FLUCTUATIONS IN A BOUNDED LIGHT BEAM PROPAGATING IN A RANDOMLY INHOMOGENEOUS MEDIUM

V. M. KOMISAROV

Acoustics Institute

Submitted to JETP editor August 24, 1966

J. Exptl. Theoret. Phys. (U.S.S.R.) 52, 711—714 (March, 1967)

The propagation of a spatially-bounded light beam in a medium with random inhomogeneities is considered. The space-angle intensity-distribution function in the aperture plane of a detector arbitrarily oriented relative to the unperturbed beam is obtained. Some statistical characteristics of the beam parameters are calculated. The possibility of taking regular refraction into account in this problem is indicated.

THE rapid development of quantum electronics makes it possible to use laser techniques both for communication purposes and to investigate the properties of the medium in which the propagation takes place. Since the mean value of the refractive index is frequently subject to random deviations, great interest attaches to the influence of these deviations on the fluctuations of the parameters of the propagating light beam. In this communication we propose to determine the statistical characteristics of the beam parameters by using the space-angle distribution function of the intensity. This function is obtained by solving a transport equation of the same type as the Fokker-Planck diffusion equation. We present the results of some of the calculations.

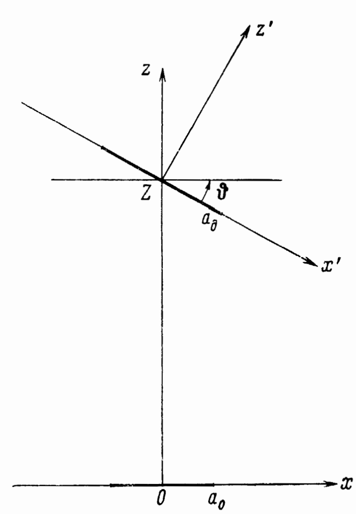
We assume that the refractive index in an inhomogeneous half-space $z > 0$ is $n = \langle n \rangle + \alpha\mu(\mathbf{r})$, where $\alpha\mu(\mathbf{r})$ are small random deviations from the mean value ($\alpha \ll 1$, $\langle n \rangle = 1$). Assume that plane wave, with initial field distribution $U_0(x_0, y_0)$ in the plane $z = 0$, is formed at the output of the optical system that focuses a laser beam and propagates in the positive z direction[1)] (see the figure; the y axis is directed perpendicular to the plane of the figure; a_0 and a_d characterize the dimensions of the apertures at the output of the system and of the detector). However, a wave with bounded transverse cross section cannot be strictly plane. Its spatial Fourier expansion contains components with wave vectors of different directions. It can then be assumed that an initial intensity distribution with respect to the directions and in space, $I_0(\rho_0, q_{0x}, q_{0y}) \times \delta(q_{0z} - k_0)$, is specified on the boundary of the inhomogeneous medium. The small diffraction angles θ_{0x}, $\theta_{0y} \sim 1/k_0 a_0 \ll 1$ are connected with the components of the vector q_0 by the relations[1]

$$q_{0x} = k_0\theta_{0x}, \quad q_{0y} = k_0\theta_{0y}. \tag{1}$$

Here $k_0 = \omega/c_0$ is the wave number in the homogeneous medium.

The final intensity distribution function can be obtained from the initial one if one knows the probability of the transition from the initial state ρ_0, q_0 to a certain interval of final states. The equation for the transition probability can be derived in the following manner:

Assuming that the conditions for the validity of the quasistatic approach and geometrical optics are satisfied[2], we write out in first approximation in α the following system of differential equations for the ray[3]:



[1)]The time factor $e^{-i\omega t}$ is omitted throughout.

$$d\mathbf{r}/d\sigma = \mathbf{S} + O(\alpha^2), \quad d\mathbf{S}/d\sigma = \alpha\nabla_\perp\mu + O(\alpha^2). \quad (2)$$

$$\mathbf{r}(0) = \mathbf{r}_0, \quad \mathbf{S}(0) = \mathbf{S}_0. \quad (3)$$

Here $\nabla_\perp$ is the component perpendicular to $\mathbf{S}_0$ of the vector ∇, $\mathbf{S}_0$ is a unit vector tangent to the ray, and σ is the path along the ray.

The problem (2)—(3) has stochastically the same form as the problem of motion of a Brownian particle.[4] The role of the time is assumed by the path σ, and the role of the velocity by the vector **S**. As is well known, processes such as Brownian motion are Markoff processes. Thus, the rays are regarded as trajectories of a Markoff process and we can obtain the probability of the transition from the state $\mathbf{r}_0$, $\mathbf{S}_0$ to some interval of final states after the ray traverses a path σ which is large compared with the correlation radius of the refractive-index fluctuations.

In accordance with (2)—(3), assuming that the random process $\mu(\mathbf{r})$ is homogeneous and isotropic, we obtain for the transition probability the equation

$$\frac{\partial W}{\partial\sigma} + \mathbf{S}\nabla_{\mathbf{r}}W = D\Delta_{\mathbf{S}}W - \sum_{i,k}\frac{\partial^2 S_i S_k W}{\partial S_i \partial S_k}, \quad i, k = x, y, z, \quad (4)$$

under the condition $W = \delta(\mathbf{S} - \mathbf{S}_0)\delta(\mathbf{r} - \mathbf{r}_0)$ at $\sigma = 0$. Here D is the diffusion coefficient of the rays.[2,3]

Inasmuch as we have in the experiment an averaging over the phase volume of a detector whose aperture plane can be inclined at a certain angle ϑ to the plane z = const, we are interested in the intensity distribution functions $I_\vartheta(\rho', q', z)$ in the aperture plane (see the figure). We therefore introduce in lieu of the path σ the distance Z to the aperture plane, in accordance with the formula

$$\sigma = \frac{Z\cos\vartheta - x_0\sin\vartheta}{S_{0x}\sin\vartheta + S_{0z}\cos\vartheta} \approx \frac{Z\cos\vartheta}{S_{0x}\sin\vartheta + S_{0z}\cos\vartheta}. \quad (5)$$

It is assumed that $\tan\vartheta \ll Z/a_0$ and $Z/a_0 \gg 1$. Then, for small deviations from the initial propagation direction ($D\sigma \ll 1$) we get for the transition probability

$$W = \frac{3}{4\pi^2 D^2\sigma^4 S_{0z}^2}\exp\Big\{-\frac{(1-S_{0y}^2)}{D\sigma S_{0z}^2}\Big[\xi_1^2 - \frac{3(x-X_0)\xi_1}{\sigma} + \frac{3(x-X_0)^2}{\sigma^2}\Big] - \frac{(1-S_{0x}^2)}{D\sigma S_{0z}^2}\Big[\xi_2^2 - \frac{3(y-Y_0)\xi_2}{\sigma} + \frac{3(y-Y_0)^2}{\sigma^2}\Big] - \frac{S_{0x}S_{0y}}{D\sigma S_{0z}^2}\Big[2\xi_1\xi_2 - 3\frac{\xi_2(x-X_0)+\xi_1(y-X_0)}{\sigma} + 6\frac{(x-X_0)(y-Y_0)}{\sigma^2}\Big]\Big\}$$
$$\times\delta\Big(\xi_3 + \xi_2\frac{S_{0y}}{S_{0z}} + \xi_1\frac{S_{0x}}{S_{0z}}\Big)\delta\Big(\xi_6 + \xi_5\frac{S_{0y}}{S_{0z}} + \xi_4\frac{S_{0x}}{S_{0z}}\Big);$$

$$\mathbf{R}_0 = \mathbf{r}_0 + \mathbf{S}_0\frac{Z\cos\vartheta - x_0\sin\vartheta}{S_{0x}\sin\vartheta + S_{0z}\cos\vartheta}, \quad \xi_1 = S_x - S_{0x},$$
$$\xi_2 = S_y - S_{0y}, \quad \xi_3 = S_z - S_{0z}, \quad \xi_4 = x - \xi_1\sigma - X_0,$$
$$\xi_5 = y - \xi_2\sigma - Y_0, \quad \xi_6 = z - \xi_3\sigma - Z_0. \quad (6)$$

We change to new coordinate systems $\mathbf{r}'$, $\mathbf{S}'$, rotated through an angle α relative to the old ones (see the figure), and integrate $W(\mathbf{r}', \mathbf{S}')$ with respect to the variables z′ and S'_z. We denote the result of the integration by $W(\rho', S', \rho_0, S_0)$. We can now find the distribution

$$I_\vartheta(\rho', k_0 S', Z) = \int I_0(\rho_0, k_0 S_0)\, W(\rho', S', \rho_0, S_0)\, d\rho_0\, dS_0. \quad (7)$$

If $U_0(\rho_0) = \exp\{-\rho_0^2/2a_0^2\}$ and $\tan^3\vartheta \ll k_0 a_0$, then

$$I_\vartheta = \frac{\text{const}}{\cos^2\vartheta\sqrt{\Delta_0\Delta_\vartheta}}\exp\Big\{-\frac{A_0 y^2 - 2H_0 y q_y + B_0 q_y^2}{\Delta_0} - \frac{A_\vartheta x'^2 - 2H_\vartheta x'(q_{x'} + k\sin\vartheta) + B_\vartheta(q_{x'} + k\sin\vartheta)^2}{\Delta_\vartheta}\Big\}, \quad (8)$$

where

$$A_\vartheta = \frac{k_0^2 DZ}{\cos^2\vartheta} + \frac{1}{4a_0^2\cos^2\vartheta}, \quad B_\vartheta = \frac{DZ^3}{3\cos^2\vartheta} + \frac{a_0^2}{4\cos^6\vartheta} + \frac{Z^2}{4k_0^2 a_0^2\cos^6\vartheta}, \quad H_\vartheta = \frac{DZ^2 k_0}{2\cos^2\vartheta} + \frac{Z}{4k_0 a_0^2\cos^4\vartheta},$$

$$\Delta_\vartheta = 4\cos^4\vartheta(A_\vartheta B_\vartheta - H_\vartheta^2).$$

It is interesting to note that in the particular case $\vartheta = 0$ formula (8) coincides (after some transformations) with the distribution function describing the multiple scattering of a beam of charged particles as a function of the deflection angle and the transverse particle displacement, obtained in[5] by solving Schrödinger's equation.

With the aid of (8) we can find the different statistical characteristics of the beam parameters. Thus, the width of the beam in the aperture plane can be characterized by the quantities

$$\langle x'^2\rangle = 2B_\vartheta\cos^4\vartheta = \frac{a_0^2}{2\cos^2\vartheta} + \frac{Z^2}{2k_0^2 a_0^2\cos^2\vartheta} + \frac{2}{3}DZ^3\cos^2\vartheta, \quad \langle y^2\rangle = 2B_0. \quad (9)$$

In these formulas, the first terms characterize the major and minor axes of the ellipse formed when the unperturbed beam intersects the observation plane. As to the third terms, which predominate at sufficiently large distances, they determine the broadening due to the fluctuations of the refractive index of the medium. The fact that the last term decreases with increasing angle ϑ is connected with the character of the action of these fluctuations. In fact, as follows from (2), random ''jolts'' perturbing the beam act in a plane perpendicular to the initial propagation direction, which in our case coincides essentially with the z axis.

We can also obtain, by integrating (8) over the area of the detector aperture, the experimentally observed brightness coefficient, we can calculate the rms beam divergence angle (fluctuations of the arrival angle), etc.

Further, using the procedure described earlier[3], we can take into account the influence of regular refraction on the statistics of the rays and obtain, under certain assumptions, the intensity distribution function in a beam propagating in a medium with variable $\langle n \rangle$.

The author is grateful to L. A. Chernov for a number of useful discussions.

[1] L. D. Landau and E. M. Lifshitz, Teoriya polya (Field Theory), Fizmatgiz, 1962, p. 189.

[2] L. A. Chernov, Rasprostranenie voln v srede so sluchaĭnymi neodnorodnostyami (Wave Propagation in a Medium with Random Inhomogeneities), AN SSSR, 1958, p. 17.

[3] V. M. Komissarov, Izv. Vuzov Radiofizika **9**, 292 (1966).

[4] S. Chandrasekhar, Stochastic Problems in Physics and Astronomy, Russ. transl. IIL, 1947, p. 39. [Revs. Modern Phys. **15**, 1 (1943)].

[5] N. P. Kalashnikov and M. I. Ryazanov, JETP 50, 459 (1966), Soviet Phys. JETP **23**, 306 (1966).

Translated by J. G. Adashko
89