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DYNAMICS OF QUASILINEAR RELAXATION IN AN UNSTABLE INHOMOGENEOUS PLASMA

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We consider the dynamics of the quasilinear relaxation of an unstable distribution function for the case of potential drift waves at $T_e \gg T_i$. We show that the narrow wave packet that results from the buildup in accordance with the linear theory shifts into the region of shorter wavelengths. The law governing the time variation of the average wave number of the packet is derived. It is shown that the packet remains narrow. The property of the nonlinear increment, consisting in the fact that the resultant disturbance with a given value of the wave vector suppresses the instability of oscillations of longer wavelength and, to the contrary, has little effect on the short-wave part of the spectrum, is valid also for other types of drift instabilities, particularly when $T_e \sim T_i$.

An inhomogeneous plasma in a strong magnetic field can be unstable even when the electron velocity distribution function is Maxwellian[1,2]. In this case waves that travel almost perpendicular to the density gradient and to the magnetic field build up in the plasma. These are called drift waves.

On the other hand, it is known that nonlinear effects can alter the distribution function in such a way that a new stable state is established (quasilinear relaxation[3,4]).

An investigation of such effects for the case of drift waves was carried out for the first time by Galeev and Rudakov[5], and then by Hoch[6]. The results of these investigations, in spite of the different character of the approximations made, are as follows: A stable state is formed quite rapidly, and the particle coordinates remain practically unchanged during the relaxation process.

In the present paper we call attention to the fact that the state established as a result of the quasilinear relaxation described in[5,6] is not final. On the other hand, the results obtained in[5,6] can be regarded as corresponding to the initial stage of the quasilinear relaxation process. We shall show that no stable state is attained in the quasilinear approximation, and the spectrum of the instability shifts towards the shorter wavelengths. The system of equations describing the process of quasilinear relaxation of an unstable distribution function of an inhomogeneous plasma was obtained in[5]:

$$\frac{\partial f}{\partial t}=\frac{(2\pi)^3}{G}\iiint d\mathbf{k}\left(\frac{\partial}{\partial v}-\frac{k_y}{k_z\omega_{He}}\frac{\partial}{\partial x}\right)\frac{k_z^2e^2}{m^2n_0}$$
$$\times\pi\delta(\omega_k-k_zv)\,|\varphi_k|^2\left(\frac{\partial}{\partial v}-\frac{k_y}{k_z\omega_{He}}\frac{\partial}{\partial x}\right)f, \qquad (1)$$

$$\frac{\partial N_k}{\partial t}+\frac{\partial}{\partial x}\left(\frac{\partial\omega_k}{\partial k_x}N_k\right)=2\gamma_k(x,t)N_k, \qquad (2)$$

$$\frac{dk_x(x,t)}{dt}=-\frac{\partial\omega_k}{\partial x} \qquad (2')$$

(the magnetic field is directed along the z axis and the plasma is inhomogeneous in the x direction). The electric field of the oscillations is assumed potential:

$$\mathbf{E}=-\nabla\varphi,$$

$$\varphi=\iiint d\mathbf{k}\varphi_k\exp\left\{i\int k_x\,dx+ik_y y+ik_z z-i\omega_k t\right\},$$

f is the electron distribution function,

$$N_k=\frac{\partial\varepsilon_k}{\partial\omega_k}k^2\frac{|\varphi_k|^2}{8\pi}$$

is the "number of waves" in the state with the given wave vector $\mathbf{k}$ (ϵ_k is the dielectric constant for the potential oscillations), G is the normalization volume, $\omega_{He} = eH/mc$, and ω_k and γ_k are the frequency and the "local" instability increment, determined from the distribution function at a given instant of time and at the given point x.

Equation (2) simplifies in two limiting cases. If

$$\gamma_k N_k \gg \frac{\partial}{\partial x}\left(\frac{\partial\omega_k}{\partial k_x}N_k\right),$$

then the so-called "local approximation" is valid, when the wave packet has time to build up before it is appreciably displaced in the x direction, and its amplitude in the vicinity of each point x increase at a rate determined by the local increment $\gamma_k(x, t)$. In the case when the second term in the left side of (2) is much larger than the remaining terms, the increment $\gamma_k(t)$ can be obtained by successive approximation. In the zeroth approximation we have

$$\frac{\partial}{\partial x}\left(\frac{\partial\omega_k}{\partial k_x}N_k\right)=0,\ \text{i.e. } N_k(x,t)=\text{const}(t)\cdot\left(\frac{\partial\omega_k}{\partial k_x}\right)^{-1} \quad (3)$$

Let us further integrate (2) over that region of x where $N_k(x)$ differs from zero (in the concrete case of drift waves, which is considered below, the region of integration is bounded by the turning points $k_x = 0$, where

$$k_x=\frac{1}{\rho}\left[1+k_y^2\rho^2-\frac{1}{\omega_k}\frac{cT_e}{eH}k_y\frac{d\ln n_0}{dx}\right]^{1/2},\quad \rho=\frac{c\sqrt{T_eM}}{eH},$$

T_e is the electron temperature). We use formula (3) and obtain

$$\gamma_k(t)=\frac{1}{2N_k}\frac{\partial N_k}{\partial t}=\frac{\int\gamma_k(x,t)\,(\partial\omega_k/\partial k_x)^{-1}dx}{\int(\partial\omega_k/\partial k_x)^{-1}dx}. \quad (3')$$

This result can be obtained also with the aid of the WKB method[7].

Just as in Galeev and Rudakov's paper[5], as well as in the later paper by Hoh[6], where the main results are the same, let us consider the limiting case, when we can use the "local" approximation, i.e., we neglect the term

$$\frac{\partial}{\partial x}\left(\frac{\partial\omega_k}{\partial k_x}N_k\right)$$

in Eq. (2). A more exact locality condition will be written out later.

Let us consider one of the simplest types of drift instability, when the electron temperature T_e is much higher than the ion temperature, and the phase velocity of the waves along the field satisfies the conditions

$$c_s\ll\omega_k/k_z\ll v_A.$$

When the left-side inequality is satisfied we can neglect the displacement of the ions along the magnetic field, whereas the right-side inequality constitutes the potentiality condition. Moreover, we assume that

$$m/M\ll 8\pi n_0T_e/H^2\ll 1,$$

i.e., the plasma is not too rarefied (the electron thermal velocity $v_{Te} = (2T_e/m)^{1/2}$ exceeds the Alfven velocity v_A). When these conditions are satisfied, the frequency of the drift oscillations and the local increment take the form[8]

$$\omega_k=-k_y\frac{cT_e}{eH}\frac{1}{n_0}\frac{dn_0}{dx}\frac{1}{1+(k_\perp\rho)^2}, \quad (4)$$

$$\gamma_k=-\pi\omega_k^2\left(k_y\frac{cT_e}{eH}\frac{1}{n_0}\frac{dn_0}{dx}\right)^{-1}\frac{T_e}{mn_0}k_z\int_{-\infty}^{\infty}dv\,\delta(\omega_k-k_zv)$$
$$\times\left(\frac{\partial f}{\partial v}-\frac{k_y}{\omega_{He}k_z}\frac{\partial f}{\partial x}\right), \quad (5)$$

where $k_\perp^2 = k_x^2 + k_y^2$.

For the case considered here, the initial equations can be written in the form

$$\frac{\partial f}{\partial t}=\int dk_x\,dk_y\,L_k\left(\pi\frac{v_{Te}^2}{mn_0}\frac{\omega_k^2}{1+k_\perp^2\rho^2}\frac{w_k}{|v|^3}L_kf\right),$$
$$L_k=\frac{\partial}{\partial v}-\frac{k_yv}{\omega_{He}\omega_k}\frac{\partial}{\partial x}, \quad (6)$$

$$\frac{1}{w_k}\frac{\partial w_k}{\partial t}=-\pi\omega_k^2\left(k_y\frac{cT_e}{eH}\frac{d\ln n_0}{dx}\right)^{-1}\frac{v_{Te}^2}{n_0}L_kf,$$
$$\frac{(2\pi)^3}{G}\omega_kN_k=w_k. \quad (7)$$

In this formulation, we have eliminated k_z with the aid of the relation $\omega_k = k_zv$, so that the spectral density of the noise energy $w_k(t)$ is now a function of the variables k_x, k_y, v, and x, with v assuming values in the region $c_s \ll v \ll v_A$. Galeev and Rudakov[5] assumed in their investigation of this system of equations that the linear buildup of the noise at the instant of the start of the quasilinear relaxation will cause w_k to become a function with a sharply pronounced maximum at the point $k_\perp = \bar{k}_\perp$, corresponding to the maximum increment γ_k determined from the initial distribution function. This caused Eq. (6) to simplify to

$$\frac{\partial f}{\partial t}=L_{\bar{k}}\left(\pi\frac{v_{Te}^2}{mn_0}\frac{\omega_{\bar{k}}^2}{1+(\bar{k}_\perp\rho)^2}\frac{w_{\bar{k}}(v)}{|v|^3}\Delta_{k_x}\Delta_{k_y}L_{\bar{k}}f\right) \quad (8)$$

(Δk_x and Δk_y are the widths of the wave packet in the direction of k_x and k_y). If we now make the substitution

$$\eta=v,\qquad \xi=\frac{v^2}{2}+\frac{\omega_{He}\omega_{\bar{k}}}{\bar{k}_y}x,$$

then the system of equations (7) and (8) becomes even simpler:

$$\frac{\partial f}{\partial t}=\frac{\partial}{\partial\eta}\left(\pi\frac{v_{Te}^2}{mn_0}\frac{1}{|\eta|^3}\frac{\omega_{\bar{k}}^2w_{\bar{k}}}{1+(\bar{k}_\perp\rho)^2}\Delta_{k_x}\Delta_{k_y}\frac{\partial f}{\partial\eta}\right), \quad (9)$$

$$\frac{1}{w_{\bar{k}}}\frac{\partial w_{\bar{k}}}{\partial t}=-\pi\omega_{\bar{k}}^2\left(\bar{k}_y\frac{cT_e}{eH}\frac{d\ln n_0}{dx}\right)^{-1}\frac{v_{Te}^2}{n_0}\frac{\partial f}{\partial\eta}. \quad (10)$$

It follows from (9) and (10) that, first, the stationary state is the state with $\partial f/\partial\eta = 0$, and, second, the particles move during the time of the relaxation process along the characteristic ξ = const. Since the maximum possible particle displacement in velocity space is $\Delta v \ll v_A$, the displacement of the particle along x can be estimated in the following fashion:

$$\left|\Delta x\frac{1}{n_0}\frac{dn_0}{dx}\right|\ll\frac{v_A^2}{v_{Te}^2}(1+\bar{k}_\perp^2\rho^2).$$

For a Maxwellian distribution function $\bar{k}_\perp \rho \sim 1$. Therefore the displacement along the x coordinate is negligibly small. It is precisely this first stage of the process which is considered in[5,6].

However, this does not terminate the time variation of the particle distribution function f and of the noise $w_k(v)$. Indeed, assume that a stationary state in the sense indicated above has been established for $k_y = \bar{k}_y$ and $k_x = \bar{k}_x$. The distribution function has changed in this case in such a way that $\partial f/\partial\eta = 0$, i.e.,

$$\frac{\partial f}{\partial v} = \frac{\bar{k}_y v}{\omega_{He}\,\omega_{\bar{k}}} \frac{\partial f}{\partial x} \tag{11}$$

Since the particle displacement along the x coordinate is negligibly small in this process, we can replace in (11) $\partial f/\partial x$ by $\partial f_0/\partial x$. We now substitute $\partial f/\partial v$ in (7) and obtain

$$\frac{1}{w_k}\frac{\partial w_k}{\partial t} = 2\pi|\omega_k||v|\frac{(k_\perp^2 - \bar{k}_\perp^2)}{1 + k_\perp^2\rho^2}\rho^2\frac{1}{n_0}\frac{\partial f_0}{\partial x}\left(\frac{d\ln n_0}{dx}\right)^{-1}. \tag{12}$$

For both the initial distribution function and for the function satisfying condition (11), the increment is maximal when $k_x = 0$. Of course, in our "local" analysis method k_x cannot vanish, and the formally obtained results must be understood in the sense that

$$|d\ln n_0/dx| \ll k_x \ll k_y.$$

We shall therefore replace henceforth $k_\perp^2$ by k_y^2.

We see from (12) that oscillations with $k_y^2 > \bar{k}_y^2$ grow, whereas oscillations with $k_y^2 < \bar{k}_y^2$ attenuate (see Fig. 1). The solution should therefore have the form of a packet with respect to k_y, whose center $k_m(v, t)$ shifts toward larger k_y. Let us assume that the packet is narrow, i.e., $\Delta k_y/k_m \ll 1$. We shall verify the correctness of this assumption later. We shall also consider values of k_y satisfying the inequality $k_m^2\rho \lesssim v_{Te}^2/v_A^2$. Then we have $\partial f/\partial x = \partial f_0/\partial x$ at all instants of time, and the wave packet in the vicinity of a certain point v is of the form

$$w_k = w_k(t_0)\exp\left\{2\gamma_0(v)\left|\int_{t_0}^{t}\frac{k_y\rho(k_y^2 - \bar{k}_y^2)\rho^2}{(1 + k_y^2\rho^2)^2}dt'\right\}\right.,$$

$$\gamma_0(v) = \pi\frac{|v|}{\rho}\frac{cT_e}{eH}\left|\frac{1}{n_0}\frac{\partial f_0}{\partial x}\right|. \tag{13}$$

This relation is obtained as a result of integrating (12). Here $t_0(v)$ is the instant of time when quasilinear relaxation of the oscillation with $k_y = k_0$ in the vicinity of the point v took place. When $t = t_0$, by assumption, the packet is narrow in the k_y direction. This means that the function $w_k(t_0)$ has a sharp maximum at $k_y = k_0$, with a certain width Δ_0,

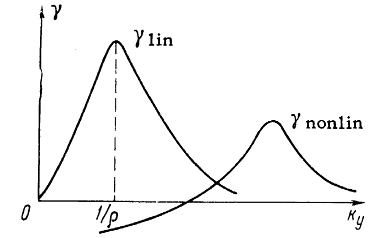

FIG. 1.

and at values of k_y satisfying the conditions $|k_y - k_0| \geq \Delta_0$ the function $w_k(t_0)$ is equal to w_0—the spectral energy density of the noise of the initial state. According to the very formulation of the quasilinear relaxation problem, described by Eqs. (1), (2), and (2′), w_0 is regarded as a specified quantity which is much larger than the value w_k for thermal noise. However, the quantity $\ln(w_{k_m}(t_0)/w_0)$ can be much greater than unity. In particular, it is of the order of the Coulomb logarithm if the final noise energy density is of the order of $n_0 T_e$.

Since the packet is assumed narrow, the quantities $k_m(t)$ and $\bar{k}_y(t)$ are equal with accuracy to terms $\Delta_{k_y}(t)/k_m(t)$. Let us now determine the time variation of $k_m(t)$ or of $\Delta_{k_y}(t)/k_m(t)$. To be able to get along without specifying the concrete form of $w_k(t_0)$ in the region $|k_y - k_0| < \Delta_0$, and to simplify the calculations, let us consider the process starting from the instant of time when the center of the packet has been displaced by an amount larger than Δ_0 and is situated in the region $k_m^2\rho^2 \gg 1$.

Differentiating the argument of the exponential in (13) with respect to k_y and equating the result to zero, and also equating k_y to $k_m(t)$, we obtain an equation for the position of the center of the packet $k_m(t)$

$$k_m^2(t - t_0) = 3\int_{t_0}^{t} k_m^2\,dt'. \tag{14}$$

The solution of this equation is

$$k_m\rho = C(t - t_0), \tag{15}$$

where C is a constant.

We now determine the constant C and the time variation of the quantity $\Delta_{k_y}(t)/k_m(t)$. To this end we expand w_k near the point $k_y = k_m(t)$ with allowance for relation (14):

$$w_k = w_0\exp\left\{2\gamma_0(v)\left[\frac{2}{3}\frac{t - t_0}{k_m\rho} - \frac{t - t_0}{k_m\rho}\frac{(k_y^2 - k_m)^2}{k_m^2}\right]\right\}. \tag{16}$$

To determine the constant C we put in formula (16) $k_y = k_m = C(t - t_0)$. As a result we get

$$C = {}^4/_3\,\gamma_0(v)/\ln\frac{w_{k_m}}{w_0}.$$

We can now determine the width of the packet $\Delta_{k_y}(t)$

$$\frac{\Delta^2_{k_y}(t)}{k_m^2(t)} = \frac{k_m\rho}{2\gamma_0(t-t_0)} = \frac{C}{2\gamma_0(v)} = 2/3\ln\frac{w_{k_m}}{w_0},$$

i.e., the relative width of the packet is small and does not vary with time. Thus, the packet, remaining narrow, shifts toward larger k_y like

$$k_m\rho = 4\gamma_0(v)(t-t_0)/3\ln(w_{k_m}/w_0). \qquad (17)$$

The quantity w_{k_m} can be determined from the conservation of the energy of the particle wave system. At each instant of time

$$\int w_k\,dk = \pi\Delta_{k_x}\Delta_{k_y}w_{k_m}\frac{\Delta v}{v_2^2}\omega_{k_m}$$

$$= \int\frac{mv^2}{2}(f-f_0)\,dv \approx \frac{mv_2^2}{2v_{Te}^2}n_0\frac{(\Delta v)^3}{v_{Te}}.$$

(For this estimate we used the subsequently obtained results concerning the character of the relaxation with respect to v; $\Delta v \sim t$ is the dimension of the region in which relation (11) was established for $\bar{k}_y = k_m(v, t)$.) With logarithmic accuracy (by assumption, $\ln(w_{k_m}/w_0) \gg 1$) we can replace w_{k_m}/w_0 in (17) by the constant

$$A = \frac{n_0T_e}{w_0}\left(\frac{v_2}{v_{Te}}\right)^5\frac{v_2}{\Delta_{k_x}\Delta_{k_y}\omega_{\bar{k}}}.$$

(More details on the motion of the center of the packet for $k_m\rho \sim 1$ are given in the Appendix.)

We have determined the time variation of $w_k(v, t)$ under the assumption that a quasilinear relaxation has occurred in the vicinity of the given point v at $k_y = k_m$. Let us consider now the initial stage of the quasilinear process, which leads to the occurrence of noise in the entire interval of phase velocities ω_k/k_z. Assume that the initial noise w_0 is small at the instant of time t = 0, and is distributed over all the k, while the initial distribution function of the resonant electrons is of the form $f_0 = n_0(x)\pi^{-1/2}/V_{Te}$ The subsequently obtained result can be generalized to include the case of any initial function, without introducing any complications of fundamental nature We have chosen f_0 in a form which is simplest for calculation. As to the nonresonant electrons, we shall assume, as before, that they have an approximate Maxwellian distribution with temperature T_e (see Fig. 2). In this case we can assume that the distribution function is unstable in the region $v_1 \le v \le v_2$ ($v_2 \ll v_A$) against buildup of potential drift waves, the frequency and wave vector of which satisfy relation (4) and the increment is determined by formula (7). From (7) it follows that the increment is maximal when $k_y = 1/\rho$, $k_x \ll k_y$, and $v = v_2$. Consequently, noise will build up most rapidly pre-

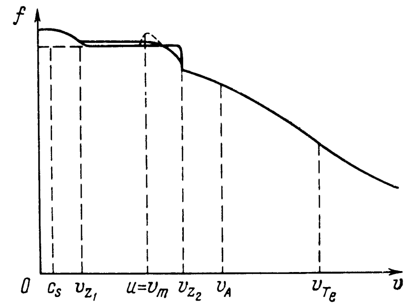

FIG. 2

cisely in the region of $v = v_2$ with $k_y = 1/\rho$, $k_x \approx 0$.

After determination of the linear phase of the instability, which lasts for a time equal to $\sim\gamma \ln A$, quasilinear relaxation of the distribution function will take place in the vicinity of the point $v = v_2$. This process is described by the system of equations (9) and (10) if the noise spectrum is narrow with respect to k_y. Temporal solutions of a system of equations of this type were investigated by Ivanov and Rudakov[9], who have shown that the distribution function has the form of a step whose front propagates into the region of smaller values of η. In our case, therefore, some sort of wave should propagate in η space. Since the distribution of the particles with respect to x remains practically unchanged here, this wave will travel in velocity space. Relation (11) will be satisfied behind the front of this wave, whereas the distribution function in front of the wave will remain equal to the initial one. It will follow from the particle conservation law that the distribution function has the form shown by the dashed line in Fig. 2.

It would be possible to calculate the velocity of the front u(t) with the aid of the formulas given in the paper of Ivanov and Rudakov[9]. Actually, the relaxation process should proceed in somewhat different fashion. If a section with $\partial f/\partial v > 0$ has formed on the distribution function (as must be the case), then ion sound will be excited immediately on the front of the wave, near v = u(t), at a frequency in the interval $\omega_{Hi} \ll \omega_s \ll \omega_{He}$, and consequently, $\omega_s \gg \omega_k$ [1)]. Accordingly we have for the increment $\gamma_s \gg \gamma_k$. Since this high-frequency instability leads to the formation of a plateau in velocity space, the slight peak on the distribution function, represented by the dashed line in Fig. 2, becomes instantaneously smeared over the entire region of instability when measured in the time scale under consideration. Therefore the distribution function will actually have the form shown by the solid line in Fig. 2.

1)This circumstance was called to our attention by A. B. Mikhailovskii.

Let us now determine the time dependence of u, i.e., the law governing the motion of the front. To this end we note that the distribution function in the region $v_1 \le v \le u(t)$, determined accurate to v_2^2/v_{Te}^2 from the particle-number conservation law, is equal to the initial distribution function. Therefore, a linear growth of the drift noise takes place in the region $v_1 \le v \le u(t)$, with an initial increment

$$\gamma_{0k} = \pi\omega_k v/v_{Te} \qquad (18)$$

(the increment γ_{0k} is maximal when $k_y = 1/\rho$, $v = u$, $k_x = 0$).

The time necessary for the drift noise at the point v to grow to the level of the final noise will therefore be determined by the duration of the linear stage

$$\frac{1}{2\gamma_0(v)} \ln A,$$

since the subsequent linear stage, during which the distribution function has an inclination from zero to a final value

$$\frac{\partial f}{\partial v} = -\frac{\bar{k}_y v}{\omega_k - \omega_{He}} \frac{\partial f_0}{\partial x} \qquad \left(\bar{k}_y = \frac{1}{\rho},\ \bar{k}_x \approx 0\right),$$

lasts for a time on the order of $1/\gamma_0$. The latter can be verified if we substitute the final noise level in relations (9) and (10).

Thus, the law governing the motion of the drift-noise front and of the point where the derivative on the distribution function changes (Fig. 2) can be written in the form

$$u(t) = \frac{v_2}{2\gamma_{0\bar{k}}(v_2)\, t} \ln A, \qquad \bar{k}_y = \frac{1}{\rho}, \qquad \bar{k}_x = 0. \qquad (19)$$

From this we can determine the time required for the wave front to traverse the instability region

$$t - t_0(v_2) = \frac{1}{2\gamma_{0\bar{k}}(v_2)} \frac{v_2 - v_1}{v_1} \ln A,\ \bar{k}_y = \frac{1}{\rho}, \qquad \bar{k}_x = 0. \qquad (20)$$

Of course, to find the form of the distribution function in a small vicinity $u^{-1}\Delta v \sim 1/\ln A$ of the point u it would be necessary to solve a system of quasilinear equations with simultaneous allowance for the drift and ion-sound oscillations. We have shown that the law of motion u(t) can be determined without solving such a complicated system of equations. It is interesting that the ion-sound oscillations, which determine the form and velocity of the motion of the front of the wave in v-space, vanish immediately behind the front, where the relation (11) is satisfied, i.e., $\partial f/\partial v < 0$.

Formula (19) is valid also when account is taken of the fact that during the time before the wave front reaches the point $v = v_1$, the center of the wave packet in the vicinity of the point v_2 shifts toward larger k_y. Indeed, in the time interval determined by (20), the value of k_y at the point $v = v_2$ shifts from $\bar{k}_y \sim \rho^{-1}$ to $\bar{k}_y \sim \rho^{-1} v_2/v_1$. This, however, has a negligible effect on the variation of f(x, v) ahead of the front, owing to the small parameter v_2^2/v_{Te}^2, and therefore the increment ahead of the front will be determined as before by the initial distribution function, and the velocity of the front will be determined by (19).

Thus, in the region $v_1 \le v \le u(t)$ ahead of the front the noise builds up with a linear increment which is maximal at $k_y = 1/\rho$, $k_x = 0$, and $v = u(t)$. To the right of the point u(t), the noise is approximately equal to the final value, but the center of the packet has already shifted toward larger k_y, and the larger the distance from the point u(t), the larger the shift. When, for example, the front reaches the point v_1, then at the point v_1 the center of the packet will be at $k_y = 1/\rho$, whereas at the point v_2 the center will have time to move to $k_y \approx v_2/\rho v_1$.

In conclusion, let us examine the locality criterion. For the drift instability considered by us, we can write this criterion in the form

$$\left|\frac{n_0}{\rho\, dn_0/dx}\right| \gg \frac{k_x \rho}{1 + (k_y\rho)^2} \sqrt{\frac{\beta M}{m}},$$

i.e., k_x should not be too large. This condition is satisfied automatically, since at all instants of time the increment is maximal when $k_x = 0$.

Let us discuss also certain consequences of the obtained solutions. First, if initially at $k_m \sim 1/\rho$ the resonant particles with respect to x have practically remained in place, then after a time

$$\frac{1}{\gamma_{0\bar{k}}} \frac{v_{Te}}{v_2} \ln A$$

the quantity $k_m^2\rho^2$ becomes of the same order as v_{Te}^2/v_2^2, and consequently the displacement of the resonant particles will be of the order of the characteristic length of the inhomogeneity. Second, the previously noted property of the nonlinear increment, consisting in the fact that the arising perturbation with a given value of the wave number suppresses the instability of oscillations with longer wavelengths, and to the contrary has little influence on the short-wave part of the spectrum, should take place also for other types of drift instability, particularly when $T_e = T_i$. To this end it is necessary that the frequency build up at a slower rate than k_y raised to the first power. As a result of the indicated effect, the shorter waves should predominate in the steady-state spectrum of the oscillations of the inhomogeneous plasma in a magnetic field.

APPENDIX

Let

$$w_k(t_0) = w_0 \exp\left\{2\gamma_0 \frac{k_y^3\rho^3}{(1+k_y^2\rho^2)}\tau\right\}.$$

Then we obtain in lieu of (15) the following relation for the determination of $k_m(t)$:

$$(t - t_0 + \tau)\frac{k_m^4\rho^4 - 3k_m^2\rho^2}{3k_m^2\rho^2 - 1} = \int_{t_0}^{t} k_m^2\rho^2 dt'. \quad (A.1)$$

Let us differentiate the obtained equation with respect to t:

$$(t - t_0 + \tau)\frac{d\varkappa}{dt} = \frac{2(\varkappa+1)(3\varkappa-1)\varkappa}{(3\varkappa^2 - 2\varkappa + 3)}, \qquad \varkappa = k_m^2\rho^2.$$

The solution of the obtained differential equation is of the form

$$\frac{(\varkappa+1)(3\varkappa-1)}{\varkappa^{3/2}} = C(t - t_0 + \tau), \quad C = \text{const.} \quad (A.2)$$

When $t = t_0$ the quantity κ is equal to κ_0, where κ_0 is determined from the equation $\partial w_k(t_0)/\partial j_y = 0$ (in our case $\kappa_0 = 3$). Thus, finally

$$\frac{(\varkappa+1)(3\varkappa-1)}{\varkappa^{3/2}} = \frac{(\varkappa_0+1)(3\varkappa_0-1)}{\tau\varkappa_0^{3/2}}(t - t_0 + \tau). \quad (A.3)$$

The form of the initial function $w_k(t_0)$ chosen by us corresponds to the distribution of noise that has grown linearly during a time τ from w_0 to the start of the quasilinear relaxation from k_y, corresponding to the maximum of the increment at the initial Maxwellian velocity distribution.

At any rate, it can be stated that the process of quasilinear relaxation, which leads to establishment of relation (11) in the vicinity of the point $v = \omega_k/k_z$, lasts for a time of the order of $1/\gamma_0(v)$. Therefore, accurate to quantities on the order of $\ln^{-1} A$, we can equate t_0 to τ. Consequently, formula (A.3) takes the form

$$\frac{(\varkappa+1)(3\varkappa-1)}{\varkappa^{3/2}} = \frac{(\varkappa_0+1)(3\varkappa_0-1)}{\varkappa_0^{3/2}}\frac{t}{t_0}$$

$$t_0 = \frac{8}{3\sqrt{3}\gamma_0}\ln A = \tau. \quad (A.4)$$

It follows from (A.4) that with good accuracy, we get $k_m\rho \sim t$ even when $t \gtrsim t_0$.

By a derivation similar to that given in the text, we can show that the relative width of the packet is small at all instants of time:

$$\frac{\Delta^2_{k_y}{}'(t)}{k_m^2(t)} \approx \frac{1}{\ln A} \ll 1.$$

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