

*TWO-DIMENSIONAL FLOW OF AN IDEALLY CONDUCTING GAS IN THE VICINITY OF  
THE ZERO LINE OF A MAGNETIC FIELD*

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The motion of a conducting gas in the vicinity of the zero line of a magnetic field is considered within the framework of the equations of magnetohydrodynamics. A class of exact solutions of the nonstationary two-dimensional problem is obtained. The time dependence is such that after a definite time interval a singularity sets in, corresponding to unlimited growth of the velocity, of the magnetic field intensity, of the gas density, and of the electric current density. The current density increases more rapidly than the gas concentration; this corresponds to realization of the conditions of dynamic dissipation of the magnetic field. The solution obtained for the immediate vicinity of the zero line can be regarded as the principal term in the general solution of the rigorous boundary-value problem.

## 1. INTRODUCTION

**I**N connection with the problem of collisionless dissipation of magnetic energy and the acceleration of fast particles in a plasma, it is of interest to investigate the behavior of an ideally conducting gas in the vicinity of the zero line of a magnetic field. Special attention is paid presently to this question in astrophysics (see<sup>[1-3]</sup> and the literature cited there). Dungey<sup>[1]</sup> has concluded, on the basis of a qualitative investigation of the equilibrium state of a plasma near the zero-field line, that such a state is unstable. The exact particular solution of the equations of magnetohydrodynamics for an incompressible liquid was subsequently obtained by Chapman and Kendall.<sup>[2]</sup> This solution has a perfectly defined character. Ultimately, a cumulative effect is developed and arbitrarily large energy densities are attained. With this, a fixed mass of the gas near the zero line receives energy from the outside in the form of an electromagnetic-field energy flux.

It must be noted that the interpretation of the obtained results in terms of the instability of the equilibrium state of the plasma does not seem very appropriate. As shown in<sup>[3]</sup>, under definite boundary condition there develops in the plasma a unique motion which also has essentially a cumulative character. These boundary conditions and the behavior of the plasma sufficiently far from the zero line were investigated in<sup>[3]</sup>. However, the method used there is not suitable for the immediate vicinity

of the zero line. Until appropriate numerical calculations that permit inclusion of the immediate vicinity of the zero line into consideration become available, it is of interest to find particular exact solutions. In this paper we find a class of such solutions within the framework of the magnetohydrodynamics of an ideally conducting gas. These solutions are similar to those obtained in<sup>[2]</sup> for an incompressible liquid, and were pointed out to us by M. A. Leontovich. Allowance for compressibility leads to a greater variety of initial conditions. In addition, in this case a finite mass of gas becomes compressed in a region of limited dimensions, whereas in the case considered in<sup>[2]</sup> the dimension of this region was unbounded in one direction.

A characteristic property of the solutions obtained in the present work is that a singularity corresponding to an unbounded growth of the magnetic field intensity, of the electric current density, and of the gas concentration is attained after a finite time interval. With this, the ratio of the current density to the gas-particle concentration also increases without limit, corresponding to realization of the conditions discussed in<sup>[3]</sup> for dynamic dissipation of magnetic energy.

## 2. FORMULATION OF THE PROBLEM AND ITS REDUCTION TO A SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS.

We write down the system equations of magnetohydrodynamics for planar flow (in the  $x, y$  plane)

of a non-viscous infinitely-conducting plasma (see Note added in proof):

$$-\frac{\partial}{\partial x} \frac{dA}{dt} = 0, \quad \frac{\partial}{\partial y} \frac{dA}{dt} = 0, \quad (1)$$

$$\begin{aligned} \rho \frac{dv_x}{dt} &= -\nabla_x p - \frac{1}{4\pi} \frac{\partial A}{\partial x} \Delta A, \\ \rho \frac{dv_y}{dt} &= -\nabla_y p - \frac{1}{4\pi} \frac{\partial A}{\partial y} \Delta A, \end{aligned} \quad (2)$$

$$\frac{d\rho}{dt} + \rho \left( \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} \right) = 0. \quad (3)$$

Here A is the z-component of the vector potential, and the components of the magnetic field are

$$B_x = \frac{\partial A}{\partial y}, \quad B_y = -\frac{\partial A}{\partial x}, \quad B_z = 0. \quad (4)$$

The operators d/dt and Δ in Eqs. (1)–(3) are defined in the usual manner:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

We assume that the pressure p is a function of the density only: p = p(ρ). This condition is satisfied by any polytropic equation of state, and in particular by isentropic and isothermal plasma motions. Thus, the results that follow can be extended to include also a plasma that loses energy by radiation from its volume.

It will be shown later that, for the class of solutions of interest to us, the plasma density, and consequently (by virtue of the foregoing assumption) the plasma pressure, depends only on the time. Therefore the spatial derivatives of the pressure in (2) vanish, and will no longer be written out.

We seek a solution of the system (1)–(3) under the following initial conditions:

1) The density of the matter is constant:

$$\rho(x, y, 0) = \rho_0, \quad (5)$$

2) The magnetic field is hyperbolic:

$$A(x, y, 0) = A_0(x^2 - y^2). \quad (6)$$

Such a vector potential corresponds to the vicinity of the zero line of a potential magnetic field (current density j<sub>z</sub> = 0). Accordingly, there are no ponderomotive forces in the initial state.

3) The initial velocity depends linearly on the coordinates, so that there is no flow of gas across the coordinate axes:

$$v_x(x, y, 0) = Ux, \quad v_y(x, y, 0) = Vy. \quad (7)$$

Thus, the initial conditions are defined by the four independent quantities ρ<sub>0</sub>, A<sub>0</sub>, U, and V. We can construct from them three independent combinations

with the dimension of time:

$$U, \quad V, \quad t_0 = (\pi\rho_0)^{1/2}/|A_0| \quad (8)$$

and not even one combination with the dimension of length. We introduce new variables (with dimension equal to a certain power of the length):

$$\tau = t/t_0, \quad u_x = t_0 v_x, \quad u_y = t_0 v_y, \quad \sigma = \rho/\rho_0, \quad a = A/A_0. \quad (9)$$

In terms of these variables, Eqs. (1)–(3) take the form

$$\frac{\partial}{\partial x} \frac{da}{d\tau} = 0, \quad \frac{\partial}{\partial y} \frac{da}{d\tau} = 0, \quad (10)$$

$$\sigma \frac{du_x}{d\tau} = -\frac{1}{4} \frac{\partial a}{\partial x} \Delta a, \quad \sigma \frac{du_y}{d\tau} = -\frac{1}{4} \frac{\partial a}{\partial y} \Delta a, \quad (11)$$

$$\frac{d\sigma}{d\tau} + \sigma \left( \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} \right) = 0. \quad (12)$$

The initial conditions (5)–(7) then become

$$\begin{aligned} \sigma(x, y, 0) &= 1, \quad a(x, y, 0) = x^2 - y^2, \\ u_x(x, y, 0) &= \gamma_0 x, \quad u_y(x, y, 0) = \delta_0 y, \end{aligned} \quad (13)$$

where

$$\gamma_0 = U \frac{(\pi\rho_0)^{1/2}}{|A_0|}, \quad \delta_0 = V \frac{(\pi\rho_0)^{1/2}}{|A_0|}. \quad (14)$$

The problem is thus completely determined by the two dimensionless parameters of (14). As to the choice of the unit of length, Eqs. (10)–(13) impose no limitations whatever. The unit of length can be chosen arbitrarily and the coordinates x and y, together with all the variables in (9), can be chosen dimensionless.

It is easy to verify that our problem has the following solution:

$$\begin{aligned} a(x, y, \tau) &= \alpha(\tau)x^2 - \beta(\tau)y^2, \quad \sigma(x, y, \tau) = \sigma(\tau), \\ u_x(x, y, \tau) &= \gamma(\tau)x, \quad u_y(x, y, \tau) = \delta(\tau)y. \end{aligned} \quad (15)$$

In fact, when (15) is substituted in (10)–(13), the latter reduce to a system of ordinary differential equations for the functions α, β, γ, and δ (the dot denotes differentiation with respect to τ):

$$\begin{aligned} \dot{\alpha} + 2\alpha\gamma &= 0, \quad \dot{\beta} + 2\beta\delta = 0, \quad \sigma(\dot{\gamma} + \gamma^2) = \alpha(\beta - \alpha), \\ \sigma + \sigma(\gamma + \delta) &= 0, \quad \sigma(\dot{\delta} + \delta^2) = \beta(\alpha - \beta), \end{aligned} \quad (16)$$

with initial conditions

$$\alpha(0) = 1, \quad \beta(0) = 1, \quad \gamma(0) = \gamma_0, \quad \delta(0) = \delta_0, \quad \sigma(0) = 1 \quad (17)$$

The system (16) has one integral. Namely, eliminating the functions γ and δ from the first two and last equations of this system, we get

$$\frac{\dot{\alpha}}{\alpha} + \frac{\dot{\beta}}{\beta} - 2\frac{\dot{\sigma}}{\sigma} = 0. \quad (18)$$

From this, using the initial conditions (17), we get

$$\sigma = (\alpha\beta)^{1/2}. \quad (19)$$

In obtaining the integral (19) we have assumed that  $\alpha$ ,  $\beta$  and  $\sigma$  are not equal to zero. Since the initial values of these quantities are positive, the subsequent results will pertain to a time interval  $\tau$  for which these quantities remain positive.

The foregoing method of obtaining the particular solution (15) is unsatisfactory from the point of view that it is similar to an accidental guess. In particular, it raises the natural question whether there are no other solutions of this type. To answer such questions we can obtain the solution (15) by another more consistent method, by using dimensionality theory<sup>[4]</sup>. In our problem there are only two parameters,  $\rho_0$  and  $A_0$ , that enter into the initial conditions with independent dimensionalities. Under these circumstances we can construct from dimensionality considerations a self-similar solution<sup>[4]</sup> which, using the established terminology, belongs to the first type<sup>[5]</sup> <sup>1)</sup>. It is more general than the solution (15), because the vector potential of the magnetic field is represented in it by a general quadratic form, and the velocity components are represented by general linear expressions in terms of the coordinates  $x$  and  $y$ . But the role of the self-similar variable is also played by the dimensionless time  $\tau$ . If we rotate the coordinate frame through an arbitrary angle  $\varphi$  in the  $x, y$  plane, then the solution (15) is obviously transformed in precisely such a way that the velocity components become dependent on both coordinates, and the vector potential becomes dependent on the product  $xy$  besides the squares of the coordinates  $x^2$  and  $y^2$ . It is easy to show that such a transformation of the expression is contained in the self-similar solution, but does not constitute all of it. For example, the self-similar solution contains a rather simple solution in which all the dependences are obtained in finite form. According to this solution, the magnetic field force lines are concentric circles, and the entire plasma rotates with a constant angular velocity around the axis of the system. The densities of the axial current and of the plasma are constant. At a certain ratio of the angular velocity of rotation to the magnitude of the magnetic field, the centrifugal

force is balanced at all points by the ponderomotive force.

We shall not present here the self-similar solution, a particular case of which is the solution (15), since, first, its additional physical applications are not clear, and, second, it is very cumbersome. From dimensionality considerations we can also conclude that in the compressible case under consideration it would be impossible to take into consideration, within the framework of the self-similar solutions, the gradient of the gas pressure. In fact such a problem would contain a third parameter with independent dimension—the speed of sound—and in this case there can be no self-similar solution<sup>[4,5]</sup>. Therefore the solutions for an incompressible liquid<sup>[2]</sup> and solutions of the type (15) for the compressible case cannot be combined into one more general self-similar solution, and must be considered independently.

### 3. QUALITATIVE ANALYSIS OF THE ORDINARY DIFFERENTIAL EQUATIONS OF THE PROBLEM

To investigate the behavior of the solutions of (16), it is convenient to introduce new functions  $\xi(\tau)$  and  $\eta(\tau)$  such that

$$\alpha = 1/\xi^2, \quad \beta = 1/\eta^2. \quad (20)$$

Without loss of generality, we shall assume that these functions are positive. From the first two equations of the system (16) and from (19) we get

$$\dot{\gamma} = \dot{\xi}/\xi, \quad \dot{\delta} = \dot{\eta}/\eta, \quad \sigma = (\xi\eta)^{-1}. \quad (21)$$

The system (16) then reduces to two second-order differential equations for  $\xi(\tau)$  and  $\eta(\tau)$ :

$$\ddot{\xi} = -\eta \left( \frac{1}{\xi^2} - \frac{1}{\eta^2} \right), \quad \ddot{\eta} = \xi \left( \frac{1}{\xi^2} - \frac{1}{\eta^2} \right), \quad (22)$$

with initial conditions

$$\xi(0) = 1, \quad \dot{\xi}(0) = \gamma_0, \quad \eta(0) = 1, \quad \eta(0) = \delta_0. \quad (23)$$

Equations (22) are invariant to the replacement of  $\eta$  by  $\xi$ :

$$\begin{aligned} \xi &\rightarrow \eta, & \eta &\rightarrow \xi, \\ \alpha &\rightarrow \beta, & \beta &\rightarrow \alpha, & \gamma &\rightarrow \delta, & \delta &\rightarrow \gamma. \end{aligned} \quad (24)$$

Such a replacement simply corresponds to rotation of the axes of the spatial coordinate system through an angle  $\pi/2$  and to reflection. It is therefore sufficient to consider only the initial conditions with  $\gamma_0 \geq \delta_0$ .

Let  $\gamma_0 > \delta_0$ . Then for small time intervals  $\tau > 0$  we have  $\dot{\xi} > \dot{\eta}$  and  $\xi > \eta$ , and by virtue of (22) we

<sup>1)</sup>The self-similarity exponent for it is determined from dimensionality considerations, and not during the course of solving the ordinary differential equations as for the second type of self-similar solution.

have  $\ddot{\xi} > 0$  and  $\ddot{\eta} < 0$ . It follows therefore that  $\xi > \eta$  always, and the second derivatives cannot change the sign until the singular point  $\tau = \tau_0$ , at which  $\eta(\tau_0)$  vanishes, is reached, with  $\xi(\tau) > 0$ . Since  $\ddot{\eta}$  is always rigorously smaller than zero, such a singular point is reached after a finite time  $\tau_0$ .

The instant  $\tau = \tau_0$  when the singularity is attained can be determined only by numerical calculation (see Sec. 4), but the character of the behavior of the solution near the singularity can be determined analytically. Namely, recognizing that when  $\tau \rightarrow \tau_0$  the quantity  $\xi(\tau)$  tends to a finite value  $\xi(\tau_0)$ , and  $\eta(\tau) \rightarrow 0$ , we retain in (22) only the principal terms:

$$\ddot{\eta} = -\xi/\eta^2, \quad \ddot{\xi} = 1/\eta. \tag{25}$$

The solution of these equations, for the region  $\tau < \tau_0$  of interest to us is

$$\eta(\tau) = (9/2\xi(\tau_0))^{1/3}(\tau_0 - \tau)^{2/3} + \dots, \quad \xi(\tau) = \xi(\tau_0) + \dots, \tag{26}$$

where the terms of higher order of smallness in  $\tau_0 - \tau$  have been omitted. Therefore, returning to the variables (20) and (21), we obtain the asymptotic behavior of the unknown functions as  $\tau \rightarrow \tau_0$ :

$$\begin{aligned} \alpha &\rightarrow \alpha_1, & \beta &\rightarrow \left(\frac{2}{9}\right)^{2/3} \frac{\alpha_1^{1/3}}{(\tau_0 - \tau)^{1/3}}, \\ \gamma &\rightarrow \gamma_1, & \delta &\rightarrow -\frac{2}{3(\tau_0 - \tau)}, & \sigma &\rightarrow \left(\frac{2}{9}\right)^{1/3} \left(\frac{\alpha_1}{\tau_0 - \tau}\right)^{2/3}. \end{aligned} \tag{27}$$

Here the quantities  $\tau_0$ ,  $\alpha_1 = \alpha(\tau_0)$ , and  $\gamma_1 = \gamma(\tau_0)$  depend on the initial conditions of the problem and their values should be determined by integrating the complete system of equations (16) and (17).

We shall discuss the solutions (27) later, together with the results of the numerical integration. We turn now to the case  $\gamma_0 = \delta_0$ .

In this case we have  $\dot{\xi} = \dot{\eta}$  and  $\dot{\xi} = \dot{\eta}$  when  $\tau = 0$ , and by virtue of (22) we have  $\ddot{\xi} \equiv 0$  and  $\ddot{\eta} \equiv 0$ . Hence, using the initial conditions (23), we obtain the singular solution

$$\begin{aligned} \xi = \eta = 1 + \gamma_0\tau, & \quad \alpha = \beta = \frac{1}{(1 + \gamma_0\tau)^2}, \\ \gamma = \delta = \frac{\gamma_0}{1 + \gamma_0\tau}, & \quad \sigma = \frac{1}{(1 + \gamma_0\tau)^2}. \end{aligned} \tag{28}$$

The solution (28) has a simple physical meaning. It corresponds to unlimited cylindrically-symmetrical expansion (when  $\gamma_0 > 0$ ) or contraction (when  $\gamma_0 < 0$ ) of the plasma without excitation of an electric current in the plasma ( $j_z \propto \alpha - \beta \equiv 0$ ).

The solution (28) delineates in natural fashion

two regions of solutions with opposite current directions: 1)  $\alpha - \beta < 0$  when  $\gamma_0 > \delta_0$  and 2)  $\alpha - \beta > 0$  when  $\gamma_0 < \delta_0$ ; the latter case reduces formally to the former with the aid of the transformation (24). When  $\gamma_0 = \delta_0 > 0$  the solution (28) is unique in the sense that the plasma density vanishes asymptotically as  $\tau \rightarrow \infty$ . For all other values of  $\gamma_0$  and  $\delta_0$  the density becomes infinite after a finite time  $\tau_0$ . For the solution (28) with  $\gamma_0 = \delta_0 < 0$ , this instant is

$$\tau_0 = 1/|\gamma_0|. \tag{29}$$

#### 4. NUMERICAL SOLUTION OF TIME-DEPENDENT EQUATIONS

The numerical solution of the problem entails no difficulty. The main results of the numerical calculation are contained in the table. With the aid of this table and the asymptotic relations (27) we can describe the motion of the gas and the variation of the magnetic field near the instant of singularity.

For each pair of the defining parameters  $\gamma_0$  and  $\delta_0$  the table lists the instant of the singularity  $\tau_0$  and the value of  $\alpha_1$ . When  $\gamma_0 > \delta_0$  the motion near the singularity is determined by (27), and in this case  $\alpha \rightarrow \alpha_1$  and  $\beta \rightarrow \infty$ . For the values  $\gamma_0 < \delta_0$  the asymptotic behavior of the solutions is determined by the equations (27) transformed in accord with (24); in this case we have near the singularity  $\alpha \rightarrow \infty$  and  $\beta \rightarrow \alpha_1$ . Thus, the quantity that increases without limit is the one ( $\alpha$  or  $\beta$ ) corresponding to the algebraically smaller initial velocity  $\gamma_0$  or  $\delta_0$ .

For convenience, the table lists not  $\delta_0$  but the ratio  $\delta_0/\gamma_0$ , and therefore the condition  $\gamma_0 > \delta_0$  corresponds to  $\delta_0/\gamma_0 > 1$  if  $\gamma_0 < 0$  (upper right part of the table) and  $\delta_0/\gamma_0 < 1$  if  $\gamma_0 > 0$  (lower left part of the table).

For the critical case  $\gamma_0 = \delta_0$  the table includes the results of the analytical solution (28). When  $\gamma_0 = \delta_0 < 0$  the time of the singularity is determined from (29), whereas when  $\gamma_0 = \delta_0 > 0$  (lower part of the table) we have  $\tau_0 = \infty$ . The value of  $\alpha_1$  for the critical case is taken to be the common value of  $\alpha = \beta$  when  $\tau \rightarrow \tau_0$ .

The eleven variants marked in the table with an asterisk were not calculated separately, but were determined on the basis of the relations of the invariant transformation (24). The same relations were used to check on the calculation accuracy. In the last two lines, for  $\delta_0/\gamma_0$  equal to 0.9 and 1.11, the value of  $\tau_0$  was not determined in the numerical calculation, which extended only to  $\tau = 10.0$ .

We note the monotonic dependence of  $\tau_0$  and  $\alpha_1$  on the ratio  $\delta_0/\gamma_0$  on the two sides of the critical

Values of  $\tau_0$  (upper line) and  $\alpha_1$  (lower line)

$\gamma_0$	$\delta_0/\gamma_0$									
	-2.0	-1.0	-0.5	0	0.5	0.9	1.0	1.11	1.5	2.0
-2.0	0.360	0.370*	0.390*	0.400	0.420*	0.460	0.500	0.420	0.300	0.230
	0.159	0.370	0.466	0.873	2.44	22.30	$\infty$	26.42	5.44	3.17
-1.0	0.560	0.590*	0.620	0.650	0.710	0.820	1.000	0.760	0.550	0.420
	0.196	0.335	0.472	0.769	1.63	7.51	$\infty$	8.92	3.48	2.44
-0.5	0.800*	0.870	0.910	0.970	1.08	1.31	2.000	1.23	0.900	0.710*
	0.244	0.355	0.464	0.652	1.07	2.74	$\infty$	3.16	1.91	1.63
0.5	0.620*	0.870*	1.08	1.45	2.25	5.01	$\infty$	5.35	3.11	2.41*
	0.472	0.355	0.289	0.206	0.117	0.0354	0	0.0274	0.0464	0.0505
1.0	0.390	0.590	0.800	1.22	2.41	9.55	$\infty$	12.05	4.67	3.20
	0.466	0.335	0.244	0.144	0.0505	0.00408	0	0.00223	0.00880	0.0119
1.5	0.280	0.460	0.650	1.10	2.73	>10	$\infty$	>10	7.94	4.66
	0.469	0.313	0.218	0.109	0.0241	—	0	—	0.00166	0.00307
2.0	0.220	0.370	0.560*	1.02	3.20*	>10	$\infty$	>10	10.08	7.34
	0.465	0.370	0.196	0.0869	0.0119	—	0	—	0.000723	0.000772

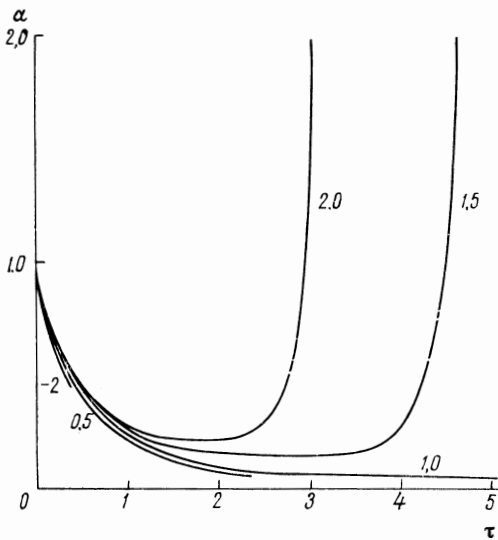


FIG. 1.

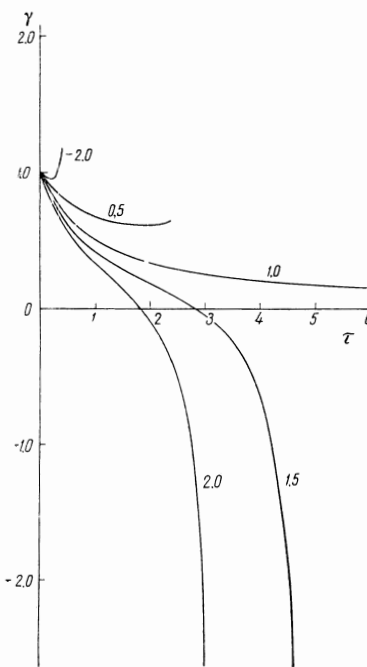


FIG. 3.

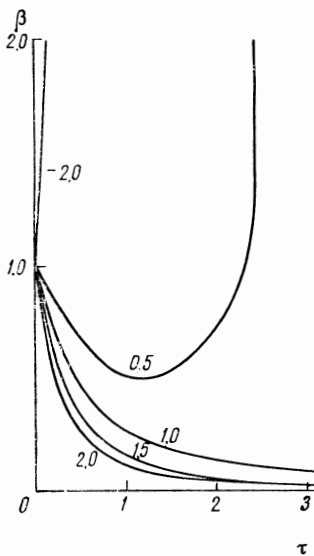


FIG. 2.

value  $\delta_0/\gamma_0 = 1$ . At a fixed value of  $\delta_0/\gamma_0 > 0$ , the dependence of  $\tau_0$  and  $\alpha_1$  on  $\gamma_0$  also has a monotonic character, whereas when  $\delta_0/\gamma_0 \leq 0$  the monotonic behavior is lost:  $\tau_0$  reaches a maximum near  $\gamma_0 \approx 0$ , and the dependence of  $\alpha_1$  is even more complicated.

To illustrate the time variation of all the functions, Figs. 1-5 show plots of  $\alpha(\tau)$ ,  $\beta(\tau)$ ,  $\gamma(\tau)$ ,  $\delta(\tau)$ , and  $\sigma(\tau)$  for  $\gamma_0 = 1$  and for several different values of the parameter  $\delta_0$ . The main properties of these plots are clear from the foregoing qualitative analysis of the equations. We call attention here to the change in the density  $\sigma(\tau)$ . When  $\gamma_0 = 1$  and  $\delta_0 > -1$  the density decreases before starting to increase

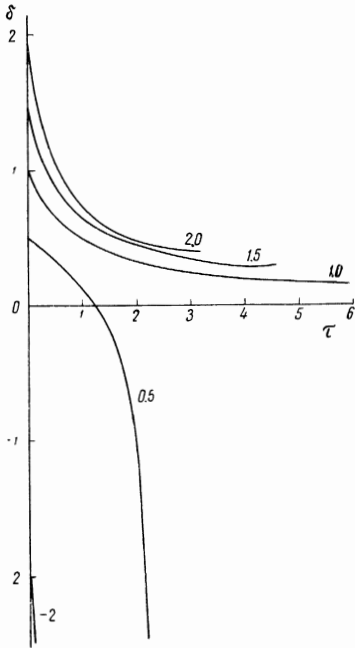


FIG. 4.

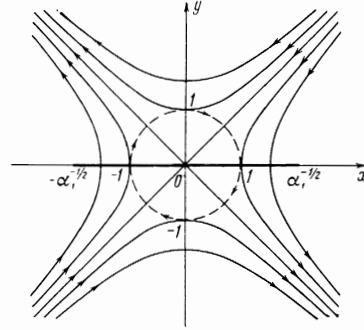


FIG. 6.

For the gas motion described by the obtained solution, an arbitrary function in the form  $\Phi(\alpha(\tau)x^2, \beta(\tau)y^2) = \text{const}$  determines the Lagrangian line on which the same gas particles are located during the course of the entire motion. Namely, as follows from (15) and (16), the total derivative is

$$\frac{d\Phi}{d\tau} = \frac{\partial\Phi}{\partial\tau} + u_x \frac{\partial\Phi}{\partial x} + u_y \frac{\partial\Phi}{\partial y} \equiv 0.$$

In other words, the gas particles which at the initial instant  $\tau = 0$  were located on the line  $\Phi(\alpha(0)x^2, \beta(0)y^2) = C$ , will be located at any subsequent instant  $\tau$  on the line  $\Phi(\alpha(\tau)x^2, \beta(\tau)y^2) = C$ , where  $C$  is the same constant.<sup>2)</sup> In particular, the Lagrange lines are obviously the magnetic force lines  $\alpha(\tau)x^2 - \beta(\tau)y^2 = \text{const}$  (lines of constant values of the vector-potential  $A(x, y, \tau)$ ).

Let us consider the fraction of the gas that is located during the initial instant  $\tau = 0$  within a circle of radius equal to unity. The corresponding Lagrange line is the line  $\alpha(0)x^2 + \beta(0)y^2 = 1$ . Therefore, at any subsequent instant of time this gas will be located inside the ellipse

$$\alpha(\tau)x^2 + \beta(\tau)y^2 = \frac{x^2}{\xi^2(\tau)} + \frac{y^2}{\eta^2(\tau)} = 1, \quad (30)$$

where the functions  $\xi(\tau)$  and  $\eta(\tau)$  introduced above have the simple meaning of semi-axes of this deforming ellipse. We recall that the choice of the unit length, and consequently, of the radius of the initial circle, is arbitrary. Therefore Eq. (30) describes the behavior of an arbitrary gas cylinder with initial circular cross section and with an axis coinciding with the zero line.

As follows from the obtained solution, the semi-axis whose direction corresponds to a smaller initial velocity vanishes at the instant  $\tau_0$ ; at the same time, the second semi-axis remains different from

<sup>2)</sup>It must be stipulated that the derivatives of  $\Phi$  with respect to both arguments must exist in the entire domain.

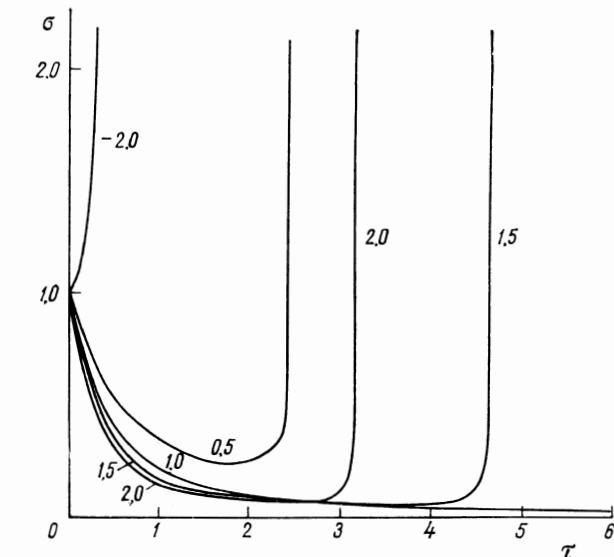


FIG. 5.

without limit. It is easy to verify with the aid of (21) that in the general case this occurs when  $\gamma_0 + \delta_0 > 0$ . The density minimum is attained at instants of time close to  $\tau_0$ , and the closer we are to the critical case  $\delta_0/\gamma_0 = 1$  the deeper the minimum.

5. SOME PROPERTIES OF THE SOLUTION

We shall assume for concreteness that  $A_0 > 0$ . Then the force lines of the magnetic field corresponding to the potential (6) have the form shown in Fig. 6.

zero and bounded. Thus, any initial circle is transformed at the instant  $\tau_0$  into a segment of the x-axis (if  $\gamma_0 > \delta$ ) or y axis (if  $\gamma_0 < \delta_0$ ), with ends at the points  $\pm \alpha_1^{-1/2}$  (see Fig. 6). It is seen from the table (see Sec. 4) that the length of this segment can be smaller as well as larger than the diameter of the initial circle. The area bounded by the Lagrange curve (30) is equal to  $\pi\xi\eta$  and tends to zero as  $\tau \rightarrow \tau_0$ , and the gas density  $\sigma = (\xi\eta)^{-1}$ , as already noted, becomes infinite.

Let us examine the behavior of the magnetic field intensity  $\mathbf{B} = 2A_0\{-\beta y, -\alpha x\}$  (see (4), (9), and (15)). As  $\eta \rightarrow 0$  we have on the line (30)  $y = \pm \eta$ , with the exception of the nearest vicinity of the points  $x = \pm \xi$ . Therefore in the limit as  $\tau \rightarrow \tau_0$  the magnetic field is equal to

$$\mathbf{B} = 2A_0\{\mp 1/\eta, -x/\xi^2\}, \quad (31)$$

where the minus and plus signs correspond to the regions  $y > 0$  and  $y < 0$ , respectively. As follows from (31), when  $\tau \rightarrow \tau_0$  the magnetic field is always tangent to the x-axis segment into which the ellipse degenerates, increases in magnitude without limit, and experiences a discontinuity on the x axis:

$$B_x(y = +\eta) - B_x(y = -\eta) = -4A_0/\eta \rightarrow -\infty.$$

The appearance of the discontinuity in the magnetic-field component transverse to the compression corresponds to an unbounded increase in the density of the electric current. Indeed, calculating the current density

$$j_z = \frac{c}{4\pi} (\text{rot } \mathbf{B})_z = -\frac{c}{4\pi} \Delta A,$$

we get from (9), (15), and (20):

$$j_z = \frac{cA_0}{2\pi} (\beta - \alpha) = \frac{cA_0}{2\pi} \left( \frac{1}{\eta^2} - \frac{1}{\xi^2} \right). \quad (32)$$

From this and from (26) it follows that when  $\tau \rightarrow \tau_0$  the current increases like

$$j_z \propto (\tau_0 - \tau)^{-1/2}. \quad (33)$$

An essential circumstance here is that the current density increases more rapidly than the gas density and accordingly the particle density is  $n \propto \sigma \propto (\tau_0 - \tau)^{-2/3}$ . The specific (per particle) current density is

$$\frac{j_z}{n} = \frac{cA_0}{2\pi n_0} \left( \frac{\xi}{\eta} - \frac{\eta}{\xi} \right) \rightarrow \frac{cA_0}{2\pi n_0} \left( \frac{2}{9a_1} \right)^{1/2} (\tau_0 - \tau)^{-2/3}, \quad (34)$$

where  $n_0$  is the initial gas concentration.

Within the framework of the obtained solution, the ratio  $j_z/n$  tends to infinity when  $\tau \rightarrow \tau_0$ . Actually, however, when a sufficiently high current

density is attained, new effects arise, not accounted for by magnetohydrodynamics. First, when

$$j \gtrsim j_{\text{cr}}^{(1)} = \sigma' E_{\text{cr}}^{(1)},$$

(where  $\sigma'$  is the plasma conductivity and  $E_{\text{cr}}^{(1)}$  the critical particle-runaway field) an intense runaway of electrons begins, and causes current instability in the plasma. This process leads to a decrease in the effective conductivity of the plasma, but apparently still does not impose any essential limitations on the applicability of magnetohydrodynamics to the description of the macroscopic plasma motions (see Note added in proof).

However, as the current is increased further, and the value

$$j \gtrsim j_{\text{cr}}^{(2)} = \kappa n e c$$

is attained, ( $c$  is the speed of light and  $\kappa$  a numerical coefficient to be determined later) direct acceleration of the particles by the strong electric field sets in (see<sup>[3]</sup> for details). In this case the magnetohydrodynamic description of the processes in the plasma is no longer suitable. It is important to emphasize, however, that the magnetohydrodynamic flow considered above leads in the vicinity of the zero line of the magnetic field to realization of the conditions of dynamic dissipation of the magnetic field, which were discussed in<sup>[3]</sup>.

## 6. REGION OF APPLICABILITY OF THE SOLUTION

It is quite difficult to determine the exact conditions under which the derived plasma motion occurs. It is actually necessary for this purpose to solve the essentially more complicated problem of the establishment of an assumed velocity profile for a sufficiently broad class of physically acceptable boundary and initial conditions. In fact, the most difficult question is that of the realization of the assumed initial linear velocity distribution.

Such a distribution could be realized in practice as the principal part of a small perturbation of an initial stationary state, a part which does not change the position of the zero point ( $|\gamma_0| \ll 1$ ,  $|\delta_0| \ll 1$ ). One might therefore assume, as was done by Chapman and Kendall<sup>[2]</sup>, that the entire process has the same character as ordinary instability.

It can be shown by linearizing (16) that a solution increasing linearly in time exists. If the initial perturbations of the magnetic field are equal to zero and  $\gamma_0$  and  $\delta_0$  are sufficiently small, then the linearized Eqs. (16) have the following solution:



$$\begin{aligned}\alpha(\tau) &= 1/4(\delta_0 - \gamma_0)(e^{2\tau} - e^{-2\tau}) - (\gamma_0 + \delta_0)\tau + 1, \\ \beta(\tau) &= 1/4(\gamma_0 - \delta_0)(e^{2\tau} - e^{-2\tau}) - (\gamma_0 + \delta_0)\tau + 1, \\ \gamma(\tau) &= 1/4(\gamma_0 - \delta_0)(e^{2\tau} + e^{-2\tau}) + 1/2(\gamma_0 + \delta_0), \\ \delta(\tau) &= 1/4(\delta_0 - \gamma_0)(e^{2\tau} + e^{-2\tau}) + 1/2(\gamma_0 + \delta_0).\end{aligned}\quad (35)$$

We can consider a more general linearized problem, without confining ourselves to an initial velocity as given by (7). In an unbounded system, it is reasonable to take as the small perturbations only quadratic dependence of the increment of the vector potential on the radius, although an arbitrary dependence on the angle is admissible. We can then, for example, obtain a solution independent of (35), in which the dependence of the velocity on the coordinates has a form more complicated than in (7),

$$v_x \propto x \cos \varphi, \quad v_y \propto -y \cos \varphi,$$

and an exponential increase with time takes place again. Such a solution no longer has a nonlinear continuation. In all probability there exist also other solutions of a similar kind.

Thus, in this situation there is a certain outward analogy with the universally accepted instability concept. The initial velocity perturbations (if sufficiently small) and the potentials increase exponentially in time. The qualitative departure from the initial character of the motion increases with time continuously, and the dependence of the motion on the initial conditions is not very strong after a sufficiently long time interval. The nonlinear continuation of the solution (35) corroborates this conclusion. During the last stage of the motion, near the instant of the singularity, only the constant coefficients depend on the initial conditions (in accordance with (27)). These are the features of the outward similarity to the ordinary concepts of instability development.

The solution, however, has also significant peculiarities, which are quite unusual from the point of view of these concepts. First, motion of this type develops in an infinitely-dimensional system under a very special linear law of velocity-perturbation. The perturbation energy density for the initial instant increases without limit at infinity ( $|x|, |y| \rightarrow \infty$ ). When the initial velocity has a linear distribution one might then think that the instability is more likely to be characteristic of this plasma current, and not to the stationary initial equilibrium state. As shown by M. A. Leontovich, in a system of finite dimensions (in a circular cylinder with rigid walls) the linearized equations no longer have exponentially growing solutions. In addition, energy considerations also argue against treatment of the

obtained solution as proof of the instability of a stationary initial equilibrium state.

Indeed, let us calculate the kinetic and magnetic energies in a cylindrical volume bounded by the Lagrange curve (30). With the aid of (8), (9), (15), (20), and (21) we obtain the energy per unit length of the cylinder

$$w = \int \left( \frac{\rho v^2}{2} + \frac{B^2}{8\pi} \right) dx dy = \frac{A_0^2}{8} \left( \xi^2 + \eta^2 + \frac{\eta}{\xi} + \frac{\xi}{\eta} \right).\quad (36)$$

It follows therefore that the energy inside the specified Lagrange line (meaning the energy per unit mass) tends to infinity like  $(\tau_0 - \tau)^{-2/3}$  as  $\tau \rightarrow \tau_0$ . The internal energy of the gas, which is not taken into account in (31), also increases without limit. For example, in an adiabatic process the internal energy is

$$\varepsilon = \frac{1}{\kappa_0 - 1} \frac{p_0}{\rho_0} (\xi\eta)^{1-\kappa_0},$$

where  $p_0$  and  $\rho_0$  are the initial spatially-homogeneous pressure and density of the gas, and  $\kappa_0$  is the adiabatic exponent. It is seen therefore that  $\varepsilon$  increases in this case more slowly than  $w$  if  $\kappa_0 < 2$ .

Thus, the motion under consideration can be realized only as the result of influx of energy from the outside. It can be shown that the influx of kinetic and magnetic energy is assured by the Poynting vector of the electromagnetic field. In this connection, the solution under discussion, so long as we are not considering a rigorous boundary-value problem including the sources of the electromagnetic field, is more likely to have the properties of a cumulative effect, and not instability in the usual sense.

The possible occurrence of a cumulative effect in a plasma situated in a hyperbolic magnetic field between two identical parallel currents was discussed in<sup>[3]</sup>. In this case the symmetry line passing between two such currents is the zero line of the magnetic field and formula (6) approximates in a vicinity sufficiently close to it the exact expression for the vector potential of such currents. A characteristic cumulative effect arises, according to<sup>[3]</sup>, when the currents producing the hyperbolic magnetic field change or move rapidly. However, the method used in<sup>[3]</sup> did not make it possible to include the immediate vicinity of the zero line into consideration.

One can imagine that the linear velocity distribution assumed above occurs in the vicinity of the zero line after the converging shock wave, produced for example by displacement of the currents,



is reflected from the symmetry axis. In this case the initial values of  $\gamma_0$  and  $\delta_0$  need not necessarily be regarded as small perturbations. The obtained solution can be regarded in this connection as the principal term in the expansion of the still-unknown rigorous solution of the problem in powers of the small distance  $r$  from the zero line. Indeed, accurate to terms of highest order in  $r^2$ , the expression for the potential, corresponding to the obtained solution, is quite general. We can therefore assume that the obtained solution is the limiting expression for a sufficiently broad class of inhomogeneous solutions of the boundary-value problems as  $r^2 \rightarrow 0$ . It may turn out to be useful also in solving such problems by numerical methods, as a limiting expression in the direct vicinity of the zero line.

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Note added in proof (March 15, 1967). The class of solutions obtained above remains also in force if the coordinate-independent viscosity and conductivity of the plasma are taken into account. This can be easily verified by recognizing

that in this solution the magnetic field and the velocity depend linearly on the coordinates, and therefore the corresponding terms in the magnetohydrodynamic equations vanish. For an incompressible liquid this circumstance was noted in [6,7].

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