ON THE ASYMPTOIC BEHAVIOR OF THE GREENS' FUNCTIONS IN QUANTUM FIELD THEORY

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On the basis of the coupled system of equations of quantum field theory, a system of renormalized equations for the nucleon and meson Green's functions and the vertex part is obtained in the form of expansions in powers of the coupling constant λ. An investigation of the asymptotic region, taking into account terms $\sim\lambda^3$, shows that the "anti-Lehmann" behavior of the Green's functions in the pseudoscalar meson theory and in electrodynamics found in first approximation remains essentially the same when higher approximations are taken into account.

1. INTRODUCTION

IT is known[1] that the exact propagation functions for quantized fields (Green's functions) have a representation in the form of a linear superposition of the corresponding free functions with a certain mass density $\rho(\chi^2)$. The Green's functions behave on the light cone like the corresponding free functions if the integral $\int_0^\infty \rho(\chi^2)d\chi^2$ converges, or they are more singular than the free functions if this integral diverges. In the derivation of the spectral representation no specific assumptions about the form of the interaction are made. For the renormalization constants in the field theory the following relations have been established:

$$0 \leqslant Z_2 < 1, \quad 0 \leqslant Z_3 < 1,$$

$$Z_3^{-1} = \int_0^\infty \rho(\chi^2)\,d\chi^2, \quad Z_2^{-1} = \int_0^\infty \rho_1(\chi^2)\,d\chi^2. \tag{1}$$

It is of interest to investigate whether the renormalization constants of electrodynamics and pseudoscalar meson theory satisfy the Lehmann conditions (1).

The asymptotic form of the Green's functions in electrodynamics (in the unrenormalized theory) has been studied by Landau, Abrikosov, and Khalatnikov,[2] and in the unrenormalized pseudoscalar meson theory by Abrikosov, Galanin, and Khalatnikov.[3] In the renormalized meson theory with weak pseudoscalar coupling and in electrodynamics, the asymptotic form of the Green's function has been investigated by Galanin, Ioffe, and Pomeranchuk[4] and by Fradkin.[5, 6]

In these papers, the object of study has been the "three-Γ" equation for the vertex part, which takes account of the leading logarithmic terms of the perturbation series. The asymptotic form of the meson and nucleon Green's functions found in this approximation has a nonphysical singularity of noninteger degree in the region of momenta $p^2 \sim e^{1/\lambda}$. Thus near the singularity and beyond it (for $p^2 \gg e^{1/\lambda}$), the asymptotic form of the Green's function loses its Lehmann properties. The question now arises whether this behavior of the Green's functions is a consequence of the insufficiency of the approximation used, i.e., whether the asymptotic form of the Green's function changes when higher orders in the coupling constant are included in the system of renormalized equations.

In the present paper, we start from the system of coupled equations for the meson and nucleon Green's functions and the vertex functions of different orders (different numbers of external meson lines) to derive a system of renormalized equations for the nucleon and meson Green's functions and the lowest order vertex part (with one external meson line) in the form of an expansion in powers of the coupling constant $\lambda = g^2/4\pi$. We investigate the contribution of the terms up to order λ^3 to the asymptotic Green's function ("seven-Γ" approximation for the vertex part). This investigation leads to the conclusion that the account of higher order terms does not change the asymptotic form of the Green's function beyond the singularity (for $p^2 \gg e^{1/\lambda}$), found in the "three-Γ" approximation. Thus the asymptotic form of the Green's function beyond the singularity retains its "anti-Lehmann" character. One of the possible reasons for this behavior may be an unsuitable choice of the basic Hamiltonian.

However, it must be noted that the above-mentioned "anti-Lehmann"-type asymptotic form belongs to the class of slowly varying functions

(functions of the logarithmic type). The renormalized system of equations for the Green's functions may in general have solutions in the class of rapidly varying functions, which may change our conclusion about the "anti-Lehmann" character of the asymptotic Green's functions beyond the singularities. The possibility of the existence of such solutions is not investigated in the present paper.

2. EQUATIONS FOR THE GREEN'S FUNCTIONS IN THE PSEUDOSCALAR MESON THEORY

In the presence of external sources the equations for the Green's functions in the pseudoscalar symmetric meson theory have the form[7,8]

$$(\hat{\mathbf{p}} - m_0) G(\mathbf{p}, \mathbf{q}) + ig_0\gamma_\sigma \int \langle\varphi_\sigma(\mathbf{p}-\mathbf{k})\rangle G(\mathbf{k}, \mathbf{q})\, d^4k - \int \sum (p, k) G(k, q)\, d^4k = \delta(p - q),$$

$$(k^2 - \mu_0^2) D_{\mu\nu}(\mathbf{k}, \mathbf{k}_1) - \int \Pi_{\mu\sigma}(\mathbf{k}, \mathbf{p}_1) D_{\sigma\nu}(\mathbf{p}_1, \mathbf{k}_1)\, d^4p_1 = \delta(k - k_1),$$

$$\Gamma_\sigma(p, p', k) = \frac{\delta G^{-1}(p, p')}{\delta[ig_0\langle\varphi_\sigma(k)\rangle]} = \gamma_\sigma\delta(p - p' - k) - \frac{\delta\Sigma(p, p')}{\delta[ig_0\langle\varphi_\sigma(k)\rangle]}, \tag{2}$$

where $\gamma_\sigma = \gamma_5\tau_\sigma$ and

$$\Sigma(p, p') = \frac{g_0^2}{4\pi^3 i}\int \gamma_\mu G(p + s, p_1) \times \Gamma_\nu(p_1, p', s_1) D_{\mu\nu}(s_1, s)\, d^4s\, d^4s_1 d^4p_1,$$

$$\Pi_{\sigma\rho}(k, k') = -\frac{g_0^2}{4\pi^3 i}\mathrm{Sp}\int \gamma_0 G(k + p, p_1)\Gamma_\rho(p_1, p_2, k') G(p_2, p)\, d^4p\, d^4p_1 d^4p_2.$$

Here g_0 is the bare charge.

From (2) and (3) we can obtain an infinite system of coupled equations for the Green's functions of the nucleon and the meson and for the higher vertex functions, by successive functional differentiation of the mass and polarization operators Σ and $\Pi_{\sigma\rho}$ with respect to the external sources, which are thereafter set equal to zero. This system of equations has the form

$$[(\hat{p} - m_0) - \hat{M}(\hat{p})] G_0(p) = 1, \qquad D_{\mu\nu} = \delta_{\mu\nu}D,$$

$$[k^2 - \mu_0^2 - \hat{P}_{\mu\rho}(k^2)] D_{\rho\nu} = \delta_{\mu\nu}, \qquad \lambda_0 = g_0^2/4\pi,$$

$$\hat{M}_0(\hat{p}) = \frac{g_0^2}{4\pi^3 i}\int \gamma_\mu G(p - k)\Gamma_\nu(p - k, p, k) D_{\mu\nu}(k^2)\, d^4k,$$

$$P_{\mu\rho}(k^2) = -\frac{g_0^2}{4\pi^3 i}\mathrm{Sp}\int \gamma_\mu G(\hat{p})\Gamma_\rho(p, p + k, k) G(p + k)\, d^4p,$$

$$\Gamma_\sigma(p, p - k, k) = \gamma_\sigma - \frac{i\lambda_0}{\pi^2}\int \gamma_\mu M_{\rho\mu}(p + s, p - k, s, k)\, d^4s'$$

$$\Gamma^{(1)}_{\rho\sigma}(p + s, p - k, s, k) = -\frac{i\lambda_0}{\pi^2}\int \gamma_\mu M^{(1)}_{\rho\sigma\mu}(p + s + s', p + k, s', s, k)\, d^4s',$$

$$\Gamma^{(2)}_{\rho_1\rho\sigma}(p + s + s', p - k, s', s, k) = -\frac{i\lambda_0}{\pi^2}\int \gamma_\mu M^{(2)}_{\rho_1\rho\sigma\mu} \times (p + s + s' + s'', p - k, s'', s', s, k)\, d^4s'' \tag{4}$$

etc. Here we have introduced the notation

$$\Gamma^{(1)}_{\sigma\mu}(p, p', k, k') = -\left.\frac{\delta\Gamma_\sigma(p, p', k)}{[ig_0\delta\langle\varphi_\mu(k')\rangle]}\right|_{\langle\varphi_\mu\rangle=0},$$

$$\Gamma^{(2)}_{\rho_1\rho\sigma}(p, p', k, k', k'') = -\left.\frac{\delta\Gamma^{(1)}_{\rho_1\rho}(p, p', k, k')}{[ig_0\delta\langle\varphi_\sigma(k'')\rangle]}\right|_{\langle\varphi_\sigma\rangle=0}$$

etc. The quantities Γ_σ, $\Gamma^{(1)}_{\rho\sigma}$, $\Gamma^{(2)}_{\rho_1\rho\sigma}$ are the vertex functions with one, two, three, etc., external meson lines. Let us write down the kernal $M^{(0)}_{\sigma\mu}$ of the integral equation for Γ_σ:

$$M^{(0)}_{\sigma\mu}(p + s, p - k, s, k) = G(p + s) \times \Gamma_\sigma(p + s, p + s - k, k) G(p + s - k) \times \Gamma_\mu(p + s - k, p - k, s) D(s) + G(p + s) \times \Gamma^{(1)}_{\sigma\mu}(p + s, p - k, s, k) D(s) + G(p + s) \times \Gamma_\nu(p + s, p - k, s - k)\left.\frac{\delta D_{\mu\nu}}{ig_0\delta\langle\varphi_\sigma\rangle}\right|_{\langle\varphi_\sigma\rangle=0}. \tag{5}$$

For brevity we shall not write down explicitly the terms of the type

$$\left.\frac{\delta D_{\mu\nu}}{ig_0\delta\langle\varphi_\sigma\rangle}\right|_{\langle\varphi_\sigma\rangle=0}.$$

The kernels $M^{(1)}_{\rho\delta\mu}$ etc., will not be given explicitly because of their complicated structure. It is seen from (4) that Γ_σ is coupled to $\Gamma^{(1)}_{\sigma\mu}$, $\Gamma^{(1)}_{\sigma\mu}$ is coupled to $\Gamma^{(2)}_{\rho\sigma\mu}$, etc.

In Eqs. (4) we renormalize the charge and the meson and nucleon masses in the usual manner.[7,8] We define the renormalized quantities Γ^*_μ, G^*, D^*, and g^2 in the following way:

$$\Gamma_\mu^* = Z_1\Gamma_\mu, \quad G^* = Z_2^{-1}G, \quad D^* = Z_3^{-1}D,$$

$$g^2 = Z_1^{-2}Z_2^2Z_3g_0^2. \tag{6}$$

Then the system of renormalized equations has the form

$$[\hat{p} - m - \Sigma^*(\hat{p})] G^*(\hat{p}) = 1,$$

$$[k^2 - \mu^2 - \Pi^*(k^2)] D^*(k^2) = 1, \quad \lambda = g^2/4\pi,$$

$$\Sigma^*(\hat{p}) = M_0^*(\hat{p}) - M_0^*(m) - (\hat{p} - m) M_0'^*(m),$$

$$\Pi^*(k^2) = P_0^*(k^2) - P_0^*(\mu^2) - (k^2 - \mu^2) P_0'^*(\mu^2), \tag{7}$$

$$M_0^*(\hat{p}) = \frac{3\lambda}{\pi^2 i} Z_1\gamma_5 \int G^*(p-k)\Gamma_5^*(p-k,p,k)D^*(k)\,d^4k,$$

$$P_0^*(k^2) = -\frac{2\lambda}{\pi^2 i} Z_1 \,\mathrm{Sp} \int \gamma_5 G^*(\hat{p})\Gamma_5^*(p,p-k,k)G^*(p-k)\,d^4p,$$

$$\Gamma_\sigma^*(p,p-k,k) = Z_1\gamma_\sigma - \frac{i\lambda}{\pi^2}\int Z_1\gamma_\mu M_{\sigma\mu}^{(0)*}$$

$$\times\ (p+s,p-k,s,k)\,d^4s,$$

$$\Gamma_{\rho\sigma}^{(1)*} = -\frac{i\lambda}{\pi^2}\int Z_1\gamma_\mu M_{\rho\sigma\mu}^{(1)*}\,d^4s' \tag{8}$$

etc. All masses have their experimental values. The constants Z_2 and Z_3 have the form

$$Z_2 = 1 + M_0'^*(m), \quad Z_3 = 1 + P_0'^*(\mu^2). \tag{9}$$

In (7) and (8) we must still eliminate the constant Z_1. To this end we proceed in the following fashion. We define an operator $R_{\sigma\mu}$ which is connected with the quantity $M_{\sigma\mu}^{(0)*}$ by an integral equation:

$$\gamma_\mu R_{\sigma\mu}(p+s,p-k,s,k) = \gamma_\mu M_{\sigma\mu}^{(0)*}\ (p+s,p-k,s,k)$$

$$+\frac{i\lambda}{\pi^2}\int \gamma_\rho M_{\mu\rho}^{(0)*}\ (p+s,\,p+s_1,\,s,\,s_1)$$

$$\times\ R_{\sigma\mu}(p+s_1,\,p-k,\,s_1,k)\,d^4s_1.$$

Multiplying both sides of this equation by $i\lambda\pi^{-2}Z_1$, integrating over d^4s and taking account of the first equation (8), we obtain an equation for Γ_σ^*:

$$\Gamma_\sigma^*(p,p-k,k) = Z_1\gamma_\sigma - \frac{i\lambda}{\pi^2}\int d^4s\Gamma_\mu^*(p,p+s,s)$$

$$\times\ R_{\sigma\mu}(p+s,p-k,s,k). \tag{10}$$

If we set $\Gamma^{(1)*} = 0$ and substitute $R_{\sigma\mu}$ in (10), we easily see that this equation describes the "three-Γ" vertex plus all possible insertions in the left corner. However, when the exact expression for $R_{\mu\sigma}$ is substituted in (10), as will be discussed below, only irreducible graphs remain in the equation for the vertex part.

The elimination of Z_1 in the equations for $\Gamma^{(1)*}$, $\Gamma^{(2)*}$, etc., is achieved by substituting the quantity $Z_1\gamma_\sigma$ obtained from (10):

$$Z_1\gamma_\sigma = \Gamma_\sigma^* + \frac{i\lambda}{\pi^2}\int d^4s\Gamma_\mu^* R_{\sigma\mu}.$$

Thus we have

$$\Gamma_{\sigma\rho}^{(1)*}(p+s,\,p-k,\,s,\,k) = -\frac{i\lambda}{\pi^2}\int\Big[\Gamma_\mu^*(p+s,\,p+s+s',\,s')$$

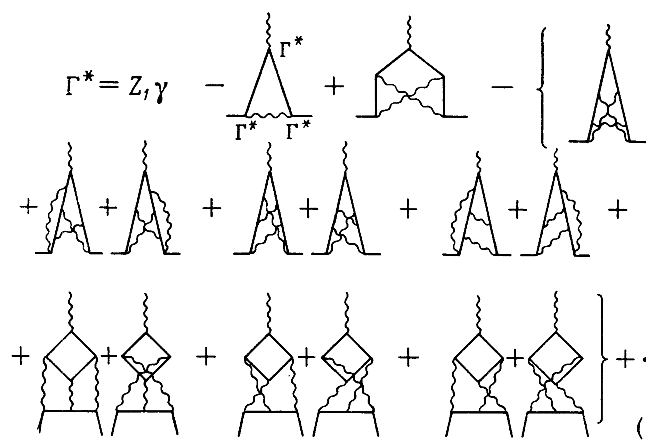
$$+\frac{i\lambda}{\pi^2}\int d^4s_1\Gamma_\nu^*(p+s,\,p+s+s_1,\,s_1)$$

$$\times\ R_{\mu\nu}(p+s+s_1,\,p+s+s',\,s_1,\,s')\Big]$$

$$\times\ M_{\sigma\rho\mu}^{(1)*}\ (p+s+s',\,p-k,\,s',\,s,\,k)\,d^4s',$$

$$\Gamma_{\rho_1\rho\sigma}^{(2)*} = -\frac{i\lambda}{\pi^2}\int\Big[\Gamma_\mu^* + \frac{i\lambda}{\pi^2}\int \Gamma_\nu^* R_{\mu\nu}d^4s_1\Big]M_{\rho_1\rho\sigma\mu}^{(2)*}d^4s'',\ \text{etc.} \tag{11}$$

Let us further expand all quantities $R_{\sigma\mu}$, $\Gamma_{\sigma\mu}^{(1)*}$, $\Gamma_{\rho_1\rho\sigma}^{(2)*}$, etc., in (10) and (11) in powers of Γ_σ^* (i.e., in powers of λ) and substitute the expansions for $\Gamma^{(i)*}$ in (10). On the right-hand side of (10) we then obtain a series consisting only of irreducible vertex parts containing Γ_σ^* at each corner. All reducible vertex parts drop out. We write the graphical equation for Γ^* up to terms of order $\sim\alpha^3$ ($\alpha = i\lambda/\pi^2$):



(12)

In all corners of the graphs we have here the vertices Γ_σ^*. Let us write the equation for Γ_σ^* in the form

$$\Gamma_\sigma^*(p,p-k,k) = Z_1\gamma_\sigma + M_{1\sigma}^*(p,k), \tag{13}$$

where $M_{1\sigma}^*$ contains all irreducible graphs. The constant Z_1 is defined by the relation

$$Z_1\gamma_\sigma = \gamma_\sigma - M_{1\sigma}^*(p_0,k_0), \quad p_0^2 = m^2, \quad k_0^2 = \mu^2,$$

and the renormalized equation for the vertex has the form

$$\Gamma_\sigma^*(p,p-k,k) = \gamma_\sigma + M_{1\sigma}^*(p,k) - M_{1\sigma}^*(m,\mu). \tag{14}$$

For the following it is convenient to write Eq. (13) for Γ_σ^* in the following form:

$$\Gamma_\sigma^*(p,p-k,k) = Z_1\gamma_\sigma - \frac{i\lambda}{\pi^2}\int\Gamma_\mu^*(p,\,p-q,\,q)G^*(p-q)$$

$$\times\ R_{\sigma\mu}^{(1)}(p,k,q)\,d^4q, \tag{15}$$

where

$$R_{\sigma\mu}^{(1)}(p,\ k,q) = \Gamma_\sigma(p-q,\,p-q-k,\,k)G(p-q-k)$$

$$\times\ \Gamma_\mu(p-q-k,\,p-k,\,q)$$

$\times$ terms $\sim\lambda$ and higher orders.

Let us now consider the derivatives of the mass and polarization operators. We have from (8)

$$\frac{dM_0^*(\hat{p})}{dp_\nu} = \frac{\lambda}{\pi^2 i} \int Z_1\gamma_\sigma \left[\frac{dG^*(p-k)}{dp_\nu} \Gamma_\sigma^*(p-k,p,k) D^*(k) + G^*(p-k) \frac{d\Gamma_\sigma^*(p-k,p,k)}{dp_\nu} D^*(k) \right] d^4k. \quad (16)$$

In (16) we must substitute the expression for $Z_1\gamma_\sigma$ found from (15) and the expression for the derivative $d\Gamma_\sigma^*/dp_\nu$:

$$\frac{d\Gamma_\sigma^*(p-k,p,k)}{dp_\nu} = -\frac{i\lambda}{\pi^2} \int \left\{ \frac{dR_{\mu\sigma}^{(1)}(p,q,k)}{dp_\nu} G^*(p-q)\Gamma_\mu^* \times (p-q,p,q) + R_{\mu\sigma}^{(1)}(p,q,k) \frac{dG^*(p-q)}{dp_\nu} \Gamma_\mu^*(p-q,p,q) + R_{\mu\sigma}^{(1)}(p,q,k) G^*(p-q) \frac{d\Gamma_\mu^*}{dp_\nu} \right\} D(q)\, d^4q. \quad (17)$$

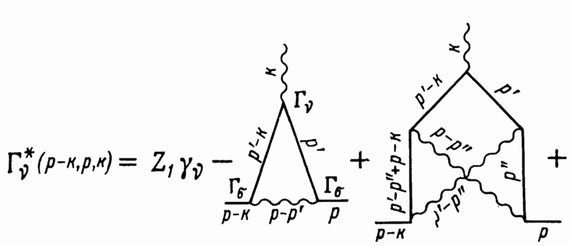
As a result, the renormalized expression for $M_0^{*\prime}(\hat{p})$ takes the following form after the necessary cancellations:

$$\frac{dM_0^*(\hat{p})}{dp_\nu} = \frac{\lambda}{\pi^2 i} \left\{ \int \Gamma_\sigma^*(p,p-k,k) \frac{dG^*(p-k)}{dp_\nu} \Gamma_\sigma^* \times (p-k,p,k) D^*(k)\, d^4k - \frac{i\lambda}{\pi^2} \int d^4k d^4q \Gamma_\sigma^*(p,p-k,k) G^*(p-k) \times D^*(k) \frac{dR_{\mu\sigma}^{(1)}}{dp_\nu} G^*(p-q)\Gamma_\mu^*(p-q,p,q) D^*(q) \right\}. \quad (18)$$

Let us now turn to the derivative of the polarization operator. We have

$$\frac{dP_{\mu\nu}(k^2)}{dk^2} = -\frac{\lambda}{\pi^2 i} \frac{1}{2k_\sigma} \mathrm{Sp} \int Z_1\gamma_\mu \left[\frac{dG^*(p-k)}{dk_\sigma} \Gamma_\nu^* \times (p-k,p,k) \times G^*(\hat{p}) + G^*(\hat{p}-\hat{k}) \frac{d\Gamma_\nu^*(p-k,p,k)}{dk_\sigma} G^*(\hat{p}) \right] d^4p. \quad (19)$$

The equation for $\Gamma_\nu^*(p-k, p, k)$ has the following graphical form:



Here we cannot write the equation for Γ_ν^* in compact form in analogy to (15), since the vertex $\Gamma_\nu^*(p'-k, p', k)$ is not in the left-hand corner but in the center of the graph. We write this equation in the form

$$\Gamma_\nu^*(p-k,p,k) = Z_1\gamma_\nu + \sum_i (-\alpha)^i \int dp_1 dp_2 \ldots dp_i \cdot \times \{[\hat{R}_{2i}(p,k,p_1,p_2\ldots p_i)] G^*(p_1-k)\Gamma_\nu^*(p_1-k,p_1,k) G^* \times (p_1)[\hat{R}_{1i}(p,p_1\ldots p_i)]\}. \quad (20)$$

Only the operator $\hat{R}_{2i}$ depends on the momentum k; this operator represents the part of the graph to the left of the vertex $\Gamma_\nu^*(p_1-k, p_1, k)$.

The further calculations are analogous to the ones for $M_0^{\prime *}(\hat{p})$: from (20) we find the quantities $Z_1\gamma_\nu$ and $d\Gamma_\nu/dk_\sigma$, which are substituted in (19). As a result we obtain the following expression for the derivative of the polarization operator:[1)]

$$\frac{dP_{\mu\nu}(k^2)}{dk^2} = -\frac{\lambda}{\pi^2 i} \frac{1}{2k_\sigma} \mathrm{Sp} \int \left[\Gamma_\mu(p,p-k,k) \frac{dG(p-k)}{dk_\sigma} - \Gamma_\nu(p-k,p,k) \times G(\hat{p}) + G(\hat{p})\Gamma_\mu(p,p-k,k) G(p-k)\hat{K}_\nu(p,k) \right] d^4p, \quad (21)$$

$$\hat{K}_\nu(p,k) = \sum_i (-\alpha)^i \int dp_1 dp_2 \ldots dp_i \ldots \frac{d\hat{R}_{2i}(p,k,p_1\ldots p_i)}{dk_\sigma} \times G(p'-k)\Gamma_\nu(p'-k,p',k) G(p')\hat{R}_{1i}(p,p_1\ldots p_i). \quad (22)$$

For example, we have for the "three-Γ" vertex

$$R_{21}(p,k,p_1) = \Gamma(p-k, p_1-k, p-p_1),$$
$$R_{11}(p,p_1) = \Gamma(p,p_1,p-p_1) D(p-p_1) \text{ etc.}$$

Let us now write the equations for the Green's functions of the nucleon and the meson:

$$dG^{-1}(\hat{p})/dp_\sigma = \gamma_\sigma - [M'(\hat{p}) - \gamma_\sigma M'(m)],$$
$$dD_{\mu\nu}^{-1}(k^2)/dk^2 = 1 - [P_{\mu\nu}'(k^2) - P_{\mu\nu}'(\mu^2)]. \quad (23)$$

The derivatives of the mass and polarization operators entering in these equations are given by (18) and (21). Equations (14) and (23) will be used in the following for the investigation of the asymptotic behavior of the functions G, D, and Γ.

3. ASYMPTOTIC BEHAVIOR OF THE GREEN'S FUNCTIONS IN THE PSEUDOSCALAR MESON THEORY WITH WEAK COUPLING

We see from (9) that the renormalization constants Z_2 and Z_3 satisfy the Lehmann conditions (1) only if the derivatives $M_0'(m)$ and $P_0'(\mu^2)$ are finite. If in this case $Z_2 \neq 0$ and $Z_3 \neq 0$, then the functions G and D behave asymptotically like the free functions G_0 and D_0:

$$G(\hat{p}) \underset{p^2\to\infty}{\to} Z_2^{-1}\frac{\hat{p}}{p^2}, \quad D(k^2) \underset{k^2\to\infty}{\to} \frac{Z_3^{-1}}{k^2}. \quad (24)$$

If $Z_2 = 0$ and $Z_3 = 0$, then G and D are more singular than the corresponding free functions. It is

[1)]In the following we shall omit the symbol * in all renormalized quantities.

of interest to investigate whether the case $Z_2 \neq 0$, $Z_3 \neq 0$ is realized. If so, the asymptotic behavior of the vertex part can be found from (14), where the functions G and D are asymptotically replaced by the free functions G_0 and D_0, according to (24). This asymptotic form of the function Γ is then substituted in the expressions for the derivatives $M'(\hat{p})$ and $P'(k^2)$ (18) and (21). If these derivatives are finite, then the constants Z_2 and Z_3 are determined by (9).

We note that if the asymptotic behavior of the vertex Γ for large momenta is given by a decreasing function, then the integrals on the right-hand size of (14) converge separately, and it follows that $Z_1 \neq 0$. Equation (14) is in this case equivalent to the homogeneous equation

$$\Gamma_\sigma(p, p-k, k) = M_{1\sigma}(p, k). \tag{25}$$

When substituted in $M'(m)$, the values of $\Gamma(p, p-k, k)$ for $k^2 \gg p^2$ make the most important contribution, while the most important contribution to $P'(\mu^2)$ comes from the values of Γ for $p^2 \gg k^2$. We shall seek the asymptotic form of Γ for large momenta in the form

$$\Gamma_\mu = \gamma_\mu \Gamma_0(f^2), \tag{26}$$

where f is the largest vector on which Γ depends. Small asymptotic corrections to Γ_0 of the form

$$\frac{\hat{k}\hat{p}}{k^2+p^2} S_1(\hat{p}, \hat{k}), \qquad \frac{k^2}{p^2+k^2} S_2(\hat{p}, \hat{k})$$

etc., which can be easily found,[4] do not give significant asymptotic contributions to the derivatives $M'(m)$ and $P'(\mu^2)$.

Following [2], we shall consider the space-like vectors p_μ and go over to a Euclidean metric. We seek the asymptotic solution of (14) for the vertex part Γ in the form of a series

$$\Gamma = \Gamma_1 + \alpha\Gamma_2 + \alpha^2\Gamma_3 + \ldots \tag{27}$$

As the first approximation Γ_1 we take the asymptotic solution of the "three-Γ" equation, which is represented by the first graph on the right-hand side of (12). It takes account of the leading logarithmic terms of perturbation theory. The equation for the asymptotically important part Γ_{10} has the following form (in one-dimensional writing):

$$\Gamma_{10}(x) = 1 - \lambda_1 \int_1^x \frac{\Gamma_{10}^3(x')\,dx'}{x'}, \tag{28}$$

where $\lambda_1 = Z_2^{-2} Z_3^{-1} \lambda$, $x = -p^2/m^2$ (the lower limit is $\sim m^2$). Because of the subtraction $M_1(p, k) - M_1(m, \mu)$, the integration is taken to the upper limit $\sim x$. In (28) we have omitted the term $M_1(p, k)$, which is asymptotically small in comparison with $M_1(m, \mu)$, as can be seen by direct substitution of the solution. The solution of (28) has the form

$$\Gamma_{10}(x) = (1 + 2\lambda_1 \ln x)^{-1/2}. \tag{29}$$

Thus $\Gamma_{10}(x)$ is a slowly decreasing function.

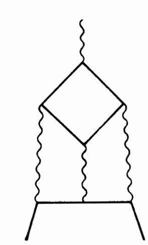
It is seen from (28) that the integral is logarithmically divergent if we set $\Gamma_{10} = 1$ under the integral. The term $\sim \alpha^2$ in the equation for Γ (the "five-Γ" term) does not contain a "doubly logarithmic" singularity, i.e., the integral will not diverge twice in the integration over the internal momenta. This is seen by considering the typical "five-Γ" integrals

$$\int_1^x \frac{\Gamma_{10}^4(x')\,dx'}{x'^2} \int_1^{x'} \Gamma_{10}(x'')\,dx'', \quad \int_1^x \Gamma_{10}(x')\,dx' \int_{x'}^\infty \frac{\Gamma_{10}^4(x'')\,dx''}{x''^2}, \tag{30}$$

which arise in the method of successive approximations. It is easy to see by integration by parts that the main asymptotic contribution from the "five-Γ" terms is of order

$$\lambda_1^2 \int_1^x \Gamma_{10}^5(x')\,x'^{-1}\,dx' \approx \lambda_1 (1 + 2\lambda_1 \ln x)^{-3/2},$$

i.e., it decreases more rapidly than Γ_{10}. The calculation shows that the irreducible vertices $\sim\alpha^3$, except for the terms with closed nucleon loops, also contain "singly logarithmic" singularities, i.e., decrease like $(1 + 2\lambda_1 \ln x)^{-5/2}$. The same holds also for the higher-order terms. However, the terms with closed nucleon loops of the type



behave asymptotically like a "five-Γ" term, i.e., like $(1 + 2\lambda_1 \ln x)^{-3/2}$. The same asymptotic behavior is shown by the vertex parts containing a "parquet," i.e., a large number of closed nucleon loops. The calculations show that the account of such a "parquet" only changes the numerical value of the coefficient of the "five-Γ" term. This numerical coefficient has been calculated by Dyatlov, Sudakov, and Ter-Martirosyan[9] for all reducible graphs of the "parquet" type.

Let us now write down the asymptotic solution of the equation for the vertex part in the form of the series (27) obtained by the method of successive approximations:

$$\Gamma_0 = \xi^{-1/2}\left(1 + \lambda_1 \frac{a_1 \ln \xi}{\xi} + \lambda_1^2 \frac{a_2 \ln^2 \xi}{\xi^2} + \ldots\right), \quad (31)$$

where $\xi = 1 + 2\lambda_1 \ln x$. It is seen from this that the main asymptotic contribution comes from the solution of the "three-Γ" equation.

Let us now obtain $M'(m)$ from (18). The first term on the right-hand side of (18) leads to an integral of the form

$$-\frac{3}{2}\lambda Z_2^{-1} Z_3^{-1} \int_1^\infty \frac{\Gamma_0^2(x')\,dx'}{x}. \quad (32)$$

We see that this integral diverges like $\ln \ln x$ for $x \to \infty$. If we take the second term on the right-hand side of (18), then the account of the "three-Γ" approximation in $R^{(1)}$ leads to the quantity

$$\lambda^2 \int_1^\infty \frac{\Gamma_0^4(x')\,dx'}{x'} \approx \lambda Z_2^{-1} Z_3^{-1}.$$

The inclusion of the "five-Γ" approximation gives $\lambda^2 Z_2^{-3} Z_3^{-2}$. The next approximations contribute terms $\sim \lambda^3$ and higher. Thus the second term in (18) containing $dR_{\mu\nu}/dp_\sigma$ is a finite quantity and cannot therefore compensate the divergence of the first term. An analogous situation obtains in the calculation of $P'_{\mu\nu}(\mu^2)$ with the help of (21).

For an illustration we note that if we replace one of the vertices Γ_μ in the "three-Γ" equation by γ_μ, then its asymptotic solution becomes equal to $(1 + \lambda_1 \ln x)^{-1}$. Then all integrals in $M'(m)$ and $P'(\mu^2)$ converge, and the solution of the equations (9) leads to the following values of Z_2 and Z_3: $Z_2 = {}^2/_5$ and $Z_3 = {}^1/_5$. Hence, the values of these constants satisfy the Lehmann conditions. However, the exact equations do not contain solutions of the type (24) and (29), because Γ_0 does not decrease rapidly enough. As already noted, we must not, however, forget that this conclusion refers only to solutions in the class of slowly varying functions.

Since we were not successful in finding solutions for G and D which behave asymptotically like the free functions G_0 and D_0, we turn now to the general equations (23) and seek the asymptotic functions G and D in the form[4]

$$G(\hat{p}) = \frac{\hat{p}}{p^2} F(p^2), \quad D(k^2) = \frac{\varphi(k^2)}{k^2}, \quad (33)$$

where F and φ are slowly varying functions. Then the equations for the determination of F and φ take the form

$$\frac{1}{F(p^2)} - \frac{2p^2 F'(p^2)}{F^2(p^2)} = 1 - \left[\gamma_\sigma \frac{dM(\hat{p})}{dp_\sigma} - M'(m)\right],$$
$$\frac{1}{\varphi(k^2)} - \frac{k^2 \varphi'(k^2)}{\varphi^2(k^2)} = 1 - [P'(k^2) - P'(\mu^2)]. \quad (34)$$

We saw already that the "three-Γ" approximation for the vertex part and the first terms on the right-hand sides of the expressions for $M'(\hat{p})$ and $P'(\mu^2)$ give the main asymptotic contribution. We now take these terms as first approximations for the solution of Eqs. (14) and (34) in the form of series for Γ, F, and φ, in analogy to (27). This system of first order equations has the form

$$\Gamma_0(x) = 1 - A\lambda \int_1 \frac{\Gamma_0^3(x') F^2(x') \varphi(x')\,dx'}{x'},$$
$$F^{-1}(x) = 1 - B\lambda \int_1 \frac{\Gamma_0^2(x') F(x') \varphi(x')\,dx'}{x'},$$
$$\varphi^{-1}(x) = 1 - C\lambda \int_1^x \frac{\Gamma_0^2(x') F^2(x')\,dx'}{x'}, \quad (35)$$

where $A = 1$, $B = {}^3/_2$, and $C = 4$. The solution of the system (35) has the form

$$\Gamma_0(x) = (1 - 5\lambda \ln x)^{1/5}, \quad F(x) = (1 - 5\lambda \ln x)^{-3/10}, \quad \varphi(x) = (1 - 5\lambda \ln x)^{-4/5}. \quad (36)$$

The solutions of the system of first-order equations (35) have nonphysical singularities, and the asymptotic behavior of the solutions beyond the singularity (for $5\lambda \ln x \gg 1$) has "anti-Lehmann" character, since in this region $\Gamma_0(x)$ is an increasing function, and F and φ are decreasing. In the calculation of the next approximations the substitution of the solutions (36) in the original integral equations leads to singular integrals. However, the behavior of the asymptotic solutions near the singularity does not coincide exactly with the behavior which follows from the form of the solutions (36). This can be seen already from the fact that in the region $5\lambda \ln x \sim 1$ the terms with the derivatives F' and φ' on the left-hand side of (34), which we omitted earlier, are larger than the terms F^{-1} and φ^{-1}. Therefore, in studying the behavior of the asymptotic solutions near the singularity one must take into account more terms than in Eqs. (35).

Let us assume that the character of the singularities of the asymptotic solutions is such that all singular integrals exist if they are interpreted as principal value integrals.[2)] Then the asymptotic contribution of such integrals for $\lambda \ln x \gg 1$ can be obtained by integration by parts, as in the case of the solutions without singularities (29). Let us show this on the example of single and dou-

[2)]We note that according to the Feynman rules the contour of integration must bypass the singularity, with

$$\ln\left[-\frac{p^2}{m^2(1 - i\varepsilon)}\right] = \ln\left(-\frac{p^2}{m^2}\right) + i\varepsilon, \quad \varepsilon > 0.$$

ble integrals similar to the integrals arising in the next approximation. Consider the integral

$$I_1=\int_1^x \frac{dx'}{1-5\lambda\ln x'}. \quad (37)$$

Integration by parts leads to a solution in the form of a series

$$I_1=\frac{x}{1-5\lambda\ln x}\left(1-\frac{5\lambda}{1-5\lambda\ln x}+\ldots\right)\sim-\frac{x}{5\lambda\ln x}$$
$$\text{for}\quad 5\lambda\ln x\gg 1. \quad (38)$$

The exact value of the integral I_1 leads to the same asymptotic form:

$$I_1=-\frac{e^{1/5\lambda}}{5\lambda}\left[E_i\left(\ln x-\frac{1}{5\lambda}\right)-E_i\left(-\frac{1}{5\lambda}\right)\right]\sim-\frac{x}{5\lambda\ln x}.$$

Let us now consider the integral

$$I_2=\lambda^2\int_1^x\frac{dx'}{x'^2(1-5\lambda\ln x')}\int_1^{x'}\frac{dx''}{(1-5\lambda\ln x'')}.$$

The calculation, which we shall not carry out here because of its complexity, also leads to the coincidence of the results of the exact calculation and of the integration by parts:

$$I_2\sim-\frac{1}{25\ln x}\quad\text{for}\quad 5\lambda\ln x\gg 1.$$

In the region $5\lambda \ln x \sim 1$ the exact calculation leads to the following form:

$$I_2\approx-{}^1/_2(\ln\xi)^2,\qquad \xi=1-5\lambda\ln x.$$

Thus, if our assumption about the character of the singularities is correct, we can use the method of successive approximations to obtain solutions for Γ_0, $F_1 \equiv F^{-1}$, and $\varphi_1 \equiv \varphi^{-1}$ in the form of expansions analogous to (31). The calculations carried out by us up to terms $\sim \alpha^3$ lead to the following result:

$$\Gamma=\xi^{1/5}\left(1+\frac{a_1\lambda\ln\xi}{\xi}+\frac{a_2\lambda^2\ln^2\xi}{\xi^2}+\ldots\right),$$
$$F_1=\xi^{3/10}\left(1+\frac{b_1\lambda\ln\xi}{\xi}+\frac{b_2\lambda^2\ln^2\xi}{\xi^2}+\ldots\right),$$
$$\varphi_1=\xi^{4/5}\left(1+\frac{c_1\lambda\ln\xi}{\xi}+\frac{c_2\lambda^2\ln^2\xi}{\xi^2}+\ldots\right). \quad (39)$$

Here also $\xi = 1 - 5\lambda \ln x$. The factor $\ln \xi$ arises from the vanishing of the determinant of the system of first-order equations. The solutions (39), which are valid in the regions $\lambda \ln x \ll 1$ and $\lambda \ln x \gg 1$, show that in these regions the main contribution comes from the first order solutions (36).

Let us now establish the type of behavior of the solutions in the region $\xi \sim 0$. To this end we first take account of the terms with derivatives on the left-hand sides of Eqs. (35). The corresponding system of equations, after differentiation and introduction of the new variable $\xi = 1 - 5\lambda \ln x$, takes the form

$$\Gamma'=\frac{1}{5}\frac{\Gamma^3}{F_1^2\varphi_1},\qquad F_1'-10\lambda F_1''=\frac{3}{10}\frac{\Gamma^2}{F_1\varphi_1},$$
$$\varphi_1'-5\lambda\varphi_1''=\frac{4}{5}\frac{\Gamma^2}{F_1^2}. \quad (40)$$

The solution of this system of equations for small ξ will be sought in the form of expansions in powers of ξ:

$$\Gamma=A_1\xi+B_1\xi^2+C_1\xi^3+\ldots,$$
$$F_1=A_2\xi+B_2\xi^2+\ldots,\qquad \varphi_1=A_3\xi+B_3\xi^2+\ldots \quad (41)$$

Substituting these expansions in (40), we obtain systems of equations for the unknown coefficients. For example, in first approximation these equations have the form

$$A_1=\frac{1}{5}\frac{A_1^3}{A_2^2A_3},\qquad A_2-10\lambda B_2=\frac{3}{10}\frac{A_1^2}{A_2A_3},$$
$$A_3-10\lambda B_3=\frac{4}{5}\frac{A_1^2}{A_2}. \quad (42)$$

Using these equations we can express the coefficients A_1, B_2, and B_3 in terms of A_2 and A_3. These also determine the remaining coefficients of the series (41). The coefficients A_2 and A_3 must be determined from the conditions of joining the solutions (41), which are valid in the region $\xi \ll \lambda$, to the solutions (36), which are valid in the region $\xi \gg \lambda$.

Substitution of the solutions of the type (41) in the original equations (14) and (34) shows that in the terms up to order α^3 all singular integrals converge in the sense of the principal value, but a relative increase of the higher approximations like $\xi \ln \xi$, $\xi \ln^2 \xi$, etc., is observed for small ξ. The account of these relatively increasing terms leads to the following behavior of the vertex part for small ξ: $\Gamma_0 \sim \xi(\ln \xi)^{-1/2}$. Examples for such terms are the "small" corrections of the form

$$\lambda\int_1^x\frac{\Gamma_0^2(x)\Gamma_0(x')\,dx'}{xF_1(x)F_1(x')\varphi_1(x')}$$

etc. The remaining terms give a relatively small contribution [for example, $\xi(\ln \xi)^{-3/2}$]. The behavior of the functions F_1 and φ_1 remains the previous one: $F_1 \sim \xi$, $\varphi_1 \sim \xi$.

This estimate of the behavior of the solutions for small ξ indicates that the solutions (39) can be used for not too small values of $\xi(\xi \gg \lambda)$. Clearly, the inclusion of the higher approximations does not alter the "anti-Lehmann" character of the solutions for $\lambda \ln x \gg 1$. We recall in this connection that as before, this conclusion refers to the class of slowly varying solutions.

It is interesting to call attention to the following circumstance: if the coefficients A, B, and C in (35) satisfy the inequality $2|A| \geq 2|B| + |C|$, then the solutions become of the Lehmann type. For example,

$$\Gamma = [1 + (2|A| - 2|B| - |C|)\lambda \ln x]^{-|A|/[2|A|-2|B|-|C|]}$$

(for $2|A| = 2|B| + |C|$ the solutions are of the power type). The inclusion of the higher approximations does not change the Lehmann behavior of the functions Γ, F, and φ. In our opinion, this circumstance is to some extent an indication that the original Hamiltonian has not been chosen suitably.

4. ASYMPTOTIC BEHAVIOR OF THE GREEN'S FUNCTIONS IN ELECTRODYNAMICS

The investigation of the asymptotic behavior of the Green's functions in electrodynamics is analogous to that in meson theory. We seek the asymptotic form of the functions G and D in a form analogous to (33):

$$G(\hat{p}) = \frac{\hat{p}}{p^2} F(p^2), \quad D_{\mu\nu}(k^2) = \frac{\delta_{\mu\nu}}{k^2} \varphi(k^2). \quad (43)$$

We choose $D_{\mu\nu}$ of the form (43) and not in the transverse form $\sim (\delta_{\mu\nu} - k_\mu k_\nu/k^2)$, in order to preserve the spectrum of positronium[10] corresponding to the poles of the photon Green's function. The system of first-order equations in electrodynamics has the form (35) with the coefficients A = 1, B = 1, and C = 4. The solution of this system is

$$\Gamma_0(x) = (1 - 4\lambda \ln x)^{1/4}, \quad F(x) = (1 - 4\lambda \ln x)^{-1/4},$$
$$\varphi(x) = (1 - 4\lambda \ln x)^{-1}. \quad (44)$$

The behavior of the solutions for small $\xi = 1 - 4\lambda \ln x$ is analogous to the corresponding behavior in meson theory. The inclusion of the higher approximations leads to solutions in the form of expansions analogous to (39):

$$\Gamma_0 = \xi^{1/4}\left(1 + \frac{a_1\lambda \ln \xi}{\xi} + \ldots\right), F_1 = \xi^{1/4}\left(1 + \frac{b_1\lambda \ln \xi}{\xi} + \ldots\right),$$
$$\varphi_1 = \xi\left(1 + \frac{c_1\lambda \ln \xi}{\xi} + \ldots\right). \quad (45)$$

This asymptotic behavior leads us to the conclusion that, as in meson theory, the Green's functions of electrodynamics are of "anti-Lehmann" type for $\lambda \ln x \gg 1$.

We note in conclusion that if we replace λ by $-\lambda$, i.e., change the interactions between like and unlike charges to the opposite, then the solutions in the form of the series (45) can be used for arbitrary values of x, and the solutions become of the "anti-Lehmann" type for all x.

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