INVESTIGATION OF THE LAGRANGIAN FOR NONLINEAR SPINOR AND SCALAR FIELDS

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Submitted to JETP editor July 23, 1966

J. Exptl. Theoret. Phys. (U.S.S.R.) 52, 502–507 (February, 1967)

Some relations are derived which must be fulfilled in order that particle-like solutions of the simplest nonlinear spinor and scalar field equations exist. Despite the restricted validity of the Lagrangians considered (for example, they do not describe neutral particles), such relations may be useful for choosing a realistic nonlinear Lagrangian.

1. SPINOR FIELD

LET us consider a spinor field described by the Lagrangian ($\hbar = e = c = 1$)

$$L = {}^1/_2\, \bar\psi_{,\mu}\gamma_\mu\psi - {}^1/_2\bar\psi\gamma_\mu\psi_{,\mu} + k_0\bar\psi\psi + \sum_N \lambda_N (\bar\psi A_N\psi)^n, \quad (1)$$

where the last term consists of a sum of scalar, vector, and tensor terms of arbitrary form but with the same degree of nonlinearity $n > 1$; λ_N is a constant. This field has a conserved energy-momentum tensor

$$T_{\mu\nu} = {}^1/_2\bar\psi_{,\mu}\gamma_\nu\psi - {}^1/_2\bar\psi\gamma_\nu\psi_{,\mu} - \delta_{\mu\nu}L, \qquad T_{\mu\nu,\nu} = 0 \quad (2)$$

and a conserved current vector

$$J_\mu = \bar\psi\gamma_\mu\psi, \qquad J_{\mu,\mu} = 0.$$

From the point of view of classical field theory one is particularly interested in the solutions of the equations obtained from the Lagrangian by variation with respect to $\bar\psi$ and ψ

$$\begin{cases} -\gamma_\mu\psi_{,\mu} + k_0\psi + n\sum_N \lambda_N(\bar\psi A_N\psi)^{n-1}A_N\psi = 0, \\ \bar\psi_{,\mu}\gamma_\mu + k_0\bar\psi + n\sum_N \lambda_N(\bar\psi A_N\psi)^{n-1}\bar\psi A_N = 0 \end{cases} \quad (3)$$

which satisfy the following conditions:

1. The static condition. The current vector J_μ and the energy-momentum tensor $T_{\mu\nu}$ are independent of the time at any point in space. The most general form of the function satisfying this condition is

$$\psi = e^{-kx_4}\psi_0(x_1, x_2, x_3),$$

where k is a real constant.

2. The solution must be continuous and bounded everywhere.

3. The solutions must be square integrable, i.e., must fall off at infinity more rapidly then $r^{-3/2}$ ($r^2 = x_1^2 + x_2^2 + x_3^2$).

Such solutions correspond to particles regarded as clusters of the field. It is easy to find the following behavior for the desired solutions at infinity:

$$\psi_0 \underset{r\to\infty}{=} C\frac{\exp\{-r\sqrt{k_0^2 - k^2}\}}{r},$$

where, by condition 3, $k_0^2 > k^2$, or

$$-1 < \beta = k / k_0 < 1. \quad (5)$$

Let us multiply the conservation law (2) by x_l and integrate over the whole volume of the field. Then we find, using conditions 1 to 3,

$$\int T_{\mu l}dv = 0. \quad (6)$$

For $\mu = 4$ the integral on the left-hand side of (6) is the total momentum of the field. Therefore the total energy of the field $\int T_{44}dv$ will be equal to the rest mass. It is known that in the linear theory k_0 is the rest mass of a particle regarded as a singularity of the linear field, $\gamma_\mu \psi_{1\mu} - k_0\psi = 0$. Making use of a limiting transition to the linear theory, we impose yet another condition on the solutions of (3):

$$4.\ k_0 = \int T_{44}dv. \quad (7)$$

And finally, we shall assume that our field can describe particles with some charge $Q \neq 0$, and in particular with the minimal charge. We then have the condition

$$5.\ Q = \int J_4 dv = \int \psi^+\psi dv. \quad (8)$$

It turns out that the requirement of the existence of a solution satisfying the conditions 1 to 5 imposes rather strong restrictions on the choice of the nonlinear term in (1) and in particular on the degree of nonlinearity n.

We note that the field equations lead immediately to

$$\int \left[L + (n-1) \sum_N \lambda_N (\bar{\psi} A_N \psi)^n \right] dv = 0. \quad (9)$$

After summation over the spatial indices we obtain from (6) another useful equation, which holds because of the field equations and conditions 1 to 3:

$$\int \left[-2L - {}^1/_2 \bar{\psi}_{,4} \gamma_4 \psi + {}^1/_2 \bar{\psi} \gamma_4 \psi_{,4} - k_0 \bar{\psi} \psi - \sum_N \lambda_N (\bar{\psi} A_N \psi)^n \right] dv = 0. \quad (10)$$

Eliminating the nonlinear term from (9) and (10), we obtain

$$\int \left[(2n-3) L + \frac{n-1}{2} (\bar{\psi}_{,4} \gamma_4 \psi + \bar{\psi} \gamma_4 \psi_{,4}) + (n-1) k_0 \bar{\psi} \psi \right] dv = 0. \quad (11)$$

Using (11), (8), and (7), we find after some transformations

$$(2n-3) \frac{k_0}{Q} \int \psi^+ \psi dv = (3n-4) k \int \psi^+ \psi dv + (n-1) k_0 \int \bar{\psi} \psi dv. \quad (12)$$

Let us introduce the notation

$$\int \psi^+ \psi dv = \int \{ |\psi_1|^2 + |\psi_2|^2 \} dv \, \mathrm{A} + \int \{ |\psi_3|^2 + |\psi_4|^2 \} dv = a^2 + b^2,$$

$$\int \bar{\psi} \psi dv = \int \{ |\psi_1|^2 + |\psi_2|^2 \} dv - \int \{ |\psi_3|^2 + |\psi_4|^2 \} dv = a^2 - b^2.$$

Then we have from (11)

$$\{ (2n-3)/Q - \beta(3n-4) - (n-1) \} a^2 = \{ -(2n-3)/Q + \beta(3n-4) - (n-1) \} b^2. \quad (13)$$

Neither a nor b can be equal to zero, since that would otherwise lead to the trivial solution; we therefore have two possibilities: either

$$(2n-3)/Q - \beta(3n-4) - (n-1) \geqslant 0,$$
$$-(2n-3)/Q + \beta(3n-4) - (n-1) \geqslant 0,$$

or

$$(2n-3)/Q - \beta(3n-4) - (n-1) < 0,$$
$$-(2n-3)/Q + \beta(3n-4) - (n-1) < 0.$$

The first case can be excluded at once, since it implies the inequality $n \leq 1$. This leaves the second possibility

$$(2n-3)/Q - (n-1) < \beta(3n-4) < (2n-3)/Q + (n-1). \quad (14)$$

Let us show that for degrees of nonlinearity $1 < n \leq 4/3$ there are no solutions for charges Q equal to or smaller than the elementary charge: $0 < Q \leq 1$. Let $1 < n \leq 4/3$. We multiply (5) by $3n - 4 \leq 0$:

$$-3n + 4 \geqslant \beta(3n-4) \geqslant 3n-4 \quad (15)$$

Subtracting the second inequalities of (14) and (15), we obtain

$$(2n-3) < (2n-3)/Q.$$

In the region $1 < n \leq 4/3$ we have $2n - 3 < 0$; therefore $1/Q < 1$, $Q > 1$. In the region $n > 4/3$ we have

$$\beta(3n-4) < 3n-4, \quad 4-3n < \beta(3n-4). \quad (16)$$

Combining the first and second inequalities of (14) and (16) pairwise, we find

$$4n-5 > (2n-3)/Q > 5-4n.$$

For $Q = 1$ solutions satisfying the conditions 1 to 5 exist only if $n > 4/3$, $1 > \beta > (n-2)/(3n-4)$. For $Q = 1/3$ we must have

$$7/5 < n < 2.$$

Spherically symmetric solutions of (3) which satisfy conditions 1 to 3 have been investigated by Finkelstein et al.[1,2] using a nonlinear term of the form

$$\sum_N \lambda_N (\bar{\psi} A_N \psi)^n = \frac{6-\lambda}{8} (\bar{\psi} \psi)^2 + \frac{2+\lambda}{8} \sum_1^5 (\bar{\psi} \gamma_\mu \psi)^2 \quad (17)$$

and by Chepurnykh[3] using a nonlinear term of the form

$$\sum_N \lambda_N (\bar{\psi} A_N \psi)^n = \lambda (\bar{\psi} \psi)^3 \quad (18)$$

In this work four nodeless solutions have been computed numerically. On the basis of these calculations it was concluded (although with poor accuracy) that

$$\int T_{44} dv = 2k_0 S \quad (19)$$

for $1/2 \leq S \leq 7/2$ (in the spherically symmetric solutions considered, the spin $S = Q/2$). Relation (19) coincides with condition 4 for $Q = 1$. These results as well as the results of [1,2], which lead to the conclusion that solutions obeying conditions 1 to 5 evidently exist for nonlinearities of the form (17) and (18), indicate that condition 4, which is natural in classical field theory, is a reasonable condition to impose.

2. SCALAR FIELD

Let us now consider a nonlinear scalar field with the Lagrangian

$$L = \varphi_{,\mu}^{+}\varphi_{,\mu} + k_0^2\varphi^{+}\varphi + \lambda(\varphi^{+}\varphi)^n, \quad n > 1.$$

Varying with respect to φ^+ and φ, we obtain the field equations

$$\varphi_{,\mu\mu} - k_0^2\varphi - \lambda n(\varphi^{+}\varphi)^{n-1}\varphi = 0,$$
$$\varphi_{,\mu\mu}^{+} - k_0^2\varphi^{+} - \lambda n(\varphi^{+}\varphi)^{n-1}\varphi^{+} = 0. \quad (20)$$

The field has a conserved energy-momentum tensor

$$T_{\mu\nu} = \varphi_{,\mu}^{+}\varphi_{,\nu} + \varphi_{,\nu}^{+}\varphi_{,\mu} - \delta_{\mu\nu}L, \quad T_{\mu\nu,\nu} = 0 \quad (21)$$

and a current vector

$$J_\mu = \varphi_{,\mu}^{+}\varphi - \varphi^{+}\varphi_{,\mu}, \quad J_{\mu,\mu} = 0.$$

As in the previous section, we impose the following conditions on the solutions of (20): 1) the static condition $\varphi = e^{-kx_4}\varphi_0(x_1, x_2, x_3)$; continuity and boundedness everywhere; and 3) square integrability.

In the case of the scalar field these three conditions are already sufficient for determining the region of values of the degree of nonlinearity n, for which such solutions do not exist.

Let us assume the existence of solutions satisfying conditions 1 to 3. These will then have an asymptotic behavior at infinity analogous to (4), and will satisfy relations similar to (6) and (9):

$$\varphi_0 \underset{r\to\infty}{=} C\frac{\exp\{-r\sqrt{k_0^2 - k^2}\}}{r}, \quad k_0^2 - k^2 > 0; \quad (22)$$

$$\int T_{\mu l}\,dv = 0; \quad (23)$$

$$\int [\varphi_{,l}^{+}\varphi_{,l} - k^2\varphi^{+}\varphi + k_0^2\varphi^{+}\varphi + n\lambda(\varphi^{+}\varphi)^n]\,dv = 0. \quad (24)$$

Summing over the spatial indices, we find from (23)

$$\int [2\varphi_{,l}^{+}\varphi_{,l} - 3L]\,dv$$
$$= \int [-\varphi_{,l}^{+}\varphi_{,l} + 3k^2\varphi^{+}\varphi - 3k_0^2\varphi^{+}\varphi - 3\lambda(\varphi^{+}\varphi)^n]\,dv = 0. \quad (25)$$

Eliminating the nonlinear term from (24) and (25), we obtain

$$(3-n)\int \varphi_{,l}^{+}\varphi_{,l}\,dv = 3(k_0^2 - k^2)(n-1)\int \varphi^{+}\varphi\,dv. \quad (26)$$

Since $k_0^2 - k^2 > 0$, $n - 1 > 0$ and both integrands in (26) are positive, we must have $1 < n < 3$ if solutions satisfying conditions 1 to 3 are to exist.

Let us now impose the following conditions on the solutions:

$$4)\ k_0 = \int T_{44}\,dv = \int [2k^2\varphi^{+}\varphi - L]\,dv, \quad (27)$$

$$5)\ Q = \int J_4\,dv = \int 2k\varphi^{+}\varphi\,dv. \quad (28)$$

We easily find from (24) and (25) that

$$\int L\,dv = \frac{2(n-1)}{n-3}\int (k_0^2 - k^2)\varphi^{+}\varphi\,dv.$$

Substituting this relation in (27) and using (28), we obtain another important equation

$$k_0 = Qk - \frac{2(n-1)}{n-3}\frac{k_0^2 - k^2}{2k}Q.$$

In particular, for $Q = \pm 1$,

$$k/k_0 = \beta = \mp(n-1)/(2n-4).$$

But owing to (22) we have $-1 < \beta < 1$; for $Q = \pm 1$ we must therefore have $1 < n < 5/3$ if solutions satisfying conditions 1 to 5 are to exist.

For the nonlinear scalar equation, as for the spinor equation, only spherically symmetric solutions have been investigated. Nehari[4] has shown the existence of positive solutions of the equation

$$R'' + \frac{2}{r}R' = R - (R^2)^{n-1}R$$

for $1 < n < 5/2$. This was shown by a different method by Zhidkov and Shirikov[5] for the case $1 < n < 2$. Using a method analogous to the one considered above, the author[6] has shown the nonexistence of solutions of (29) for $n > 3$ and has determined a lower limit for the eigenvalues R(0).

I express my gratitude to V. A. Levin for useful discussions.

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Translated by R. Lipperheide
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