SOVIET PHYSICS JETP VOLUME 24, NUMBER 6 JUNE, 1967

WAVE PROPAGATION IN A JOSEPHSON TUNNEL JUNCTION IN THE PRESENCE OF VORTICES AND THE ELECTRODYNAMICS OF WEAK SUPERCONDUCTIVITY

I. O. KULIK

Physico-technical Institute for Low Temperatures, Academy of Sciences, Ukrainian S.S.R.

Submitted to JETP editor June 24, 1966

J. Exptl. Theoret. Phys. (U.S.S.R.) 51, 1952—1969 (December, 1966)

Some features of the dynamic mixed state of "weak" superconductivity (the Josephson effect) in external fields **H** ⊥ **E** are investigated. Such a state, characterized by the presence of a moving vortex chain, arises when the electric and magnetic field strengths exceed certain critical values. The equation of the critical state is derived. The magnetization curve for a Josephson superconductor $M(H_0)$, similar to the corresponding curve in the Abrikosov theory of superconductors of the second kind, is plotted. Propagation of waves in the presence of vortices is considered. Vortex motion in the mixed state is accompanied by the appearance of radiation with frequencies $\omega_n = 2en\overline{V}/\hbar$ where $\overline{V}$ is the mean barrier voltage. The model which admits of an exact solution can also be employed for a qualitative analysis of "resistive" effects in superconductors of the second kind.

1. INTRODUCTION

In this paper we investigate the electromagnetic properties of thin dielectric films placed between superconductors. Interest in objects of this type appeared after Josephson[1] discovered the phenomenon which was subsequently called "weak" superconductivity. If the thickness of the film under consideration is sufficiently small ($l \sim 10$—20 Å), then the presence of the dielectric does not prevent the superconductivity correlation of electrons between the metals, as a result of which there is a possibility of the complete passage of the superconducting tunnel current through the barrier. This kind of dielectric film with the metal regions of the order of the field penetration depth into the superconductor adjoining it on both sides has a number of unusual properties.

In sufficiently weak fields it behaves like an ideal diamagnet ($\mu = 0$) and an ideal paraelectric ($\epsilon = \infty$). It constitutes a (two-dimensional) model of a structure which can be referred to as a superconducting dielectric (at $T = 0$). Stronger electric and magnetic fields penetrate into the weak superconductor, producing a state completely analogous to the mixed state of superconductors of the second kind introduced by Abrikosov.[2] However, in the presence of an electric field the vortex lattice is moving, i.e., such a state can be referred to as a dynamic mixed state. The model considered below is very simple mathematically and admits in all cases of an exact solution. Besides being of independent interest, its study may, it seems to us, also turn out to be useful for the study of so-called "resistive" effects in superconductors of the second kind (see[3]).

The appearance of a coherent phase difference between the superconductors separated by a film, which occurs with the appearance of weak superconductivity, leads also to the possibility of the existence of additional branches of the vibrational spectrum in which the varying quantity is the difference of the phases of φ. Inasmuch as according to the Josephson relations it is the derivatives of φ with respect to the coordinates and the time which determine the magnetic field H and the potential difference V between the superconductors, such vibrations will manifest themselves in the form of electromagnetic waves propagating along the film surface. In the nonsuperconducting state of the film (in the absence of the Josephson effect) such waves were investigated by Swihart[4] who showed that the velocity of propagation c_0 of these waves is given by the formula $c_0 = c(l/2\epsilon_0\lambda_L)^{1/2}$ where l is the film thickness, ϵ_0—its dielectric constant, c—the velocity of light, and λ_L—the London penetration depth of the superconductor ($\lambda_L \gg l$). However, the appearance of weak superconductivity in thin films leads to the circumstance that such waves take on a threshold dispersion[1]:

$$\omega^2 = \omega_0^2 + c_0^2k^2, \quad \omega_0 = c_0/\lambda_j \sim 10^{10}\text{—}10^{11}\ \text{sec}^{-1}$$

where λ_j is the so-called Josephson penetration depth.[5]

In a magnetic field exceeding a certain critical value H_{c1} there appear within the Josephson tunnel junction, in complete analogy with the case of superconductors of the second kind,[2] quantum filaments of magnetic flux. At the same time, as was shown in that paper, for $H > H_{c1}$ waves which do not have the threshold frequency can propagate along the filaments. These waves can be considered to be flexural oscillations of vortex filaments. Their velocity coincides with c_0 for stationary filaments and is $c_0(1-\beta^2)^{1/2}$ for moving filaments ($\beta = v/c_0$ where v is the velocity of the motion). In stronger fields the filaments approach each other and form a periodic structure. In this case propagation of waves becomes also possible in a direction perpendicular to the magnetic field (which lies in the plane of the film). For small k these waves also have a sound dispersion, their velocity increasing with increasing magnetic field (it vanishes for $H = H_{c1}$ and tends asymptotically to c_0 for $H \gg H_{c1}$).

All the types of waves considered should also have analogs in the theory of superconductors of the second kind; however, in the latter case, their study even within the framework of the Ginzburg-Landau theory meets with considerable mathematical difficulties. The study of a model admitting of an exact solution is in this connection of definite interest.

2. BASIC EQUATIONS

We shall consider plane tunnel junctions whose dimensions are large compared with the parameter λ_j. In this case a "macroscopic" description is possible in which a weak superconductor is considered to have definite values of H, B, E, D, ϵ, and μ (the magnetic field lies in the plane of the junction, and the electric field is directed along its normal). According to the Josephson relations,[1] the local values of the fields are expressed in terms of derivatives of the phase φ with respect to the coordinates and the time by means of the relations

$$\mathbf{H} = \frac{\hbar c}{4e\lambda_L}[\mathbf{n}, \nabla\varphi], \quad V = lE_z = \frac{\hbar}{2e}\frac{\partial\varphi}{\partial t}, \quad \varphi = \varphi(x, y, t), \tag{2.1*}$$

where **n** is a unit vector along the normal to the plane of the junction (**n** ∥ z).

The function $\varphi(x, y, t)$ satisfies the following nonlinear equation[5-7]:

$$\Delta\varphi - \frac{1}{c_0^2}\frac{\partial^2\varphi}{\partial t^2} = \frac{1}{\lambda_j^2}\sin\varphi. \tag{2.2}$$

*$[\mathbf{n}, \nabla\varphi] = \mathbf{n} \times \nabla\varphi$.

Here Δ is the two-dimensional Laplace operator, and c_0 and λ_j are the basic parameters of weak superconductivity. The expression for c_0 was given above, and λ_j is of the form[5]

$$\lambda_j^2 = \hbar c^2 / 8\pi e(\lambda_{L1} + \lambda_{L2}) j_s, \tag{2.3}$$

where j_s is the density of the critical Josephson tunnel current which can according to[1] be expressed in terms of the conductivity of the tunnel contact in the normal state and the energy gaps of the two superconductors (below we consider for simplicity the case of a symmetric contact with $\Delta_1 = \Delta_2$ and $\lambda_{L1} = \lambda_{L2}$). c_0 is commonly a quantity of the order of 10^9 cm/sec, and λ_j is of the order of magnitude of 0.1 mm (see[8]).

Equation (2.2) is valid for sufficiently weak (electric and magnetic) fields with a sufficiently slow spatial and temporal variation. The conditions for the applicability of this equation can be written in the form

$$eV \ll \Delta, \quad H \ll H_{cm}; \quad \omega \ll \Delta/\hbar, \quad a \gg \xi_0, \lambda_L, \tag{2.4}$$

where H_{cm} is the thermodynamic critical field of a bulk superconductor, Δ is its energy gap, $\xi_0 \sim \hbar v_0/\Delta$ is the correlation length, a is the characteristic spatial period, and ω is the frequency of the field change along the junction. Conditions (2.4) are usually fulfilled experimentally (see[8]) because the critical parameters of weak superconductivity (H_{c1} and V_{c1}, see below) are very small[1)].

Let us draw attention to the analogy of Eq. (2.2) with the modified Ginzburg-Landau equation proposed in[3] for describing "resistive" effects in superconductors of the second kind. This equation, just as the equation proposed in[3], is a local nonlinear wave-type equation[2)].

Below we shall find useful an expression for the energy of a weak superconductor. The energy referred to unit area of the surface of the tunnel junction is of the form

$$\varepsilon = \text{const} + \frac{\hbar j_s}{2e}\left\{-\cos\varphi + \frac{\lambda_j^2}{2}(\nabla\varphi)^2 + \frac{\lambda_j^2}{2c_0^2}\left(\frac{\partial\varphi}{\partial t}\right)^2\right\}. \tag{2.5}$$

According to Josephson,[1] the first term of this equation, $(\hbar j_s/2e)(-\cos\varphi + \text{const})$, represents an addition to the energy of two weakly coupled super-

1)Larkin and Ovchinnikov obtained [9] an equation which replaces (2.2) when V and ω are not small compared with Δ. However, we shall not consider these changes and assume that the conditions for the applicability of the adiabatic approximation $eV \ll \Delta$ and $\hbar\omega \ll \Delta$ are fulfilled.

2)We take the opportunity to note here that a phenomenological derivation of a similar equation on the basis of a variational principle analogous to that cited in [3] is also contained in a previous paper by Suhl. [10]

conductors which depends on the difference of the phases φ and which is due to the interaction (the so-called tunnel Hamiltonian) H_T between them. The second term, containing $(\nabla\varphi)^2$, can be rewritten with the aid of (2.1) as $2\lambda_L H^2/8\pi$, i.e., it represents the energy of the magnetic field that penetrates inside the weak superconductor. Finally, the last term in (2.5) can be expressed in the form $l\epsilon_0 E^2/8\pi$ where $\mathbf{E} = (V/l)\mathbf{n}$ is the electric field intensity. We consider the thickness of the dielectric to be small compared with the London penetration depth λ_L ($\lambda_L \sim 10^{-5}$ cm), but large compared with the Debye length λ_D characterizing the penetration of the electrical field into the metal ($\lambda_D \sim 1$ Å). Under these conditions one can assume the electrical field to be concentrated mainly in the dielectric film (one can neglect the dependence of E on z), whereas the magnetic field can be assumed to be concentrated in the portions of the superconductor adjoining the film.

Considering the junction as a macroscopic body, we must average expression (2.5), i.e., find $\bar{\epsilon}$. The latter quantity can also be obtained in another way, by considering an (imaginary) magnetization process of a weak superconductor which leads to given values of B, H, etc. We have (see[11])[3)]

$$d\bar{\varepsilon} = 2\lambda_L \frac{H_0 d\bar{H}}{4\pi} + l\frac{\bar{E} dE_0}{4\pi}. \quad (2.6)$$

Here H_0 is the external magnetic field, E_0—the "external" electric field with the value V_0/l where V_0 is the voltage supplied to the capacitor plates. Instead of the energy $\bar{\epsilon}$, it is more convenient to use the thermodynamic potential $\tilde{\epsilon}$ which is a function of M and P (M is the magnetic and P the electric moment per unit volume), i.e., it is in fact a function of $\bar{H}$ and $\bar{E}$. For this quantity we obtain[11] (we are referring to the average value)

$$d\tilde{\varepsilon} = 2\lambda_L \frac{H_0 d\bar{H}}{4\pi} - l\frac{E_0 d\bar{E}}{4\pi}. \quad (2.7)$$

In relations (2.6) and (2.7) we make allowance for the fact (already noted above) that the magnetic field is concentrated in an effective layer of thickness $2\lambda_L$ and the electric field—in a layer of thickness l ($\lambda_L \gg l \gg \lambda_D$)[4)].

In the following, it will be more convenient to go over to the dimensionless variables

$$x' = x/\lambda_j, \quad y' = y/\lambda_j, \quad t' = c_0 t/\lambda_j.$$

We introduce also dimensionless units of measurement for the energy ϵ, the magnetic field H, and electric field E in accordance with the relations

$$\varepsilon' = \varepsilon\Big/\left(\frac{\hbar j_s}{2e}\right), \quad H' = H\Big/\left(\frac{\hbar c}{4e\lambda_L\lambda_j}\right), \quad E' = E\Big/\left(\frac{\hbar c_0}{2e\lambda_j l}\right). \quad (2.8)$$

After this, Eqs. (2.1), (2.2), (2.5), and (2.7) take on a simpler form (we omit the primes below):

$$\Delta\varphi - \partial^2\varphi/\partial t^2 = \sin\varphi, \quad (2.9)$$

$$\mathbf{H} = [\mathbf{n}, \nabla\varphi], \qquad \mathbf{E} = (\partial\varphi/\partial t)\mathbf{n}, \quad (2.10)$$

$$\bar{\varepsilon} = \text{const} + \langle -\cos\varphi + \tfrac{1}{2}(\nabla\varphi)^2 + \tfrac{1}{2}(\partial\varphi/\partial t)^2 \rangle, \quad (2.11)$$

$$\tilde{\varepsilon} = \text{const} + \langle -\cos\varphi + \tfrac{1}{2}(\nabla\varphi)^2 - \tfrac{1}{2}(\partial\varphi/\partial t)^2 \rangle, \quad (2.12)$$

$$d\tilde{\varepsilon} = H_0 d\bar{H} - E_0 d\bar{E}. \quad (2.13)$$

(the angle brackets indicate averaging).

3. DYNAMIC MIXED STATE OF WEAK SUPERCONDUCTIVITY

In this section we investigate the electromagnetic properties of weak superconductivity in fields exceeding the critical value (which will be obtained below) when there appears a structure of moving vortex filaments (dynamic mixed state); the period and velocity of this structure are determined by the values of the magnetic and electric fields applied to the junction.

Let us consider the solution of (2.9) which depends on the difference $x - \beta t = \xi$ where $\beta = v/c_0$ (v is the velocity of motion of the vortex structure, $v < c_0$). The equation for $\varphi = \varphi_0(\xi)$ (where φ_0 corresponds to the equilibrium state) is of the form

$$(1 - \beta^2)\, d^2\varphi_0/d\xi^2 = \sin\varphi_0 \quad (3.1)$$

or, after going over to the variable $\zeta = \xi/(1-\beta^2)^{1/2}$

$$d^2\varphi_0/d\zeta^2 = \sin\varphi_0. \quad (3.2)$$

This equation has the first integral

$$\frac{1}{2}\left(\frac{d\varphi_0}{d\zeta}\right)^2 = -\cos\varphi_0 + \text{const} = 2\left(\sin^2\frac{\varphi_0}{2} + \alpha^2\right), \qquad \text{const} \geqslant 1, \quad (3.3)$$

where α is some constant.

[3)]For simplicity we assume below that the dielectric constant of the film material ϵ_0 is unity.

[4)]In the usual notation (see [11]) Eq. (2.7) is of the form $d\tilde{\epsilon} = (2\lambda_L H dB - (DdE)/4\pi$. In our case $B = \bar{H}$, $E = \bar{E}$, i.e., B represents the average value of the magnetic and E of the electric field in the sample. Furthermore, we must assume $H = H_0$, since the demagnetizing factor $n_H = 0$ (it is assumed that the boundary of the film is parallel to the external magnetic field H_0). For the electric field we have $D = E_0 = V_0/l$, since the depolarizing factor $n_E = 1$ (the electric field is normal to the surface of the film).

Integrating again and assuming $\varphi_0(0) = \pi$ (which is not essential), we obtain

$$\zeta = \frac{1}{2}\int_{\pi}^{\varphi_0} d\varphi \Big/ \sqrt{\sin^2\frac{\varphi}{2} + \alpha^2}. \qquad (3.4)$$

The obtained equation determines in implicit form the ζ dependence of φ_0. The function $\varphi_0(\zeta)$ satisfies the relation

$$\varphi_0(\zeta + 2\tau) = \varphi_0(\zeta) + 2\pi \qquad (3.5)$$

with a certain value of τ, obtained from the condition

$$\tau = \int_0^{\pi/2} d\varphi \Big/ \sqrt{\sin^2\frac{\varphi}{2} + \alpha^2} = \frac{1}{\sqrt{1+\alpha^2}} K\left(\frac{1}{\sqrt{1+\alpha^2}}\right) \qquad (3.6)$$

[K(x) is the complete elliptical integral of the first kind[12]]. The origin in (3.4) has been chosen in such a way that $\varphi_0(-\tau) = 0$ and $\varphi_0(\tau) = 2\pi$ (Fig. 1a).

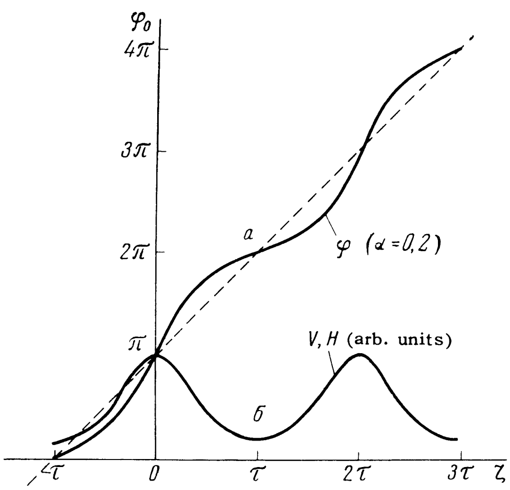

FIG. 1

The electric and magnetic fields in the junction, expressed in terms of $d\varphi_0/d\zeta$, are periodic functions of ζ with a period 2τ (Fig. 1b). On the basis of (2.10) we have

$$H = H_y = \frac{1}{\sqrt{1-\beta^2}}\frac{d\varphi_0}{d\zeta}, \qquad E = E_z = -\frac{\beta}{\sqrt{1-\beta^2}}\frac{d\varphi_0}{d\zeta},$$

$$\frac{d\varphi_0}{d\zeta} = 2\sqrt{\sin^2\frac{\varphi_0}{2} + \alpha^2}. \qquad (3.7)$$

It is readily seen that the following relations occur:

$$\int_{x_0}^{x_0+\Delta x} H dx = 2\pi, \qquad \int_{t_0}^{t_0+\Delta t} E dt = 2\pi, \qquad (3.8)$$

where the integrals are over the period [on the x or t axis respectively, $\Delta x = 2\tau(1-\beta^2)^{1/2}$, $\Delta t = 2\tau\beta^{-1}(1-\beta^2)^{1/2}$]. In the usual variables the

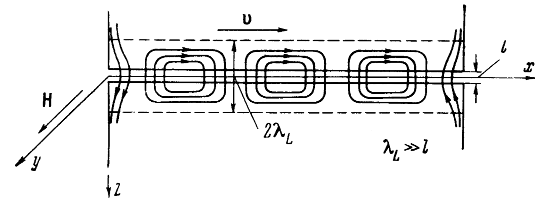

FIG. 2

first of Eqs. (3.8) expresses the quantization condition of the magnetic flux (cf.[2])

$$2\lambda_L \int_{(\Delta x)} H dx = \Phi_0, \qquad \Phi_0 = hc/2e, \qquad (3.9)$$

and the second relation gives an analogous quantum condition for the electric field

$$l \int_{(\Delta t)} E dt = h/2e = \Phi_0/c. \qquad (3.10)$$

The solution of (3.4) and (3.7) describes the periodic structure of the moving vortex filaments, each of which carries a quantum of the flux Φ_0 (see Fig. 2; the arrows show the flow lines). The period of this structure determined by the parameter α should be found on the basis of thermodynamic considerations, in analogy with Abrikosov's theory.[2] α turns out to be some function of the external magnetic field H_0.

A. Let us consider initially the stationary case when the vortices are not moving, i.e., $\beta = 0$ (V = 0).

Let us calculate the energy corresponding to a given distribution of vortex filaments. On the basis of (2.11) this quantity is

$$\bar{\varepsilon}(\alpha) = \text{const} + \frac{1}{2\tau}\int_{-\tau}^{\tau}\left[-\cos\varphi_0(\zeta) + \frac{1}{2}\left(\frac{d\varphi_0}{d\zeta}\right)^2\right]d\zeta. \qquad (3.11)$$

The integral occurring here can be readily evaluated, and this yields with account of (3.6)

$$\bar{\varepsilon}(\alpha) = (1+\alpha^2)E\left(\frac{1}{\sqrt{1+\alpha^2}}\right)\Big/K\left(\frac{1}{\sqrt{1+\alpha^2}}\right) - \frac{1}{2}\alpha^2 + C_1 \qquad (3.12)$$

[E(x) is a complete elliptical integral of the second kind, and C_1 is a constant].

Now let us make use of expression (2.13). Differentiating it with respect to α, we obtain

$$H_0 = \frac{d\bar{\varepsilon}/d\alpha}{d\bar{H}/d\alpha}. \qquad (3.13)$$

The dependence of $\bar{H}$ on α can be readily found on the basis of (3.7):

$$\bar{H}(\alpha) = \frac{1}{2\tau}\int_{-\tau}^{\tau} H d\zeta = \frac{\pi}{\tau} = \pi\sqrt{1+\alpha^2}/K\left(\frac{1}{\sqrt{1+\alpha^2}}\right). \qquad (3.14)$$

Substituting (3.12) and (3.14) in (3.13), we find the dependence of H_0 on α; this allows one subsequently to find $\alpha = \alpha(H_0)$, after which one can calculate $\bar{H} = B$ as a function of H_0. Introducing for convenience a new variable $\gamma = 1/(1+\alpha^2)^{1/2}$, we obtain

$$\overline{H} = \frac{\pi}{\gamma K(\gamma)}, \quad H_0 = \frac{4}{\pi}\frac{E(\gamma)}{\gamma}, \quad 0 \leqslant \gamma \leqslant 1. \quad (3.15)$$

These equations determine in implicit form the dependence of $\overline{H}$ on H_0, i.e., the magnetization curve of a Josephson superconductor. $E(\gamma)/\gamma$ is a monotonically decreasing function of γ, its minimum value (for $\gamma = 1$) being unity. Consequently the smallest possible value of H_0, which we shall denote by H_{c1}, is $4/\pi$: $H_{c1} = 4/\pi$. For $H_0 < H_{c1}$ Eq. (3.15) has no solution; in this case there exists only the trivial solution of Eq. (3.1) $\varphi_0 = 0$, corresponding to ideal diamagnetism (the field inside the superconductor vanishes: $B = \overline{H} = 0$, $M = -H_0/4\pi$, $\mu = 0$). A nonvanishing solution corresponding to the onset of penetration of the field into the superconductor appears first for $H_0 = H_{c1}$ [5]. Returning to the usual variables (Sec. 2), we obtain for H_{c1}

$$H_{c1} = \Phi_0 / \pi^2 \lambda_L \lambda_j, \quad (3.16)$$

which coincides with the expression which has been obtained for the onset of penetration of the field into a weak superconductor by Josephson.[7] Our Eqs. (3.15) allow one to obtain the entire course of the magnetization curve of a weak superconductor $B(H_0)$ for $H_0 > H_{c1}$.

The dependence of the magnetic moment $M = (\overline{H} - H_0)/4\pi$ on the field which follows from (3.15) is shown in Fig. 3 [6]. This curve bears a qualitative resemblance to the corresponding curve in Abrikosov's theory.[2] Just as in[2], for $H_0 = H_{c1}$ the $M(H_0)$ curve has a vertical tangent on the right. In large fields ($H_0 \gg H_{c1}$) the asymptotic of the magnetic moment is of the form

$$M(H_0) \approx -1/8\pi H_0^3. \quad (3.17)$$

The area under the obtained curve determines the "thermodynamic" critical field of weak superconductivity H_c:

$$-\int_0^\infty M(H_0)\, dH_0 = \frac{H_c^2}{8\pi}. \quad (3.18)$$

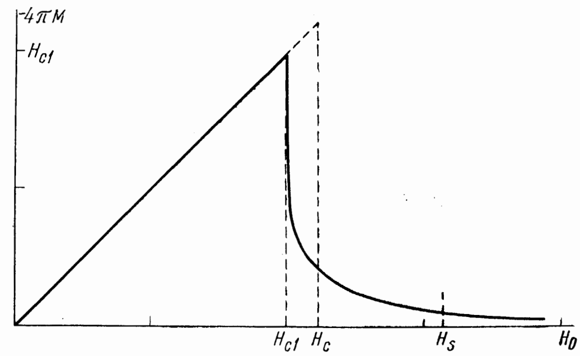

Fig. 3

Making use of relation (3.15), we obtain after some transformations

$$H_c^2 = H_{c1}^2 \left[1 + 2\int_0^1 \left(E(\gamma)K(\gamma) - \frac{\pi^2}{4}\right)\frac{d\gamma}{\gamma^3}\right]. \quad (3.19)$$

Evaluating the integral (see, for example,[13]) we obtain

$$H_c = \frac{\pi}{2\sqrt{2}} H_{c1} = 1{,}11 H_{c1}. \quad (3.20)$$

Making use of the expression for H_{c1} cited above, we find that in the adopted dimensionless variables the value of H_c is $\sqrt{2}$. [7] We note that as was to be expected H_c coincides with the field obtained from the condition (written in terms of the usual variables)

$$2\lambda_L \frac{H_c^2}{8\pi} = \frac{\hbar j_s}{2e}, \quad (3.21)$$

where the first term represents an energy increase (per unit of contact area) connected with the Meissner effect and the second term represents an energy decrease due to the superconductive correlation. However, the first-order transition that should have occurred on the basis of these considerations at $H_0 = H_c$ is actually not realized, because already for the lower field H_{c1} there is penetration of the magnetic field due to the occurrance of a periodic structure of vortex filaments analogous to Abrikosov's filaments.[2]

A distinctive feature of the given model is the very fast decrease of the magnetic moment with increasing magnetic field above H_{c1} (Fig. 3); this renders the experimental study of the region near H_{c1} difficult. We note, however, that by preparing the tunnel junction in the shape of an ellipse (in the

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5) For $H_0 = H_{c1}$, α vanishes. According to (3.6) the period of the vortex structure is then infinite. For $H_0 \to H_{c1}$ the vortex filaments are far from each other; it is for this reason sufficient to solve the problem with one vortex. In the Appendix we consider this case as an illustration (regardless of the presence of an exact solution for all H_0), since it facilitates a better understanding of the physical aspect of the phenomena under consideration.

6) The total magnetic moment of a weak superconductor is (in the usual units) $2\lambda_L S M(H_0)$ where S is the surface of contact (we take no account of effects due to the partial penetration of the magnetic field into the superconductors along their surfaces which are not in contact).

7) Introducing dimensionless variables, one could obtain for H_c the value $1/\sqrt{2}$ which coincides with that for superconductors of the second kind in the Landau variables [2] in a different manner. However, then Eqs. (2.9)–(2.13) take on a more complicated form (the ratio H_c/H_{c1} is of course independent of the choice of units).

xy plane), i.e., by producing a certain demagnetizing factor $n \neq 0$ one can extend the region of the mixed state in the field. The magnetization curve $M(H_0)$ is then obtained from the curve of Fig. 3 by means of a simple geometrical transformation[8)].

The situation with respect to the magnetic properties of weak superconductivity described above occurs in fields that do not exceed the thermodynamic critical field of a bulk superconductor H_{cm} (for the sake of definiteness we assume that the superconductors adjoining the film are of the first kind). Since H_{cm} usually exceeds H_{c1} considerably ($H_{c1} \sim 1$ Oe, $H_{cm} \sim 10^2$ Oe) and, as has already been noted, the magnetic moment decreases very rapidly with increasing H_0, then for the problem being considered in this paper the clarification of the question of how the magnetic moment will actually behave in large fields ($H_0 \sim H_{cm}$) is not very important. Under these circumstances one can consider the upper critical field (and with it the Ginzburg-Landau weak-superconductivity parameter κ) to be infinite[9)] (however, the results cited above cannot be obtained from Abrikosov's results for $\kappa = \infty$, since weak superconductivity is essentially two-dimensional; the periodic vortex structure which occurs is one-dimensional, and not a two-dimensional one as in Abrikosov's theory[2]).

In principle the nature of the magnetic properties of weak superconductivity in large fields can be of two types: 1) it may turn out that a more consistent account of the correlation effects than that in the derivation of (2.2) will lead to the disappearance of weak superconductivity at some field H_{c2} smaller than H_{cm}; 2) in principle, a situation is possible, in which weak superconductivity is preserved in fields exceeding H_{cm}. Such superconductivity should be localized near the surface of the film. The field (exceeding H_{cm}) at which such "surface" superconductivity disappears will indeed play the role of H_{c2}. However, the possibility of such a phenomenon depends considerably on the relation of magnetic and correlation energy due to the rapid decrease of the ordering parameter on moving away from the film surface (at distances of the order of ξ_0). An investigation of this problem is outside the framework of this article and will be carried out separately.

B. We shall now go over to an investigation of the non-stationary case occurring for a nonvanishing potential difference V between the superconductors. In doing this, we must consider the general solution of Eq. (3.1) characterized by two parameters α and β which specify the period of the vortex structure and its velocity respectively. In order to determine these parameters, it is necessary to calculate the thermodynamic potential $\tilde{\epsilon}$ as a function of α and β, and furthermore use relation (2.13), assuming that the functions $\overline{H}(\alpha, \beta)$ and $\overline{E}(\alpha, \beta)$ are also known. Differentiating (2.13), we obtain

$$H_0 \partial\overline{H} / \partial\alpha - E_0 \partial\overline{E} / \partial\alpha = \partial\tilde{\varepsilon} / \partial\alpha, \qquad (3.22)$$

$$H_0 \partial\overline{H} / \partial\beta - E_0 \partial\overline{E} / \partial\beta = \partial\tilde{\varepsilon} / \partial\beta.$$

Hence one can find H_0 and E_0 as a function of α and β, after which one has to transform this dependence and furthermore obtain $\overline{H}$ and $\overline{E}$ as functions of H_0 and E_0.

Proceeding to the corresponding calculation, we calculate first of all the function $\tilde{\epsilon}(\alpha, \beta)$. On the basis of (2.12) we have

$$\tilde{\varepsilon}(\alpha, \beta) = \text{const} + \frac{1}{2\tau} \int_{-\tau}^{\tau} \left[-\cos\varphi_0 + \frac{1}{2}(1-\beta^2)\left(\frac{d\varphi_0}{d\xi}\right)^2 \right] d\zeta, \qquad (3.23)$$

whence it is seen [after going over from the variable ξ to the variable $\zeta = \xi/(1-\beta^2)^{1/2}$] that this quantity does not actually depend on β and is consequently given by expression (3.12) which is valid for $\beta = 0$.

$\overline{H}(\alpha, \beta)$ is calculated with the aid of (3.7) which yields [cf. (3.14)]

$$\overline{H}(\alpha, \beta) = \pi\sqrt{1+\alpha^2} / K\left(\frac{1}{\sqrt{1+\alpha^2}}\right)\sqrt{1-\beta^2}. \qquad (3.24)$$

Finally, substituting $\varphi_0(\zeta)$ from (3.4) in the second of Eqs. (2.10), we have

$$\overline{E}(\alpha, \beta) = -\beta\overline{H}(\alpha, \beta). \qquad (3.25)$$

Turning to Eqs. (3.22) and taking into account that $\partial\tilde{\epsilon}/\partial\beta = 0$, we obtain on the basis of (3.24) and (3.25)

$$\beta = -\overline{E} / \overline{H} = -E_0 / H_0. \qquad (3.26)$$

Consequently, the velocity of motion of the vortex structure is proportional to the electric field intensity. We note that an analogous formula for the velocity of the vortices exists also in the theory of superconductors of the second kind (see[3]).

Using the remaining equation of the system (3.22) and going over for convenience to the variable $\gamma = 1/(1+\alpha^2)^{1/2}$, we obtain relations, replacing

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8)We investigated the corresponding problem for superconductors of the second kind in [14]. All the results of that paper can be transfered without change to the case considered in this paper.

9)The fact that for weak superconductivity κ is large follows from the formula $\kappa \sim \lambda/\xi_0$ where λ is the electromagnetic length (penetration depth) and ξ_0 the correlation distance. In our case λ_j (not λ_L) plays the part of λ, is a result of which κ turns out to be very large.

(3.15), that are also valid in the non-stationary case:

$$H_0 = H_{c1}^* \frac{E(\gamma)}{\gamma}, \quad \overline{H} = \frac{\pi^2}{4} H_{c1}^* \frac{1}{\gamma K(\gamma)}, \tag{3.27}$$

where

$$H_{c1}^* = \frac{H_{c1}}{\sqrt{1-\beta^2}}, \quad H_{c1} = \frac{4}{\pi}. \tag{3.28}$$

Thus the dependence of $\overline{H}$ on H_0 is given by the curve obtained above (Fig. 3) where one should, however, replace H_{c1} by $H_{c1}^* = H_{c1}/(1-\beta^2)^{1/2}$ and β is given by (3.26).

Equations (3.26)—(3.28) are the solution of the posed problem of finding $\overline{H}$ and $\overline{E}$ as a function of H_0 and E_0. Without entering into a consideration of the corresponding curves, we shall only note the following circumstance. As follows from (3.27) and (3.28), in the nonstationary case the lowest critical field for which the penetration of vortices first occurs is a function of the electric field E_0[10]. Indeed, the minimum value of the field H_0 for which a solution of (3.27) first appears is $H_0^{min} = H_{c1}^*$. Substituting then H_{c1}^* from (3.28) and β from (3.26), we obtain

$$H_0^{min} = \sqrt{H_{c1}^2 + E_0^2}. \tag{3.29}$$

When $H_0 < H_0^{min}$, then $\overline{E} = 0$ and $\overline{H} = 0$. Writing down the relations

$$B = \overline{H} = H + 4\pi M, \quad D = \overline{E} + 4\pi P$$

and introducing formally the magnetic permeability (μ) and the dielectric permittivity (ϵ), we obtain $\mu = \overline{H}/H = 0$ and $\epsilon = D/\overline{E} = \infty$, i.e., the superconductor behaves like an ideal diamagnet and an ideal paraelectric. However, for $H_0 > H_0^{min}$ these properties no longer occur. In this case there is penetration of the vortex filaments of the magnetic flux, as a result of which H becomes nonzero ($\mu = 0$). For $E_0 \neq 0$ the vortex filaments begin to move, i.e., E is also not zero ($\epsilon \neq \infty$). The state of the superconductor which appears thus was referred to above as the dynamic mixed state[11]. Relation (3.29) serves as the equation of the curve that separates the region of the superconducting state I from the mixed state II (see Fig. 4).

C. In conclusion of this section we shall dwell

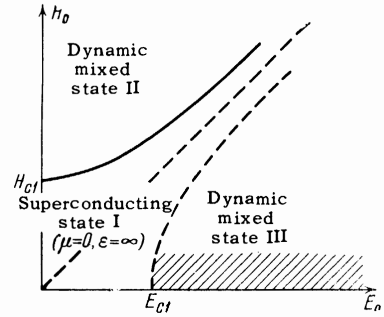

FIG. 4

on the following curious fact. Equation (2.9)

$$\frac{\partial^2\varphi}{\partial x^2} - \frac{\partial^2\varphi}{\partial t^2} = \sin\varphi \tag{3.30}$$

is invariant with respect to a replacement of x by t when φ is simultaneously replaced by $\pi + \varphi$. The thermodynamic identity (2.13) serves as an additional condition for this equation. In the transformation $x \rightleftarrows t$ the quantities H and E are interchanged: $H \rightleftarrows E$ [see (2.10)]. At the same time, $d\tilde{\epsilon}$ changes only its sign. However, as is readily seen from (2.12), the sign of $d\tilde{\epsilon}$ changes also in the replacement $x \rightleftarrows t$, and $\varphi \to \pi + \varphi$. Consequently, all Eqs. (2.9), (2.10), (2.12), and (2.13) remain invariant in the simultaneous transformation

$$x \rightleftarrows t, \quad H \rightleftarrows E, \quad \varphi \to \pi + \varphi. \tag{3.31}$$

Carrying out such a transformation and repeating all the operations presented above, we conclude that below the dashed line in the diagram of Fig. 4 we will also have a dynamic mixed state corresponding a moving vortex system[12]. In zero magnetic field ($H_0 = 0$) such a state appears first at $E_0 = E_{c1}$ where E_{c1} is the "lowest critical electric field." In the dimensionless variables adopted by us E_{c1}, as well as H_{c1}, equals $4/\pi$. In the usual units we shall have

$$E_{c1} = 2\hbar c_0 / \pi e\lambda_j l \sim 10^3 \text{ V/cm} \tag{3.32}$$

(this corresponds to a potential difference $V_{c1} \sim 0.1$ millivolt).

The obtained "vortices" are quantized in the sense of condition (3.10). In term of the coordinates $x = H_0/H_{c1}$, $y = E_0/E_{c1} = V_0/V_{c1}$ the equation of the curve separating the region of the superconducting state I from the mixed state II is of the form $x^2 = 1 + y^2$, and from the mixed state III it is of the

[10]An analogous fact was first noted by us in [3] for the dynamic mixed state of superconductors of the second kind. However, unlike the case considered in [3], where this effect is very small, for weak superconductivity it is large.

[11]Because it corresponds to nonzero resistance, such a state in the theory of superconductors of the second kind is sometimes called "resistive."[3]

[12]The velocity of the vortices in region III is larger than c_0. On the line $H_0 = 0$ it becomes infinite. Obviously, when relativistic effects are taken into account the velocity of motion cannot exceed the speed of light c ($c_0 \ll c$). Consequently, a certain portion of the diagram of Fig. 4 which lies below the dashed curve has no physical significance.

form $y^2 = 1 + x^2$. Taking into account the fact that H_0 and E_0 can be both positive and negative, we arrive at a diagram which is the corresponding generalization of Fig. 4.

The treatment presented in parts B and C of this section refers to the case when the energy dissipation in the vortex motion can be neglected. The superconductor can then be considered in the "dielectric" approximation as a body characterized by some value of ϵ. For T = 0 the dissipation mechanism connected with the nonsuperconducting ("quasi-particle") current[15] is absent. If the dissipation is weak (small T), then the behavior described above will also occur (if there is an energy source compensating for the losses due to dissipation), but a detailed analysis of this problem is outside the framework of this article.

4. WAVE PROPAGATION IN THE MIXED STATE

The behavior of the Josephson tunnel junction described in the preceding section characterizes "equilibrium" properties of weak superconductivity in the absence of energy dissipation (for T = 0). In this section we investigate small deviations from the state of equilibrium. As will be shown below, a small perturbation can propagate in a vortex lattice in the form of waves with a linear dispersion law, and there is no attenuation (this is, naturally, only so in the absence of dissipation). The posed problem can be solved exactly (in the limit of long waves) for arbitrary values of H_0 and E_0 corresponding to a moving vortex lattice. However, in order not to complicate the presentation we shall carry out the calculation for the case of a periodic structure at rest (in the absence of perturbation connected with a wave), and the final results will be cited for moving vortices.

Turning to Eq. (2.9), we set $\varphi = \varphi_0 + \varphi_1$, where φ_0 is the solution corresponding to the equilibrium state [see (3.4)], and φ_1 is a small perturbation. For φ_1 we obtain the following linear equation:

$$\Delta\varphi_1 - \partial^2\varphi_1 / \partial t^2 - \cos\varphi_0\cdot\varphi_1 = 0. \tag{4.1}$$

The solution of this equation will be sought in the form of a plane wave

$$\varphi_1 = \psi(x) e^{iky} e^{-i\omega t}. \tag{4.2}$$

Substituting (4.2) in (4.1), we obtain the following equation for ψ

$$-\frac{1}{2}\frac{d^2\psi}{dx^2} - \sin^2\frac{\varphi_0(x)}{2}\cdot\psi = \varepsilon\psi, \quad \varepsilon = \frac{\omega^2 - k^2 - 1}{2}. \tag{4.3}$$

We have, thus, obtained the Schrödinger equation with the "potential" $U(x) = -\sin^2[\varphi_0(x)/2]$ which is a periodic function of x [with a period 2τ, where τ is found from (3.6)]. The wave function of such an equation can, according to the well-known Floquet theorem, be represented in the form of a Bloch wave

$$\psi(x) = e^{iqx} u_q(x), \tag{4.4}$$

where $u_q(x)$ is a periodic function of x [q is the wave-vector component of the perturbation (4.2) in a direction perpendicular to the vortices].

Substituting (4.4) in (4.3), we find the dependence of the "energy" ϵ on q, and then find by means of the second of relations (4.3) the dependence of the frequency ω on the wave-vector components k and q.

For q = 0, Eq. (4.3) can be solved exactly, the corresponding value of the energy being $\epsilon_0 = -1/2$. Indeed, multiplying both parts of (4.3) by $d\varphi_0/dx$, and using the equation which $\varphi_0(x)$ obeys,

$$d^2\varphi_0 / dx^2 = \sin\varphi_0, \tag{4.5}$$

we obtain

$$\frac{d}{dx}\left(\frac{d\varphi_0}{dx}\frac{du_0}{dx} - \sin\varphi_0\cdot u_0\right) = 0, \quad \frac{d\varphi_0}{dx}\frac{du_0}{dx} - \sin\varphi_0\cdot u_0 = C, \tag{4.6}$$

where C is an integration constant. Dividing both parts of the latter equation by $(d\varphi_0/dx)^2$ and integrating, we have

$$u_0(x) = C\frac{d\varphi_0}{dx}\int^x \frac{dx}{(d\varphi_0/dx)^2} + C_1\frac{d\varphi_0}{dx}. \tag{4.7}$$

As can be seen from (3.3), the function $d\varphi_0/dx$ never vanishes if $\alpha \neq 0$. For $C \neq 0$ the first term in (4.7) is a monotonically increasing function of x. Since we are interested in the periodic solution of (4.3), it is clear that we should set C = 0. Therefore, accurate within an unimportant normalizing factor $u_0(x)$ can be assumed to coincide with $d\varphi_0/dx$:

$$u_0(x) = \frac{d\varphi_0}{dx} = 2\left[\sin^2\frac{\varphi_0(x)}{2} + \alpha^2\right]^{1/2}. \tag{4.8}$$

Now let us consider the case $q \neq 0$. Assuming q to be small (compared with the reciprocal distance between the vortices), we write the expansions

$$\varepsilon(q) = \varepsilon_0 + a_1 q + \frac{1}{2}a_2 q^2 + \ldots, \tag{4.9}$$

$$u_q(x) = u_0(x) + qu_1(x) + q^2 u_2(x) + \ldots \tag{4.10}$$

Substituting (4.9) and (4.10) in (4.3), we have in successive approximations

$$\frac{d^2u_1}{dx^2} - \cos\varphi_0\cdot u_1 = -2a_1u_0(x) - 2i\frac{du_0}{dx}, \tag{4.11}$$

$$\frac{d^2u_2}{dx^2} - \cos\varphi_0\cdot u_2 = -2a_1u_1(x) - 2i\frac{du_1}{dx} + (1 - a_2)u_0(x) \tag{4.12}$$

etc.

Making use of the orthogonality of the functions $u_1(x)$ and $u_2(x)$ and the solution of the homogeneous equation of $u_0(x)$, we find a_1 and a_2. It is then easy to check that $a_1 = 0$ and the coefficient a_2 is given by the expression

$$a_2 - 1 = 4 \int_{-\tau}^{\tau} f(x) \sin\varphi_0(x)\, dx \Big/ \int_{-\tau}^{\tau} (d\varphi_0/dx)^2 dx, \quad (4.13)$$

where f(x) is an arbitrary solution of the equation

$$d^2f/dx^2 - \cos\varphi_0(x)\cdot f = \sin\varphi_0(x) \quad (4.14)$$

[the solution of the corresponding homogeneous equation vanishes on substitution in (4.13)].

The solution of Eq. (4.14) can be found by a method analogous to the one described above for evaluating $u_0(x)$. As a result we obtain

$$f(x) = \frac{d\varphi_0}{dx} \int_{-\tau}^{x} \frac{C_0 - \cos\varphi_0(x)}{(d\varphi_0/dx)^2} dx + C_1 \frac{d\varphi_0}{dx}. \quad (4.15)$$

The integration constant C_0 can be found from the periodicity condition of the function f(x)[13] which gives

$$\int_{-\tau}^{\tau} \frac{C_0 - \cos\varphi_0(x)}{(d\varphi_0/dx)^2} dx = 0. \quad (4.16)$$

A simple calculation leads to the expression

$$C_0 = 1 + 2\alpha^2 \left[1 - K'\left(\frac{1}{\sqrt{1+\alpha^2}}\right) \Big/ E\left(\frac{1}{\sqrt{1+\alpha^2}}\right)\right]. \quad (4.17)$$

Substituting (4.15) and (4.17) in (4.13), we have

$$a_2 = \frac{\alpha^2 K^2((1+\alpha^2)^{-1/2})}{(1+\alpha^2)E^2((1+\alpha^2)^{-1/2})} \qquad \varepsilon(q) \approx -\frac{1}{2} + \frac{1}{2} a_2 q^2. \quad (4.18)$$

Finally, utilizing expression (4.3) for ϵ, we obtain

$$\omega^2 = k^2 + a_2 q^2. \quad (4.19)$$

When the motion of the vortex structure is taken into account, the form of the expression for ω changes as follows ($\beta = v/c_0$):

$$\omega^2 = (1-\beta^2)k^2 + (1-\beta^2)^2 a_2 q^2. \quad (4.20)$$

Here φ_1 is of the form

$$\varphi_1 = \exp\left(\frac{i\omega\beta}{1-\beta^2}\xi\right) u_q(\xi) e^{iq\xi} e^{iky} e^{-i\omega t}, \quad \xi = x - \beta t. \quad (4.21)$$

According to (4.20) the propagation velocity of the waves along the vortices (q = 0) is

$$c_l = c_0\sqrt{1-\beta^2} \quad (4.22)$$

(we are using dimensional units), whereas in the direction perpendicular to the vortices (k = 0) it is given by the expression

$$c_t = c_0(1-\beta^2)F(H_0/H_{c1}), \quad (4.23)$$

where the function F(x) is of the form [see (4.18)]

$$F(x) = \sqrt{1-\gamma^2}K(\gamma)/E(\gamma), \quad \gamma = \frac{1}{\sqrt{1+\alpha^2}}, \quad x = \frac{H_0}{H_{c1}}, \quad (4.24)$$

the dependence of γ on x being given in implicit form by relation (3.15).

A plot of the function F(x) is presented in Fig. 5.

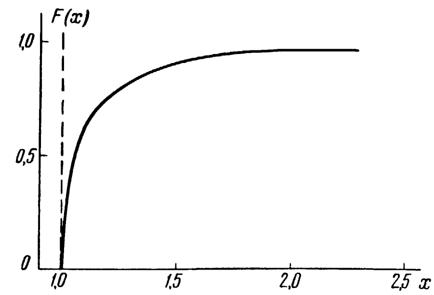

FIG. 5

As can be seen from the obtained curve, the velocity of the waves propagating across the vortices vanishes for $H_0 = H_{c1}$, increases with increasing magnetic field, and tends to c_0 for $H_0 \to \infty$ (if $\beta = 0$). The velocity of the waves along the vortices does not depend on H_0 and is c_0 (for $\beta = 0$). These waves can be considered as flexural oscillations of the vortex filaments (see the Appendix).

The quantity ω introduced above represents the frequency of the wave in a coordinate system connected with the moving vortices. In the laboratory coordinate system there appears a set of frequencies given by the formula (n is an integer)

$$\Omega_n = \frac{\omega}{1-\beta^2} + qv + 2\pi n\nu, \quad \nu = \frac{v}{2\lambda_j\sqrt{1-\beta^2}\,\gamma K(\gamma)} \quad (4.25)$$

(here we have used physical units). Such an expression is obtained with account of all time-dependent phase factors in (4.21), the existence of a discrete set of frequencies following from the periodicity of the function $u_q(\xi)$. The period of this function in t is $\Delta t = 2\tau(\gamma)(1-\beta^2)^{1/2}/\omega_0\beta$, which leads indeed to (4.25).

However, the frequencies $\omega_n = 2\pi n\nu$ are in fact not connected with vortex oscillations, but arise through the motion of the periodic structure of the mixed state as a whole[14]. Obviously these are frequencies corresponding to the so-called nonstationary Josephson effect.[1] Employing the relations (3.26)—(3.28) obtained above, we have

[13] f(x) coincides with $u_1(x)$ to within a coefficient.

[14] We considered an analogous effect for superconductors of the second kind in [3,16].

$$\omega_n = 2en\bar{V}/\hbar, \qquad \bar{V} = l\bar{E}.$$

We have thus obtained the well-known Josephson formula[1]: $2eV = \hbar\omega$; however, the role of the potential difference in this formula is actually taken on by $\bar{V}$, i.e., the average value of V (in addition, harmonics of the Josephson frequency are also present). It is seen from here that just as was to be expected, this effect is absent in the superconducting state (corresponding to the region I of the diagram of Fig. 4), because here $\bar{E} = \bar{H} = 0$ [15].

We note that the analysis carried out in this article is valid without account of the potential barriers to the entry of vortices at the edges of the tunnel junction (see[17], where an analogous problem is considered for superconductors of the second kind). If the magnetic field is sufficiently large ($H_0 > H_S$, see the Appendix) or if the edge of the junction parallel to the external field is "rough," then such effects will be of no importance.

In conclusion I take the opportunity to express my deep gratitude to A. A. Abrikosov and A. I. Larkin for a discussion of this work and valuable remarks.

APPENDIX

The solution corresponding to a single vortex (valid for $H_0 \to H_{c1}$) is obtained from (3.4) for $\alpha = 0$

$$\varphi_0(x,t) = 4\tan^{-1}\exp\frac{x-\beta t}{\sqrt{1-\beta^2}} \qquad \text{(A.1)}$$

Here $\varphi_0 = 0$ for $x \to -\infty$ and $\varphi_0 = 2\pi$ for $x \to +\infty$. The magnetic and electric field distribution corresponding to (A.1) is on the basis of (3.7) of the form

$$H = 2/\sqrt{1-\beta^2}\,\mathrm{ch}\frac{x-\beta t}{\sqrt{1-\beta^2}}, \qquad E = -2\beta/\sqrt{1-\beta^2}\,\mathrm{ch}\frac{x-\beta t}{\sqrt{1-\beta^2}}. \qquad \text{(A.2)*}$$

Both distributions have the form of curves with a maximum at $x = \beta t$, whereas at infinity the field vanishes. According to (A.2) the value of the magnetic field at the maximum is $H_S = \frac{1}{2}\pi H_{c1} \approx 1.6\,H_{c1}$ (for $\beta = 0$). It is interesting to compare this value with the corresponding value in Abrikosov's theory[2] which is $H(0) \lesssim 2H_{c1}$.

In dimensional units the expression for H_S is of the form

$$H_s = \Phi_0 / 2\pi\lambda_L\lambda_j. \qquad \text{(A.3)}$$

It is curious to note that this value coincides with the critical field of Ferrell and Prange,[5] analogous to the surface-barrier field in the theory of superconductors of the second kind introduced by Bean and Livingston[17] (see also[7]).

Let us calculate the energy of a moving filament. The energy referred to unit length of the filament is on the basis of (2.11) of the form[16]

$$\varepsilon = \int_{-\infty}^{\infty}\left\{1-\cos\varphi_0 + \frac{1}{2}\left(\frac{\partial\varphi_0}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial\varphi_0}{\partial t}\right)^2\right\}dx. \qquad \text{(A.4)}$$

Integrating, we obtain

$$\varepsilon = \varepsilon_0/\sqrt{1-\beta^2}, \qquad \text{(A.5)}$$

where in the dimensionless units adopted by us ϵ_0 is equal to eight. It is seen from (A.5) that a vortex filament can be characterized by a certain field mass m^*. In physical units this quantity, referred to unit length of the filament, is

$$m^* = \frac{\varepsilon_0}{c_0^2} = \frac{4\hbar j_s\lambda_j}{ec_0^2} \sim m_e, \qquad \text{(A.6)}$$

where m_e is the mass of the electron (with $\lambda_j \sim 10^{-2}$ cm).

On the other hand, the energy of the filament is proportional to its length, i.e., to the integral

$$\int \varepsilon[1+(\partial X/\partial y)^2]^{1/2}dy,$$

where $x = X(y, t)$ is the equation of the filament shape. Taking the velocities and deformations to be small, we represent the addition to the energy in the form

$$\varepsilon_0\int\left\{\frac{1}{2}\left(\frac{\partial X}{\partial y}\right)^2 + \frac{1}{2c_0^2}\left(\frac{\partial X}{\partial t}\right)^2\right\}dy, \qquad \text{(A.7)}$$

whence it is seen that $X(y, t)$ obeys the wave equation

$$\frac{\partial^2 X}{\partial y^2} - \frac{1}{c_0^2}\frac{\partial^2 X}{\partial t^2} = 0. \qquad \text{(A.8)}$$

Propagation of elastic flexural waves with a velocity c_0 is thus possible along the filament. The treatment is valid for long waves ($k\lambda_j \ll 1$), however, as was shown in Sec. 4 of this paper, actually such waves are also present when k is not small.

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15)We note that in a real situation it is $\bar{E}$ which is commonly given (or $\bar{V}$, which is equivalent) and not E_0 (V_0). This quantity determines the experimentally measured mean barrier voltage. As regards E_0, in order to determine it, one must consider the conditions of connecting the weak superconductor into a "dielectric" circuit (by placing it in an external capacitor).

*ch = cosh.

16)We take the constant appearing in (2.11) to be unity. It can be shown that precisely this value of this constant is obtained from the microscopic theory of the Josephson effect [18] if in the energy account is taken of the term which does not depend on the phase (subtracting the corresponding energy for a normal metal). If the constant is chosen differently, then the integral (A.4) will diverge, however the corresponding "zero energy" has no physical significance and does not influence the subsequent results.

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Translated by Z. Barnea
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