CONTRIBUTION TO THE THEORY OF FERMI STATISTICAL ACCELERATION

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It is shown that the detailed balancing principle is satisfied in Fermi acceleration and that the acceleration reduces to the diffusion of the particles in momentum space. A kinetic equation is derived for particles accelerated in a turbulent plasma, and conditions are obtained under which the effective mean free path does not depend on the momentum and the distribution function is isotropic. Nonstationary solutions without allowance for losses are considered, and it is shown that at high energies the asymptotic behavior of the spectra has a universal character. The effect of containment of fast particles in the turbulent region by collisions with the waves is investigated. Possible applications of the results to the theory of particle acceleration in solar flares and in supernova shells are discussed briefly.

FERMI pointed out an effective mechanism of particle acceleration in a turbulent plasma (collisions with moving magnetic inhomogeneities) and estimated the rate of growth of the average particle energy in such a process.[1] The estimate is based on a comparison of the probabilities of the "head-on" and "rear-end" collisions, at which the particle energy increases and decreases, respectively. At a given particle velocity, the collision probability per unit time is equal to the mean ratio of the modulus of the relative velocity of the particle and the cloud to the distance L between clouds, and in head-on collisions it is somewhat higher than in rear-end collisions. Hence, according to [1],

$$\overline{\frac{dE}{dt}} \approx \left(\frac{\overline{u^2}}{Lc^2}\right) vE, \qquad (1)$$

where E is the total particle energy (including the rest energy), and c, u, and v are the velocities of the light, cloud, and particle.

Fermi's idea was subsequently used in many papers to explain the generation of fast particles in different astrophysical processes. In most papers, the estimates were based on relation (1), It is noted in [2] (Sec. 16) that the fluctuations of the number of head-on and rear-end collisions play a certain role.

This article is devoted essentially to an analysis of the main properties of Fermi acceleration. We show that the detailed balancing principle is satisfied in collisions with magnetic inhomogeneities, and consequently the acceleration reduces to diffusion in momentum space. We consider the role of particle scattering and determine the conditions under which the distribution function can be regarded as isotropic. We calculate the diffusion coefficients in real and momentum spaces, analyze the relative role of magnetic pulsations of varying scales in the dynamics of fast particles, and derive a kinetic equation. These questions are dealt with in Sec. 1. In Secs. 2 and 3 we consider different nonstationary solutions. In Sec. 4 we investigate stationary processes with allowance for synchrotron-radiation losses. By way of illustration we consider certain questions of interest to cosmic-ray physics.

1. FUNDAMENTAL EQUATIONS

We consider a certain region of plasma in a magnetic field, in which the mean values of the intensity H, plasma density N, plasma temperature T, and turbulence characteristics can be regarded as homogeneous (i.e., independent of the coordinates). These parameters can, generally speaking, depend on the time. We shall assume that $T \lesssim \overline{H}^2/8\pi$ and $Nu^2M \lesssim H^2/8$ (M = ion mass, u = velocity of turbulent plasma pulsations). Under these conditions, the turbulence is made up of an aggregate of hydromagnetic waves of different lengths, and order of magnitude of the propagation velocity of the inhomogeneities is determined by the Alfven velocity $u_a = \overline{H}/\sqrt{4\pi MN}$. We denote by L the main turbulence scale and by H_m the magnitude of the field in waves of this scale. We assume that H_m is the maximum amplitude of the perturbation of the magnetic field.

It is natural to choose for the internal turbulence scale a length on the order of the Larmor radius of the thermal ions, for an intense cyclotron dissipation occurs at such wavelengths. The character of variation of the pulsation amplitude with decreasing scale λ is not clarified for the wave turbulence. We can only state that the amplitude decreases faster than $\lambda^{1/3}$ (the Kolmogorov-Obukhov law), since we have in the plasma, besides the nonlinear fragmentation of the waves, also shock-wave formation accompanied by a faster transfer of the energy of the large-scale motions to the small-scale pulsations.

We assume that particle collisions with individual wave packets are statistically independent, for in the case of strong plasma turbulence it is possible for the particles to be displaced rapidly transversely to the force lines as a result of the drift on the shock-wave fronts (the velocity of such a drift is of the order of the particle velocity if the discontinuity in the field is of the order of the mean field, and the width of the front is small or comparable with the Larmor radius of the accelerated charge). In addition, when the pulsations have broad spectrum, appreciable changes can take place in the picture of the field within the time elapsed between two reflections.

In addition to the Fermi acceleration, Cerenkov and cyclotron resonances with different waves should also produce, in such a plasma, diffusion of the particles in momentum space. However, in view of the significant differences between the resonant waves and the large-scale pulsations, the rms change of the momentum breaks up into two independent terms - the Fermi term connected with reflection from long strong waves, and the resonant-statistical term determined by the quasilinear theory. The present paper is devoted to a study of the Fermi acceleration. In investigations of real acceleration processes it is also necessary, in general, to take into account quasilinear terms. It can be shown, however, that under the conditions that determine the possibility of Fermi acceleration (see formula (13) and the following), it is the Fermi mechanism that determines the asymptotic behavior of the spectrum.

Particles having velocities $v \gtrsim u_a$ and Larmor radii radii $r_L \ll L$ will be effectively reflected from waves with the stronger field and will be accelerated. In the absence of scattering, however, the acceleration has a rigidly fixed limit. Let us denote by $v_{||}$ the particle velocity parallel to the field in the region where $H = \overline{H}$. If $v_{||}^2 \geq v^2 (H_m - \overline{H})/H_m$, the particles are no longer reflected or accelerated. Consequently, under the conditions formulated above, the Fermi acceleration is effective only in the presence of fast scattering of particles. The scattering is due to waves of length on the order of the Larmor radius of a particle of the given velocity.

To construct the kinetic equation describing the fast particle, one could calculate the mean and rms changes in the particle momentum per unit time and construct a Fokker-Planck equation. This is a rather cumbersome procedure. We point out, in particular, that the mean change in energy contains, besides the calculated Fermi term, also a term of opposite sign. This effect is also connected with the fact that at the specified longitudinal and transverse particle velocities the relative velocity of head-on collision is larger than that of rear-end collision. Therefore, in the case of head-on collisions, reflection is possible from waves having a somewhat higher minimal amplitude than for rear-end collision. Therefore, in the case of head-on collisions, reflection is possible from waves having a somewhat higher minimal amplitude than for rear-end collisions, and consequently the acceleration is connected with a narrower section of the wave spectrum and is less probable.

In the case of the Fermi acceleration, the derivation of the equation can be greatly simplified by recognizing that the probabilities for the momentum to change from $p_{||}$ to $p'_{||}$ and back (from $p'_{||}$ to $p_{||}$) are equal. In the nonrelativistic case this deduction is obvious. Indeed, assume that, in a head-on collision with some wave having a longitudinal $u_{||}$, the velocity $v_{||}$ has increased to a value $v_{||} + 2u_{||}$. It is clear that in the case of a rear-end collision with the same wave the velocity changes from $v_{||} + 2u_{||}$ to the initial value $v_{||}$. The probability (for equal $v_\perp$) is determined completely by the relative velocity, which in the former case is $v_{||} + u_{||}$ and in the latter $v_{||} + 2u_{||} - u_{||} = v_{||} + u_{||}$. Consequently a head-on collision of a particle with velocity $v_{||}$ is equal to the probability of a "rear-end" collision at a velocity $v_{||} + 2u_{||}$.

Let us consider the same question in the relativistic case. We denote by p and ϵ the momentum and total energy of the particle and let p be measured in units of Mc and ϵ in units of Mc^2, so that they are dimensionless quantities. We denote $u_{||}/c$ by β_0 and assume that $\beta_0 > 0$, in head-on collisions and $\beta_0 < 0$ in rear-end collisions. Transforming to a coordinate frame in which the wave is at rest, we find that after reflection $p_{||}$ and ϵ have the following values:

$$p_{||}' = -\frac{p_{||} + \beta_0\varepsilon}{\sqrt{1-\beta_0^2}}, \quad \varepsilon' = \frac{\varepsilon + \beta_0 p_{||}}{\sqrt{1-\beta_0^2}} \qquad (2)$$

Carrying out the inverse Lorentz transformation to the initial coordinate system, we find that after the collision

$$p_{\parallel}'' = -\frac{p_{\parallel}(1+\beta_0^2)+2\beta_0\varepsilon}{1-\beta_0^2}, \quad \varepsilon'' = \frac{\varepsilon(1+\beta_0^2)+2\beta_0 p_{\parallel}}{1-\beta_0^2}. \tag{3}$$

If we now choose $p_{\parallel}''$ and ϵ'' as the initial values and reverse the sign of β_0, then the final values will be $p_{\parallel}$ and ϵ. Consequently, head-on rear end collisions are direct and inverse processes, successive realization of which returns the particle to the initial state. The probabilities of these processes are determined by the quantity $|p'|$. If the initial values are p and ϵ, and if $\beta_0 > 0$, then $|p_{rel}| = (p_{\parallel} + \beta_0\epsilon)/\sqrt{1-\beta_0^2}$. But if the initial values are $|p''|$ and ϵ'' and $\beta_0 < 0$, then

$$|p_{\mathrm{rel}}| = \left|\frac{|p_{\parallel}''| - \beta_0\varepsilon''}{\sqrt{1-\beta_0^2}}\right| = \frac{p_{\parallel}(1+\beta_0^2)+2\beta_0\varepsilon - \beta_0[\varepsilon(1+\beta_0^2)+2\beta_0 p_{\parallel}]}{(1-\beta_0^2)^{3/2}} = \left|\frac{p_{\parallel}+\beta_0\varepsilon}{\sqrt{1-\beta_0^2}}\right|.$$

Thus, the direct and inverse transitions are equally probable in the relativistic case.

It follows therefore (see, e.g.[3,4]) that the change in the distribution function has a pure diffusion character, and since $p_{\perp}$ remains constant, the equation for the distribution function of fast particles is

$$\frac{\partial f}{\partial t} = \frac{1}{2}\frac{\partial}{\partial p_{\parallel}}\frac{\overline{(\Delta p_{\parallel})^2}}{\Delta t}\frac{\partial f}{\partial p_{\parallel}} \tag{4}$$

+ terms allowing for other effects.

The function f is defined here in the usual manner ($\int f d^3p = n$, where n is the number of fast particles per unit volume).

Let K(h) be a quantity inverse to the average minimum distance along the force line between the points at which $H - \overline{H} = h$. We denote by θ the angle between the particle momentum and the field at the points where $H = \overline{H}$. Then the number of reflections per unit time (when $v \gg u$) in the approximation linear in u will be[1)] $|v_{\parallel}| K(h)_{h=H\cot^2\theta}$. According to (3), the value of $(\Delta p_{\parallel})^2$ is $4\beta_0\epsilon$, accurate to terms $\sim u^2$. Hence the change in f due to Fermi collisions is

$$\left(\frac{\partial f}{\partial t}\right)_F = \frac{\partial}{\partial p_{\parallel}} 2\beta_0^2\varepsilon |v_{\parallel}| K(h)\Big|_{h=\overline{H}\operatorname{ctg}^2\theta}\frac{\partial f}{\partial p_{\parallel}}. \tag{5}$$

We now take scattering into account by introducing in (5) an operator of the form $\tau_s^{-1}(p)\hat{S}_\theta$, where $\tau_s(p)$ is the average time for scattering a particle with momentum p through an angle $\sim\pi$, and the operator $\hat{S}$ acts on the angular part of f and leads to establishment of an isotropic velocity distribution. It follows from the latter condition that the zeroth eigenvalue of the operator $\hat{S}$ is $\sigma_0 = 0$, and the corresponding eigenfunction is ψ_0 = const.

Let us write (5) with allowance for scattering in a spherical coordinate system in momentum space p, θ:

$$\frac{\partial f}{\partial t} = 2\beta_0^2 c\left\{\cos\theta\frac{\partial}{\partial p}\varepsilon p|\cos\theta|K(\theta)\left(\cos\theta\frac{\partial f}{\partial p} - \frac{\sin\theta}{p}\frac{\partial f}{\partial\theta}\right) - \frac{\sin\theta}{p}\frac{\partial}{\partial\theta}\varepsilon p|\cos\theta|K(\theta)\left(\cos\theta\frac{\partial f}{\partial p} - \frac{\sin\theta}{p}\frac{\partial f}{\partial\theta}\right)\right\} + \frac{1}{\tau_s(p)}\hat{S}_\theta f \tag{6}$$

(we took account of the fact that $|p_z| = p|\cos\theta|$ and $\epsilon v_z = cp_z$).

As already noted, the Fermi acceleration is especially effective in the case when the scattering is strong (the time of scattering of a particle with momentum p through an angle $\sim\pi$ is much shorter than the time of acceleration to this value of the momentum). The distribution function is then close to isotropic. We expand f in terms of the eigenfunctions of the operator $\hat{S}$, which we denote by $\psi_n(\theta)$. The corresponding eigenvalues will be denoted by $(-\sigma_n^2)$. Substituting into (6) the function f in the form

$$f = \bar{f}(p) + \sum_{n=1}^{\infty} a_n(p)\psi_n(\theta)$$

and integrating over the surface of the unit sphere, we get

$$\frac{\partial\bar{f}}{\partial t} = 2\beta_0^2 c\left\{\frac{\partial}{\partial p}\varepsilon p\frac{\partial\bar{f}}{\partial p}\int_0^{\pi/2}\sin\theta\cos^3\theta K(\theta)\,d\theta - \frac{1}{p}\varepsilon p\frac{\partial\bar{f}}{\partial p}\int_0^{\pi/2}\sin^2\theta\frac{\partial}{\partial\theta}\cos^2\theta K(\theta)\,d\theta\right\}, \tag{7}$$

$$a_n(p) = \frac{\beta_0^2 c\tau_s(p)}{\sigma_n^2}\int_0^{\pi}\psi_n(\theta)\sin\theta\frac{\partial}{\partial p}\varepsilon p\frac{\partial\bar{f}}{\partial p}\left(\cos^2\theta|\cos\theta|K(\theta) - \varepsilon\frac{\partial\bar{f}}{\partial p}\sin\theta\frac{\partial}{\partial\theta}\cos\theta|\cos\theta|K(\theta)\right)d\theta \tag{8}$$

(the calculation was made under the assumption that $a_n \ll \bar{f}$, i.e., assuming almost isotropic distribution). Integrating the expression with $\partial/\partial\theta$ in (7) by parts, we get

$$\frac{\partial\bar{f}}{\partial t} = D_F\frac{1}{p^2}\frac{\partial}{\partial p}p^3\varepsilon\frac{\partial\bar{f}}{\partial p}, \quad D_F = 2\beta_0^2 c\int_0^{\pi/2}\sin\theta\cos^3\theta K(\theta)\,d\theta. \tag{9}$$

1) Allowance for the change in $v_{\parallel}$ in the case of motion in an inhomogeneous field reduces somewhat the collision frequency, but introduces no fundamental changes in the equation.

From a comparison of (7) and (8) we see that the scattering can be regarded as strong if

$$\frac{D_F \tau_s(p)}{\sigma_t^2}\frac{\varepsilon}{p} \ll 1. \qquad (10)$$

An investigation of the scattering of particles by hydromagnetic waves (see, for example,[5,6]) shows that

$$\tau_s(p) \approx \frac{1}{\Omega_H}\left[\frac{\bar{H}}{h(r_L(p))}\right]^2, \qquad (11)$$

where Ω_H is the cyclotron frequency and $h(r_L(p))$ is the average amplitude of pulsations with scale on the order of the particle Larmor radius $r_L = Mc^2p/e\bar{H}$. Therefore the condition (10) can be made concrete by specifying the pulsation spectrum. We shall discuss this question somewhat later.

The pulsation spectrum must be likewise specified for the calculation of D_F. It is obvious that $K(h)$ is equal to the average number, per unit length, of waves having an amplitude larger than h. Let us assume that $K(h)$ can be represented by a power law

$$K(h) = L^{-1}(h_m/h)^\nu, \qquad (12)$$

where L is the main scale of the turbulence and $h_m = H_m - \bar{H}$. As already noted, we can assume that $\nu < 3$. Substituting (12) in (9), we see that the integral D_F is equal to

$$D_F = \frac{2\beta_0^2 c}{L}\left(\frac{h_m}{\bar{H}}\right)^\nu \int_0^{\pi/2} \sin\theta\cos^3\theta\,\mathrm{tg}^{2\nu}\theta\,d\theta$$

$$= \frac{\beta_0^2 c}{2L}\left(\frac{h_m}{\bar{H}}\right)^\nu \Gamma(\nu+1)\Gamma(2-\nu) \qquad (13)$$

when $\nu < 2$ and diverges at the upper limit when $\nu \geq 2$. The meaning of this result is fact that when $\theta \to \pi/2$ the reflection takes place from waves of arbitrarily small amplitude, and the amplitude depends sufficiently weakly on the scale, then the time between collisions tends to zero. Actually it is necessary to cut off the integral in (13) at the wavelength corresponding to the Larmor radius of the particle. When $\nu = 2$ this leads to a logarithmic dependence of D_F on p, and when $\nu > 2$ the form of the dependence will depend essentially on the form of the spectrum.

Thus, a pure Fermi acceleration (reflection of the particles from long waves) takes place only for pulsation amplitudes that decrease sufficiently rapidly with decreasing scale ($h(\lambda) \to 0$ if $\lambda \to 0$ more rapidly than $\sqrt{\lambda}$). In this case D_F does not depend on the momentum, and the diffusion coefficient is proportional to ϵp in momentum space. When ν tends to 2, D_F increases rapidly. Thus, for ν equal to 1, 1.5, and 1.7 the integral in (13) takes on the values of 0.26, 0.6 and 1.5, respectively.

As to the parameter β_0^2 in (13), its value is determined by averaging the square of the projection of the group velocity u_g of different waves on the field in all possible orientations of u_g. In the case of Alfven waves we always have $\beta_0 = u_a/c$.

Let us consider now in greater detail the condition (10) for strong scattering. Assuming that $\sigma_t \sim 1$, $h(\lambda) = h_m(\lambda/L)^{1/\nu}$, and $D_F \approx u_a^2/Lc$, we can transform (10) into

$$p \gg \beta_0\left(\frac{\sqrt{\varepsilon}\,\bar{H}}{h_m}\right)^{2\nu/(\nu+2)}\left(\frac{r_0}{L}\right)^{(\nu-2)/(\nu+2)}, \qquad (14)$$

where $r_0 = u_a/\Omega_H = c/\Omega_0$ is the internal turbulence scale and $\Omega_0 = \sqrt{4\pi e^2N/M}$ is the ion plasma frequency.

Under astrophysical conditions usually $r_0 \ll L$. Therefore when $\nu > 2$ the condition (14) is satisfied for all $p > \beta_0$. In the Fermi acceleration mode ($\nu < 2$) the condition (14) is valid above a certain interval of values $p > \beta_0$. However, if ν is close to 2, the corresponding values of p are close to β_0. For example, in the Sun's upper chromosphere, where $L \sim 10^7$ cm during flares and the plasma density is $N \sim 10^{10}$ cm^{-3}, the critical values for $h_m \approx \bar{H}$ and $\epsilon \sim 1$ are $\sim 4.5\,\beta_0$ and $\sim 2\beta_0$ for ν equal to 1.5 and 1.75 respectively, although $r_0/L \sim 3\times 10^{-5}$.

The foregoing estimates of the strong-scattering conditions pertain to heavy particles. In the case of electrons, the resonant wavelengths ($\lambda \sim r_L$) are as a rule smaller than the internal scale, although it can be assumed, inasmuch as the scattering time (for given $h(r_L)$ is smaller by a factor $M/m = 1800$ for the electron than for the ion, that there exists in this case a region of values of p for which the distribution function of the accelerated electrons is isotropic.

Intense Fermi acceleration contributes to containment of fast particles in the turbulent region, since the regular motion along the force line goes over in this case into diffusion.

The mean free path can be expressed in terms of D_F. Assuming that $D_F \approx 2\beta_0^2 c/\bar{\lambda}$, we obtain $\bar{\lambda} \approx 2\beta_0^2 c/D_F$. The average velocity is $|\bar{v}_z| = v/2 = cp/2\epsilon$. Therefore (accurate to a dimensionless factor $q \sim 1$) the diffusion coefficient in real space, $D_z = (1/2)|\bar{v}_z|\bar{\lambda}$ is given by

$$D_0 = \frac{q}{2}\frac{\beta_0^2 c^2}{D_F}\frac{p}{\varepsilon}. \qquad (15)$$

Thus, when $\nu < 2$ and for strong scattering, the change in the distribution function f as a result of Fermi collisions is given by

$$\frac{\partial f}{\partial t} = D_F\frac{1}{p^2}\frac{\partial}{\partial p}\varepsilon p^3\frac{\partial f}{\partial p} + \frac{q\beta_0^2c^2}{2D_F}\frac{p}{\varepsilon}\frac{\partial^2 f}{\partial z^2} \qquad (16)$$

where z is the coordinate along the force line of the unperturbed field (since we consider henceforth only the isotropic part of f we shall leave the bar over f and assume that $\bar{f} \equiv f$).

It is easy to obtain Fermi's main formulas from (16). In the nonrelativistic case, for example, $dp/dt = \text{const} \cdot p$. The numerical values of these constants coincide in order of magnitude with the corresponding constant in (1). At the same time, for the analysis of the finer points (for example, the form of the spectrum) it is necessary to use Eq. (16). We note that the detailed-balancing principle imposes a number of limitations on the spectrum.

It follows from the foregoing analysis that there exists a broad class of physical conditions under which Fermi acceleration takes place. It is quite probable that this includes the developed wave turbulence in a plasma in the presence of a magnetic field. Equation (16) makes it possible to investigate the asymptotic behavior of the distribution function in the region of sufficiently large values of momentum. Let us consider several solutions of (16).

2. NONSTATIONARY ACCELERATION WITHOUT ACCOUNT OF LOSSES

Let us consider the acceleration of particles under conditions of pulsed excitation and subsequence damping of the turbulence. We assume first that the diffusion of the particles along the force lines and other losses can be neglected, and we assume that $D_F = 0$ when $t < 0$ and increases abruptly at $t = 0$ to a certain final value, after which it decreases to zero in accord with a specified law. Putting $\tau = \int_0^t D_F(t')\,dt'$ and omitting the diffusion term from (16), we find that in the nonrelativistic case ($\epsilon = 1$)

$$\frac{\partial f}{\partial \tau} = p\frac{\partial^2 f}{\partial p^2} + 3\frac{\partial f}{\partial p}. \qquad (17)$$

Equation (17) has particular solutions of the form

$$f_q = e^{-\varkappa^2\tau} p^{-1} J_2(2\varkappa\sqrt{p}), \qquad (18)$$

which are bounded at zero. The general solution can be represented in the form

$$f(p,\tau) = \frac{1}{p}\int_0^\infty \psi(\varkappa) e^{-\varkappa^2\tau} J_2(2\varkappa\sqrt{p})\,\varkappa\,d\varkappa. \qquad (19)$$

Let $f(p, 0) = f_0(p)$:

$$p f_0(p) = \int_0^\infty \psi(\varkappa) J_2(2\varkappa\sqrt{p})\,\varkappa\,d\varkappa.$$

Using the Fourier-Bessel theorem, we get

$$\psi(\varkappa) = \int_0^\infty p' f_0(p') J_2(2\varkappa\sqrt{p'})\, 2\sqrt{p'}\, d(2\sqrt{p'})$$
$$= 2\int_0^\infty f_0(p') J_2(2\varkappa\sqrt{p'})\, p'\, dp'.$$

Substituting in (19) and integrating with respect to κ, we get

$$f(p,\tau) = \frac{1}{p\tau}\int_0 f_0(p') \exp\left(-\frac{p+p'}{\tau}\right) I_2\left(\frac{2\sqrt{pp'}}{\tau}\right) p'\,dp' \qquad (20)$$

(I_2 is a Bessel function of imaginary argument) Relation (20) determines the source function for Eq. (17).

It can be assumed in most astrophysical problems that the injection function f_0 differs from zero in the region $p \lesssim p_0 \ll 1$. With this, the asymptotic form of $f(p, \tau)$ has a universal form at large τ and when $p \gg p_0$. Expanding I_2 in a Laurent series and assuming that $\exp(-p'/\tau) \approx 1$, we get

$$f(p,\tau) \approx \frac{1}{2\tau^3} e^{-p/\tau} \int_0^\infty f_0(p') p'^2 dp' = \frac{1}{\tau^3} e^{-p/\tau} \frac{n_0}{2\pi}, \qquad (21)$$

where n_0 is the total number of injected particles per cm^3. The particle density in space does not change during the acceleration process, as can be seen directly from (17).

The intensity of particles with momentum $> p$ is

$$S(>p) = 4\pi c\int_p^\infty p^3 f(p,\tau)\,dp$$
$$= 3n_0 c\tau\left(1 + \frac{p}{\tau} + \frac{1}{2}\frac{p^2}{\tau^2} + \frac{1}{6}\frac{p^3}{\tau^3}\right) e^{-p/\tau} \qquad (22)$$

and has a broad plateau at small p/τ ($S > p$) $\approx 3n_0 c\tau(1 - \tfrac{1}{24} p^4 \tau^{-4} + \ldots)$). If the turbulence attenuates sufficiently rapidly, so that the integral $\tau_0 = \int_0^\infty D_F(t)\,dt$ exists, then the spectrum of the accelerated particles tends to the limit defined by (22) at $\tau = \tau_0$.

The value of $S(>0)/n_0 c = 3\tau$ determines $\bar{p}$. This result can be obtained from the Fermi formula (1) (accurate to a factor ~ 1). But allowance for the fluctuations of the number of head-on and "rear-end" collisions shows that the spectrum stretches out in the region of much larger p. The intensity decreases by one order of magnitude compared with $\bar{p}$ only when $p \approx 8\tau$. The energy of such particles is seven times higher than for $p = \bar{p}$.

In addition to the considered problem with initial conditions, it is of interest to investigate the form of f in the presence of a constant particle flux in momentum space, coming from the region of small p. Such a problem stimulates qualitatively the continuous injection of particles due to attenuation of the turbulence. With this, the solution in the region of small p is proportional to p^{-2}, namely $f = p^{-2}\varphi(p, \tau)$. The function φ should be self-similar and depend on one combination of p and τ, for there are no parameters with the dimension of time under the conditions of the problem (more accurately speaking, they can be eliminated by introducing the variable τ).

Let us put $\varphi = \varphi(\xi)$, where $\xi = p\theta(\tau)$. Substituting $f = p^{-2}\varphi(\xi)$ in (17), we find that $\theta(\tau) = 1/\tau$ and

$$\xi \frac{d^2\varphi}{d\xi^2} + (\xi - 1)\frac{d\varphi}{d\xi} = 0. \tag{23}$$

Hence

$$\varphi = A(1+\xi)e^{-\xi} \tag{24}$$

and

$$f = \frac{A}{p^2}(1+\xi)e^{-\xi} = \frac{A}{p^2}\left(1 + \frac{p}{\tau}\right)e^{-p/\tau}. \tag{25}$$

When $\xi \to 0$ the solution is stationary ($f \sim 1/p^2$), and when $\xi \gtrsim 1$ an exponential velocity spectrum is formed. In dimensional variables, the stationary spectrum is characterized by an average velocity

$$\bar{v} = cD_F t_0 = \frac{u_a^2}{2L}\left(\frac{h_m}{\bar{H}}\right)^{\nu}\tau_0\Gamma(\nu+1)\Gamma(2-\nu), \tag{26}$$

where t_0 is the characteristic damping time of the turbulence.

Thus, the velocity spectrum $f \propto \exp(-v/\bar{v})$ with $\bar{v} \propto t$ is typical for the nonrelativistic energy region in the absence of losses.

In the ultrarelativistic case ($\epsilon \approx p$) the equation takes the form

$$\frac{\partial f}{\partial \tau} = \frac{1}{p^2}\frac{\partial}{\partial p}p^4\frac{\partial f}{\partial p}. \tag{27}$$

We put $p = e^{\sigma}$ and $f = p^{-3/2}\varphi e^{-9/4\tau}$. Equation (27) takes on the form of the one-dimensional equation of heat conduction

$$\partial\varphi / \partial\tau = \partial^2\varphi / \partial\sigma^2 \tag{28}$$

with a Green's function

$$G(\sigma, \sigma', \tau) = \frac{1}{2\sqrt{\pi\tau}}e^{-(\sigma-\sigma')^2/4\tau}. \tag{29}$$

It follows therefore that when the particles are accelerated in a region with damped turbulence ($\tau \to \tau_0$ as $t \to \infty$) from small values of p, a spectrum is established in the form

$$f_\infty = Ap^{-3/2}\exp(-\ln^2 p / 4\tau_0). \tag{30}$$

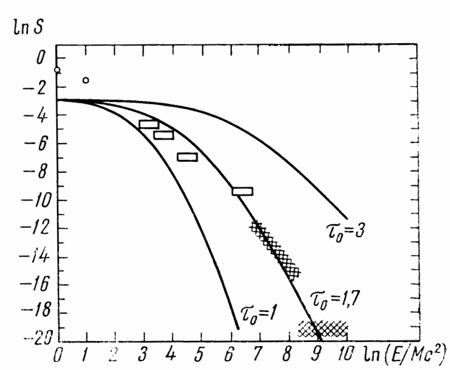
The intensity of particles with momentum > p is

$$S(>p) = c\int_p^\infty fp^2dp = B\left\{1 - \Phi\left(\frac{\ln p - 3\tau_0}{2\sqrt{\tau_0}}\right)\right\}, \tag{31}$$

where Φ is the probability integral and A and B are constants that do not depend on p. When $\tau_0 \approx u_a^2 t_0/Lc \sim 1$ the spectrum (1) becomes very hard. The figure shows the particle distributions for $\tau_0 = 1$, 1.7 and 3 (S is in $cm^{-2}\,sec^{-1}\,sr^{-1}$), and for comparison also the experimental[7] energy spectrum of cosmic rays [points, rectangles, and shaded strips). For convenience in comparison it is assumed that in (31) $B = 5 \times 10^{-4}\ cm^{-2}\ sr^{-1}$. The spectrum (31) has been obtained under the assumptions of a pulsed injection of the accelerated particles and of homogeneous turbulence characteristics. If the injection is continuous, then the number of low-energy particles increases noticeably. At the same time, the form of the spectrum for $p > \bar{p} = \exp(4\tau_0)$ remains practically constant. By suitable choice of the injection function it is possible to eliminate the discrepancy between the spectrum (31) at $\tau_0 = 1.7$ and the cosmic-ray spectrum at small p.

The results show that the Fermi acceleration can in principle give rise to the observed cosmic-ray spectrum even if losses are not taken into account. However, the shape of the spectrum depends exceedingly strongly on the value of τ_0. Therefore in attempts to explain the cosmic-ray spectrum by means of the Fermi mechanism without losses it is necessary to propose either that the acceleration has been produced during the course of one superpowerful explosion, or as a result of a series of explosions with nearly equal characteristics ($\tau_0 \approx 1$—1.7). We shall show in Sec. 4, in particular, that τ_0 can have approximately this value for the Crab nebula.

The acceleration considered in this section, without allowance for losses, can take place in good magnetic traps (for example, in a toroidal field). However, the acceleration can be effective



even when the characteristic dimension of the turbulent region is small compared with the scale of the inhomogeneity of the field. With this, the containment of the particles is ensured by the Fermi mechanism itself. This problem is dealt with in the next section.

3. NONSTATIONARY ACCELERATION WITH ALLOWANCE FOR DIFFUSION IN SPACE

We assume that a turbulence has been excited in some region of the plasma and attenuates in accord with a specified law. We shall assume that the characteristics of the plasma and of the turbulence differ from zero at $0 \leq z \leq \Lambda$, and propose for simplicity that they do not depend on z in this region. We write the coefficient D_F in the form $D_F = D_0\theta(t)$, where $\theta(0) = 1$, and we introduce in lieu of t a dimensionless variable $\tau = D_0 t$.

Having applications to processes such as solar flares in mind, we consider the nonrelativistic problem ($\epsilon = 1$). The equation of acceleration with allowance for diffusion (16) takes the form

$$\frac{\partial f}{\partial \tau} = \theta(\tau)\left(p\frac{\partial^2 f}{\partial p^2} + 3\frac{\partial f}{\partial p}\right) + \frac{q\beta_0^2 c^2}{2D_0^2}\frac{1}{\theta(\tau)}\frac{\partial^2 f}{\partial z^2}. \quad (32)$$

We seek a solution in the form

$$f = \sum_{n=1}^{\infty} A_n f_n(p,\tau)\sin\frac{n\pi z}{\Lambda}, \quad (33)$$

where $\sin(n\pi z/\lambda)$ are the eigenfunctions of the operator $\partial^2/\partial z^2$, satisfying the condition $f|_{z=0\,\Lambda} = 0$. If at $\tau = 0$ the initial function $f_0(p)$ is independent of the coordinates, then the coefficients A_n in (33) are equal to

$$A_n = \int_0^{\Lambda}\sin\frac{n\pi z}{\Lambda}dz \Big/ \int_0^{\Lambda}\sin^2\frac{n\pi z}{\Lambda}dz = \frac{2}{n\pi}(1-\cos n\pi) \quad (34)$$

and differ from zero when $n = 2m + 1$.

The equation for f_m is of the form

$$\frac{\partial f_m}{\partial \tau} = \theta(\tau)\left(p\frac{\partial^2 f_m}{\partial p^2} + 3\frac{\partial f_m}{\partial p}\right) - \frac{a_m^2}{\theta(\tau)}pf_m, \quad (35)$$

where

$$a_m^2 = \frac{(2m+1)^2 q\beta_0^2 c^2}{2\Lambda^2 D_0^2}. \quad (36)$$

We put $f_m = p^{-3/2}\varphi_m(p,\tau)$. We get

$$\frac{1}{p}\frac{\partial\varphi_m}{\partial\tau} = \theta(\tau)\frac{\partial^2\varphi_m}{\partial p^2} - \frac{a_m^2}{\theta(\tau)}\varphi_m - \frac{3}{4}\frac{\theta(\tau)}{p^2}\varphi_m. \quad (37)$$

It is seen from (37) that there exists an asymptotic solution of the type $\varphi_m = \exp[p\Theta(t)]$, valid under the condition

$$p^2\Theta^2 \gg 1. \quad (38)$$

It is obvious that the inequality (38) is satisfied when $p \gg \bar{p}$.

The function Θ satisfies the Ricatti equation

$$\dot{\Theta} = \theta(\tau)\Theta^2 - a_m^2/\theta(\tau). \quad (39)$$

Let us consider the case of exponential damping of the turbulence, $\theta = e^{-\gamma\tau}$. Putting $\Theta = e^{\gamma\tau}\psi$, we get

$$\dot{\psi} + \gamma\psi = \psi^2 - a_m^2$$

and

$$\psi = \frac{1}{2}\left(\gamma - \varkappa\frac{e^{\varkappa\tau}+\alpha}{e^{\varkappa\tau}-\alpha}\right), \quad (40)$$

where $\kappa = \sqrt{\gamma^2 + 4a_m^2}$, and α is an integration constant. It is obvious that a superposition of solutions with different α in the form

$$\varphi(p,\tau) = \int_{-1}^{+1} U(\alpha)\exp\left\{-\frac{p}{2}\left(\varkappa\frac{e^{\varkappa\tau}+\alpha}{e^{\varkappa\tau}-\alpha}\right)\right\}d\alpha \quad (41)$$

also satisfies Eq. (37) (subject to the condition (38).

When $\tau = 0$, we have

$$\varphi_0(p) = \int_{-1}^{1} U(\alpha)\exp\left\{\frac{p\gamma}{2} - \frac{p\varkappa}{2}\frac{1+\alpha}{1-\alpha}\right\}d\alpha. \quad (42)$$

We represent $\varphi_0(p)\exp(-p\gamma/2)$ in the form of a Laplace transformation:

$$\varphi_0(p)e^{-p\gamma/2} = \int_0^{\infty} e^{-\sigma p}W(\sigma)d\sigma$$

and put $\sigma = (1/2)\kappa(1+\alpha)/(1-\alpha)$. We get

$$\varphi_0(p) = \varkappa e^{p\gamma/2}\int_{-1}^{+1} W\left(\frac{\varkappa}{2}\frac{1+\alpha}{1-\alpha}\right) \times \exp\left\{-\frac{\varkappa}{2}p\frac{1+\alpha}{1-\alpha}\right)\frac{d\alpha}{(1-\alpha)^2}. \quad (43)$$

From a comparison of (42) and (43) we get

$$U(\alpha) = W\left(\frac{\varkappa}{2}\frac{1+\alpha}{1-\alpha}\right)\frac{\varkappa}{(1-\alpha)^2}, \quad (44)$$

where $W(x)$ is the inverse Laplace transform of $\varphi_0(p)\exp(-\gamma p/2)$. Formulas (42) and (43) thus yield the solution of the problem with arbitrary initial condition.

It is of interest to calculate the spectrum of F-particles emitted by the turbulent region during the entire acceleration time, The particle flux at the boundaries is proportional to $\theta^{-1}\partial f/\partial z$. Therefore we have for each z-harmonic

$$F_m(p) \propto p^{-3/2}\int_{-1}^{1} U(\alpha)\int_0^{\infty}\exp\left\{-\frac{p}{2}\left(\varkappa\frac{e^{\varkappa\tau}+\alpha}{e^{\varkappa\tau}-\alpha} + \gamma\tau\right\}d\tau d\alpha. \quad (45)$$

The calculation of the integral (45) with respect to time by the saddlepoint method leads to an exponential spectrum with a very complicated expression in the argument. However, this relation can be approximated sufficiently accurately by

$$F_m(p) \propto p^{-2} \int_{-1}^{+1} U(\alpha) \exp\left\{-\frac{\gamma p}{2}\frac{1+\nu}{\nu}\alpha^\nu\right\} d\alpha, \quad (46)$$

where

$$\nu = \gamma/\alpha = \gamma/\sqrt{\gamma^2 + 4a_m^2}. \quad (47)$$

If the acceleration starts out from small p, then the function $U(\alpha)$ should have a sharp maximum at $\alpha \to 1$ and therefore

$$F_m(p) \propto p^{-2} \exp\left(-\frac{\gamma p}{2}\frac{1+\nu}{\nu}\right). \quad (48)$$

Thus the asymptotic spectrum has in this case, too, the form of an exponential function of the momentum (cf. Sec. 2):

$$F_m \propto v^{-2} e^{-v/v_0}, \quad (49)$$

where

$$v_{0m} = \frac{2c}{\gamma}\frac{\nu}{1+\nu} = \frac{2cD_0t_0}{1 + [1 + 2(2m+1)^2\alpha\beta_0^2c^2t_0^2/\Lambda^2]^{1/2}} \quad (50)$$

and t_0 is the time required for D_F to decrease by a factor e. If we assume that t_0 is determined by the emission of waves from the turbulent volume ($u_a t_0 \approx \Lambda$), then we get ultimately ($m = 0$)

$$v_0 \approx cD_0t_0. \quad (51)$$

When $m \geq 1$ the spectrum is much softer.

Thus, the Fermi mechanism ensures a sufficiently effective containment of the particles in the turbulent volume.

The results can apparently be of great interest for the theory of generation of fast protons in solar flares. It was established in recent years that the spectrum of such protons has the form (49), and v_0 usually ranges from 10^9 to $\sim 10^{10}$ cm/sec, depending on the flare [8]. It was also established that conditions favorable for the Fermi acceleration are established in the lower corona during flares (the plasma is heated to $\sim 10^3$ eV, thus eliminating the problem of injection, and motions with velocities $\sim 10^8$ cm/sec set in). The magnetic fields in such regions are $\sim$ 10—100 Oe, and the plasma density is $\sim 10^9$—10^{10} cm^{-3}. The acceleration time is $t_0 \approx 10^2$ sec, corresponding to the time of a wave propagating with a velocity $\sim 10^7$—10^8 cm/sec over a distance on the order of the linear scale of the flare (10^9—10^{10} cm). For $v_0 = 5 \times 10^9$ cm/sec and $t_0 \sim 10^2$ sec it follows from (51) that

$$D_0 \approx u_a^2/Lc \approx v_0/ct_0 \approx 2 \cdot 10^{-3} \text{ sec}^{-1}.$$

When $H = 10$ Oe and $N_0 = 10^9$—10^{10} cm^{-3}, the pulsation scale L should amount to 10^7—10^8 cm.

We note that many recent papers attempt to attribute the acceleration in the flares to electric fields (see, for example, [9,10]). Comparison of the proton and nuclear spectra shows that the average particle momentum is proportional to the charge and not to the mass [8]. This relatively small effect can be attributed, however, to the faster escape of the nuclei from the solar system. Of much greater importance is a comparison of the proton and electron spectra. In the case of acceleration by an electric field, their mean energies should be of the same order, and in the case of Fermi acceleration, the proton energies will be three orders of magnitude higher. Neither direct nor experimental data confirm the existence in flares of electrons with energies $\sim$100 MeV—1 GeV. At the same time, investigations of the nonthermal x-ray and radio emission of flares points to electron acceleration up to energies on the order of 100 keV – 1 MeV, thus offering evidence in favor of the Fermi acceleration.

The ultrarelativistic problem with allowance for a spatial diffusion is much simpler, since the coefficient of diffusion in the z direction does not depend on the momentum. But substituting $f_n = \psi_n \exp(-a_n^2 \int_0^t dt/\theta)$ we reduce the equation to the form considered in Sec. 2 and obtain spectra of the form $\exp(-\frac{1}{4}\tau^{-1}\ln^2 p)$.

4. ELECTRON ACCELERATION WITH ALLOWANCE FOR SYNCHROTRON RADIATION

Let us consider a stationary problem in which synchrotron radiation is taken into account together with the Fermi acceleration. This problem is of interest in connection with the interpretation of nonthermal electromagnetic radiation for supernova shells. The equation obtained in this article makes it possible to determine more precisely the form of the electron spectrum. Allowance for the radiation losses leads to the appearance of a term $p^{-2}\partial(p^2\dot{p}f)/\partial p$ in the right side of (27), where

$$\dot{p} = \frac{2e^4H^2}{9m^3c^5}p^2 \quad (52)$$

is the rate of momentum loss due to synchrotron-radiation, averaged over the angle between the velocity and the field (we recall that p is measured in units of Mc, where M is the rest mass of the particle under consideration, in this case the electron). The equation takes the form

$$\frac{1}{p^2}\frac{\partial}{\partial p}p^4\left(\frac{\partial f}{\partial p} + \mu f\right) = 0, \quad (53)$$

where

$$\mu = \frac{2e^4H^2}{9m^3c^5D_F} \approx \frac{2e^4H^2L}{9m^3c^4u_a^2} = \frac{8\pi e^4}{9m^2c^4}\frac{M}{m}LN \quad (54)$$

(N is the density of the cold plasma).

The solution that leads to a bounded electron energy density is trivial:

$$f = e^{-\mu p}. \quad (55)$$

The radiation intensity is $dI \sim fp^4dp$, from which we see that the main contribution is made by electrons with $p = 4/\mu$. The emission of such electrons is concentrated in the region of frequencies $\omega = 16eH/mc\mu^2$. Thus, if we know the frequency region $\omega \sim \omega_{eff}$, in which the greatest energy is radiated and if we know the magnetic field intensity H, then we can determine the value of D_F:

$$D_F \approx \frac{1}{18}\frac{e^2}{mc^3}\omega_H{}^2\sqrt{\frac{\omega_{эфф}}{\omega_H}} \quad (56)$$

($\omega_H = eH/mc$ is the nonrelativistic cyclotron frequency).

In the case of the Crab nebula $\omega_{eff} \approx 6 \times 10^{15}\ sec^{-1}$ and $H \approx 10^{-3}$ Oe, and consequently $D_F \approx 10^{-10}\ sec^{-1}$. If we assume that the turbulence characteristics remain approximately constant during the time t_0 on the order of the age of the nebula (3×10^{10} sec), then the parameter $\tau_0 = D_Ft_0$, which determines the structure of the spectrum of the accelerated protons (see Sec. 2) will be of the order of unity. Consequently, the Crab nebula could generate protons and nuclei whose spectrum corresponds to primary cosmic rays.

In conclusion let us formulate briefly the main results of this paper. In Sec. 1 it is shown that the detailed-balancing principle is satisfied in the Fermi acceleration, and a kinetic equation was derived on this basis. It turned out that the effective mean free path does not depend on the particle velocity if the amplitude h of the field pulsations with scale λ decreases more rapidly than $\sqrt{\lambda}$ as $\lambda \to 0$. The conditions under which the angular distribution of the particles can be regarded as isotropic were determined.

Using crude estimates for the growth rate of the mean particle energy, the resultant equation leads (apart from factors on the order of unity) the same results as the Fermi formulas. However, in the investigation of the spectrum (especially the high-energy tail), it is necessary to use exact equations.

An investigation of different nonstationary problems shows that under rather general assumptions the asymptotic behavior of the spectrum at high energies has a universal form ($\exp(-v/v_0)$ in the nonrelativistic case and $\exp(-\frac{1}{4}\tau_0^{-1}\ln^2 p)$ in the relativistic one). These spectra differ only in the numerical values of the constants v_0 or τ_0, which characterize the intensity of the turbulent pulsations and the acceleration time. The cosmic-ray spectrum is of this type.

The Fermi acceleration increases the particle lifetime in the turbulent region. Allowance of the possibility of particle escape does not affect the form of the spectrum. There are serious grounds (data on the shape of the spectrum and the relation between the mean proton and electron energies) for assuming that particle acceleration during solar flares is of the Fermi type. The parameter value at which the relativistic spectrum coincides with the observed cosmic-ray spectrum agrees with data on the acceleration in the Crab nebula.

The concrete examples considered in the paper are presented mainly for illustration and do not claim a complete explanation of the phenomena. In particular, we did not consider at all such an important question as particle injection. Nonetheless, the obtained results show that a detailed development of a Fermi-acceleration theory is of great interest and is a most promising trend in cosmic-ray physics.

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