,/\,n = eEs_0(1-\lambda)I(\lambda) \approx eEs_0, \quad (43)$$

where, as follows from (31), (34), and (37), the factor $(1-\lambda)I(\lambda)$ does not depend on E and remains quite close to unity for all values of the plasma parameters. Since, however, according to (40), the velocity $\overline{v}_{ze} \gtrsim 3s_0$, the first term in (41) always prevails over the second, and when $\delta T_v/\delta t = 0$ it leads to continuous heating of the electron gas. It is interesting to note also that it follows from (42) and (43) that the ion component of the plasma is heated almost just as effectively as the electron component. It must be kept in mind here that T_i has the meaning of the ion temperature only under condition (32), for in the opposite case of sufficiently small ν_T, when condition (35) is satisfied, only a small fraction of the ions with $v > s_0$ is essentially heated, and their number, as shown by rough estimates, is $\sim ns_0v_{Ti}^{-1}\exp[-s_0^2/2v_{Ti}^2]$.[9)]

In conclusion it should be pointed out that since the characteristic time of variation of the electron temperature is of the order of $mv_{Te}^2/eE\overline{v}_{ze}$, and the time of Maxwellization as a result of electron-electron collisions is $\sim \nu_T^{-1}$, it follows obviously that the function $f_e^{(0)}(v)$ will be close to Maxwellian provided

$$E \ll \frac{v_{Te}}{4v_0}E_c \qquad E_c = \frac{mv_{Te}\nu_T}{e}. \quad (44)$$

Since the ratio $v_{Te}/v_0 \gtrsim \sqrt{M/m}$, the magnitude of the external field can greatly exceed the critical value E_c when, according to the theory of a stable plasma, all the electrons should go over into the continuous-acceleration mode[11,12].[10)]

Finally, if some additional energy loss ($\delta T_{e,i}/\delta t \neq 0$) increasing with rising temperature exists, then Eqs. (41) and (42) have stationary solutions for which the function $f_e^{(0)}(v)$, although not Maxwellian, does decrease quite rapidly (exponentially) with increasing velocity v.

5. LIMITS OF APPLICABILITY AND GENERAL CONCLUSIONS

We now discuss the limits of applicability of the results. The fundamental hypothesis on which our solution method is based is the assumption that the anisotropic part of the distribution function $f_e^{(1)}(v, \xi)$ is small compared with its isotropic part $f_e^{(0)}(v)$. Since, as follows from (38), $f_e^{(1)}$ has a maximum at $\xi = \pm 1$, the assumption will obviously be justified if

$$\frac{1}{2}\left[\frac{v_0}{4}\left(\frac{v}{v_{Te}}\right)^3 + v_0\right]\left(\frac{\partial \ln f_e^{(0)}}{\partial v}\right)\ln\frac{4u_T}{v_0} \ll 1. \quad (45)$$

Accordingly, it should be remembered that Eq. (12) for $f_e^{(0)}(v, t)$ is valid only in the region of velocities satisfying the inequality (45). On the other hand, expression (28) for the spectral density W(k, x), and the equations for the moments (40)—(42), have in some sense a wider range of applicability. Indeed, in the derivation of (28) we used expression (11) for $\partial f_e^{(1)}/\partial\xi$ only to calculate the zeroth moment, i.e., the norm of the function $f_e^{(0)}(v)$, in the derivation of (40)—(42) we used it only to calculate the third moment $\overline{v_e^3}$. Therefore in order for these equations to be valid it is sufficient to have the main contribution made to these moments by particles with velocity satisfying the condition (45). If we assume that the function $f_e^{(0)}(v)$ is close to Maxwellian (which is the case, for example, under condition (44)), or else it decreases when $v > v_{Te}$ not more slowly than $\exp[-v^2/2v_{Te}^2]$, then we can assume that the main contribution to these moments is made by particles with $v^2 < 6v_{Te}^2$. Then, condition (45) for the convergence of the employed method takes the form

$$\frac{E}{E_c} \ll \frac{v_0}{16v_{Te}}\exp\frac{v_{Te}}{5v_0}. \quad (46)$$

Since, according to (33)—(37), the value of v_0 for a strongly nonisothermal plasma exceeds the thermal velocity of the ions v_{Ti} by only a few times, this criterion depends very strongly on the degree of non-isothermy of the plasma T_e/T_i and

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9)It must be emphasized, however, that in the region of the parameters in which ion-ion collisions are insignificant a detailed analysis of the character of ion heating calls for solving also an equation for the function $f_i(v)$, since the processes of scattering of ions by the waves can lead to the anisotropy of the ion absorption.

10)For a strong non-isothermal plasma, the ratio v_{T_e}/v_0 may greatly exceed $\sqrt{M/m}$ (see (36)).

on the mass ratio m/M. Thus, for example, if condition (35) is satisfied this condition gives for a hydrogen plasma with $T_e/T_i = 25$ a value $E \ll 25E_c$, whereas when $T_e/T_i = 10^3$ it becomes $E \ll 10^{18}E_c$, i.e., it imposes practically no limitations on the field E[11]. It must be emphasized, however, that if there are no additional losses, i.e., $\delta T_{e,i}/\delta t = 0$, or if condition (44) is violated, then the function $f_e^{(0)}(v)$ can decrease for large $v \gg v_{Te}$ much more slowly than Maxwellian, and thus the criterion (36) may turn out to be weaker than (45). To estimate the limits of applicability in this case it is necessary, of course, to use the latter. We note, finally, that it follows from condition (45) that total neglect of the collision frequency corresponds obviously to the case of an infinitely strong field, when the iteration method used by us is not applicable. The problem then becomes essentially nonstationary and calls for a special investigation.

The second assumption used by us was that the change of the anisotropic part of the distribution function $f_e^{(1)}(v, \xi, t)$ is slow, and that $\partial f_e^{(1)}/\partial t$ can be neglected compared with $D_{\xi\xi} v^{-2} \partial f_e^{(1)}/\partial \xi$. Taking (38) into account, we can easily verify that this is justified if

$$\frac{F}{E_c} \ll \frac{v_0}{16 v_{Te}} \exp \frac{v_{Te}^2}{v_0 \bar{v}_{ze}}. \qquad (47)$$

Since, according to (40), $\bar{v}_{ze} \ll v_{Te}$, this condition is certainly satisfied if condition (46) is satisfied.

Neglect of the derivative $\partial W/\partial t$ in Eq. (4) is permissible, obviously, if $\partial \ln W/\partial t \ll \gamma_s^{(0)}$ or, what is the same, if [12]

$$\frac{E^2}{nT_e} \ll \frac{m}{M}\left(\frac{s_0}{s_m}\right)^4 \approx \frac{m}{M}\left(\frac{T_i}{T_e}\right)^2\left[\ln 4\frac{M}{m}\left(\frac{T_e}{T_i}\right)^3\right]^2. \qquad (48)$$

We note that the criteria (47) and (48) are meaningful only if $\partial f_e^{(0)}/\partial t \neq 0$.

Finally, it should be indicated that the quasistationary mode investigated by us is realizable, strictly speaking, only when the external field E is turned on adiabatically. If it is turned on instantaneously, then it is necessary to assume that during the rise time of the noise ($\sim (M/m)\omega_0^{-1}$) the electrons do not have time to acquire a velocity $\sim v_{Te}$. This imposes an additional limitation on the field:

$$E^2/nT_e < 4\pi(m/M)^2. \qquad (49)$$

It is possible, however, that this criterion is exaggerated and can be slightly relaxed.

As regards the allowance for effects of higher order in the nonlinearity, particularly nonlinear damping (or s-s scattering), these apparently play no important role whatever, for by virtue of relation (23) the decrement of the nonlinear damping is identically equal to zero in this approximation[9].

In conclusion, let us summarize briefly our result. Immediately after the field is turned off, the electrons begin to accelerate until their average velocity exceeds the value v_0 at which excitation of ion sound is possible. The noise energy then increases rapidly and reaches its quasistationary value, determined by formulas (22) and (28)—(30). Simultaneously, as a result of an increase in the scattering of the electrons by the ion-sound noise, the electron-collision effective frequency increases rapidly, and this limits the average velocity and consequently the current flowing through the plasma. The plasma resistance then increases strongly and turns out to be directly proportional to the applied field (see (40)). This is followed by a quasistationary stage, in which both the electron and ion temperatures rise (see (41) and (42)). It is important to emphasize here that if conditions (45)—(49) are satisfied, then the number of runaway electrons is negligibly small even for fields E that exceed E_c, and the runaway electrons have no influence on the character of the plasma[13]. On the other hand, if the process lasts sufficiently long, or if the conditions (45)—(49) are violated as a result of the increase in the temperature, then the number of runaway electrons can increase strongly, and to analyze the problem it is necessary to take into account already the interaction between the electrons and the high-frequency Langmuir oscillations.

Finally, it must be noted that the results obtained here will apparently become valid also in the presence of a magnetic field H parallel to E and satisfying the condition $H^2 \ll 4\pi nmc^2$. Then the presence of noise should lead to anomalous diffusion that increases with increasing field E.

APPENDIX

We present a solution of the integral equation (26). An analysis of this equation shows that the derivatives of the function $W_0(x)$ become infinite at

[11]The inequality (44) goes over with this into $E \ll 70E_c$.

[12]We note that when the condition (48) is satisfied the total noise energy is $\mathscr{E} \ll nT_e$.

[13]A rough estimate of the flux of runaway electrons contains too much leeway, and would be meaningless here. A more or less rigorous solution of this problem calls for a special analysis of Eqs. (3) and (4).

$x = x_0$. Therefore, in order to make the sought function regular at this point, we change over in (26) to new variables t and ζ, with a new unknown function U(t), with the aid of the relations

$$t = \frac{\sqrt{x^2 - x_0^2}}{x_0}, \quad \zeta = \frac{\sqrt{y^2 - x_0^2}}{x_0},$$

$$U(t) = \frac{A}{x_0^2} W_0(x_0\sqrt{1 + t^2}). \qquad (A.1)$$

Then Eq. (26) takes the form

$$\zeta(1 + \zeta^2) = \int_0^1 \frac{U(t, \zeta)\,t\,dt}{\sqrt{1 - t^2}} [(1 + t^2\zeta^2)^{1/2} - \lambda x_0(1 + \zeta^2)]. \quad (A.2)$$

Recognizing that U(0) = 0 by virtue of continuity, and consequently U(t) is an odd function of the argument, we shall seek a solution of Eq. (A.2) in the form of a series

$$U(t) = \sum_{k=0}^{\infty} U_k t^{2k+1}. \qquad (A.3)$$

Substituting (A.3) in (A.2) and equating coefficients at equal powers of ζ, we obtain

$$U_0 = \frac{1}{b_1(1 - \lambda x_0)},$$

$$U_1 = \frac{1}{b_2(1 - \lambda x_0)} + \frac{\lambda x_0}{1 - \lambda x_0} S_1 U_0 - \frac{\alpha_1 U_0}{1 - \lambda x_0},$$

. . . .

$$U_k = \frac{\lambda x_0}{1 - \lambda x_0} S_k U_{k-1} - \frac{1}{1 - \lambda x_0} \sum_{n=1}^{k} \alpha_n U_{k-n}, \qquad (A.4)$$

where

$$S_k = \frac{b_k}{b_{k+1}} = 1 + \frac{1}{2k + 1},$$

$$b_k = \int_0^1 \frac{t^{2k}\,dt}{\sqrt{1 - t^2}} = \frac{\sqrt{\pi}}{2} \frac{\Gamma(k + 1/2)}{k!}$$

$\Gamma(z)$ is the gamma function and

$$\alpha_0 = 1, \quad \alpha_n = (-1)^{n-1} \frac{\Gamma(n - 1/2)}{2\sqrt{\pi}\,n!}, \quad n \geqslant 1$$

are the expansion coefficients of the function $(1 + t^2)^{1/2}$ in a Taylor series

$$(1 + t^2)^{1/2} = \sum_{n=0}^{\infty} \alpha_n t^{2n}.$$

Since the quantities b_k, α_n, S_k, x_0, and λ are known, relations (A.4) enable us to find all the coefficients U_k and consequently the function U(t). However, in order to avoid summation of the series (A.3), we can proceed in somewhat form and, using relations (A.4), obtain for U(t) a differential equation. Indeed, since

$$U(t) = \sum_{k=0}^{\infty} U_k t^{2k+1} = U_0 t + U_1 t^3 + \sum_{k=2}^{\infty} U_k t^{2k+1}$$

$$= U_0 t + \frac{t^3}{b_2(1 - \lambda x_0)} + \frac{1}{(1 - \lambda x_0)}$$

$$\times \sum_{k=1}^{\infty} t^{2k+1} \Big[S_k U_{k-1} \lambda x_0 - \sum_{n=1}^{k} \alpha_n U_{k-n} \Big]$$

$$= U_0 t + \frac{t^3}{b_2(1 - \lambda x_0)} + \frac{\lambda x_0}{1 - \lambda x_0} t^2$$

$$\times \sum_{k=0}^{\infty} U_k t^{2k+1} + \frac{\lambda x_0}{1 - \lambda x_0} \sum_{k=0}^{\infty} U_k \frac{t^{2k+3}}{2k + 3} -$$

$$- \frac{1}{1 - \lambda x_0} \sum_{k=0}^{\infty} t^{2k+1} \Big[\sum_{n=0}^{k} \alpha_n U_{k-n} - U_k \Big],$$

we get the equality

$$U(t) = U_0 t + \frac{t^3}{b_2(1 - \lambda x_0)} + \frac{\lambda x_0}{1 - \lambda x_0} t^2 U(t)$$

$$+ \frac{\lambda x_0}{1 - \lambda x_0} \int_0 t U(t)\,dt - \frac{U(t)}{1 - \lambda x_0} [\sqrt{1 + t^2} - 1]. \qquad (A.5)$$

Differentiating it with respect to t, we get the following differential equation for U(t):

$$dU/dt - \varphi(t) U = F(t), \quad U(0) = 0;$$

$$\varphi(t) = \frac{t[3\lambda x_0\sqrt{1 + t^2} - 1]}{(1 + t^2)[1 - \lambda x_0\sqrt{1 + t^2}]},$$

$$F(t) = \frac{1/b_1 + 3t^2/b_2}{\sqrt{1 + t^2}[1 - \lambda x_0\sqrt{1 + t^2}]}. \qquad (A.6)$$

Thus

$$U(t) = \int_0^t F(t')\,dt' \exp \int_{t'}^t \varphi(t'')\,dt''. \qquad (A.7)$$

Recognizing now that

$$b_1 = \frac{4}{\pi}, b_2 = \frac{16}{3\pi}, \int_0^t \varphi(t')\,dt' = \ln \frac{(1 - \lambda x_0)^2}{\sqrt{1 + t^2}[1 - \lambda x_0\sqrt{1 + t^2}]^2}$$

and calculating the integral in (A.7), we get

$$U(t) = \frac{4}{\pi\sqrt{1 + t^2}[1 - \lambda x_0\sqrt{1 + t^2}]^2}$$

$$\times \Big[t\Big(1 + \frac{4}{3} t^2\Big) - \lambda x_0 t(1 + t^2)^{3/2}$$

$$- \frac{\lambda x_0}{4} \ln(t + \sqrt{1 + t^2}) \Big]. \qquad (A.8)$$

Returning finally in (A.8) to the old variables, in accord with (A.1), we obtain expression (29) for $W_0(x)$.

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