ACCELERATION OF CHARGED PARTICLES IN MOVING TRAPS PRODUCED BY ELECTROMAGNETIC WAVES

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For a certain range of initial particle states a plane electromagnetic wave forms a trap. The energy of particles in this trap increases with increasing velocity of the wave and approaches infinity when the wave velocity approaches the velocity of light in vacuum.

KRASOVITSKIĬ[1] and Bonch-Osmolovskiĭ[2] have discussed the acceleration of charged particles by plane transverse monochromatic waves with a phase velocity slowly changing along the direction of propagation of the wave according to a special law, so that the phase velocity of the wave is accurately equal to the longitudinal component of the particle velocity.

Here we have considered the acceleration of a charged particle by an arbitrary plane transverse wave whose parameters (shape, amplitude, velocity of propagation) slowly change according to a given law. For a certain interval of initial particle states, such a wave forms a unique trap which moves along the direction of propagation with the velocity of the wave. If a particle is held in this trap, an increase of the velocity of the wave results in an increase of the average value of the particle energy, which approaches infinity when the wave velocity approaches the velocity of light in vacuum. The effect is analogous to phase stability in linear accelerators but exists even in the absence of a longitudinal field.

Let a charged particle move in an electromagnetic field described by a vector potential **A**:

$$A_x = A_x(\theta, z, t), \quad A_y = A_y(\theta, z, t), \quad A_z = 0;$$
$$\theta = t - \int n(z)\,dz. \tag{1}$$

Here t is the time, the coordinates are expressed in light seconds, and $1/n(z) < 1$ indicates a field propagation velocity which slowly changes along the direction of propagation, so that it can be considered constant for distances of the order L—a certain characteristic length of the field, which can be considered as a wavelength. The dependence of **A** on z and t is also slow, i.e., in distances of the order of L the function **A** can be considered as a function only of θ.

As a consequence of the cyclic nature of the transverse coordinates x and y, there exists an integral of motion

$$\mathbf{p}_\perp + \frac{e}{mc^2}\mathbf{A} = \mathbf{C} = \text{const}, \tag{2}$$

where $\mathbf{p}_\perp$ is the transverse momentum of the particle. Here and subsequently, momenta will be expressed in units of mc, and the particle energy β is referred to the rest energy mc^2.

While n is constant, we can transform to a reference frame K′ tied to the wave, using an ordinary Lorentz transformation. Here the vectors entering into Eq. (2) are not transformed, and

$$t = \frac{nt' + z'}{\sqrt{n^2 - 1}}, \quad z = \frac{nz' + t'}{\sqrt{n^2 - 1}}, \tag{3}$$

$$\theta = -\sqrt{n^2 - 1}\, z'. \tag{4}$$

If we also neglect the slow variation of **A** with z and t, there is no electric field in the K′ system and the particle energy is the integral of motion:

$$\beta'^2 = p_\parallel'^2 + \left(\mathbf{C} - \frac{e}{mc^2}\mathbf{A}\right)^2 + 1 = \text{const}, \tag{5}$$

where $p_\parallel'$ is the z′ component of the particle momentum.

Let us further discuss the motion of the particle in the system K′ at distances of the order $L' = nL/\sqrt{n^2 - 1}$. In this case motion is possible in that region where

$$p_\parallel'^2 = \beta'^2 - \left(\mathbf{C} - \frac{e}{mc^2}\mathbf{A}\right)^2 - 1 \geqslant 0. \tag{6}$$

If the equation

$$\beta'^2 - \left(\mathbf{C} - \frac{e}{mc^2}\mathbf{A}\right)^2 - 1 = 0 \tag{7}$$

does not have real roots, then the particle will pass through the field without change of sign of its longitudinal velocity. If this equation has k real roots $z'_1 < z'_2 < \ldots < z'_k$, and the motion originates to the left of the least root or to the right of the greatest root, this corresponds to reflection of the particle. In the case when the motion originates in the region between z'_1 and z'_k, we will consider that the particle is held in a trap. Here the motion is finite along the z′ axis, and is bounded by the two neighboring roots z'_i and z'_{i+1} between which inequality (6) is satisfied; the period of motion T′ can easily be found from Eq. (5). Only this last case will be considered below.

The energy of the particle, the period of its motion, and the roots of Eq. (7) remain constant as long as the parameters of the wave can be considered constant. If the parameters change so slowly that their variation can be neglected during the period T′, then in the system K′ we can consider them functions only of t′, since in the transformations of Eq. (3) the average value of the variable z′ is constant as a result of the finiteness of the motion. Without loss of generality this average value can be set equal to zero by choice of the origin of coordinates.

Under these conditions there exists an adiabatic invariant $\oint p'_{\parallel} dz'$. Expressing $p'_{\parallel}$ from Eq. (5) in terms of z′, we find that the quantity

$$I' = \int_{z'_i}^{z'_{i+1}} \left[\beta'^2 - \left(\mathbf{C} - \frac{e}{mc^2}\mathbf{A}\right)^2 - 1\right]^{1/2} dz' \tag{8}$$

is an adiabatic invariant relating the particle energy to the parameters of the wave.

Transforming Eq. (8) to the laboratory system, we obtain

$$I = \frac{1}{n^2-1} \int_{\theta_i}^{\theta_{i+1}} \left[Y^2 - \left[\left(\mathbf{C} - \frac{e}{mc^2}\mathbf{A}\right)^2 + 1\right] (n^2-1) \right]^{1/2} d\theta, \tag{9}$$

where

$$Y = n\beta - p_{\parallel}. \tag{10}$$

The quantity Y for constant wave parameters is the integral of motion corresponding to that given by Eq. (5) in the system K′. For a slow variation of the wave parameters, Y varies in accordance with Eq. (9).

A number of conclusions follow from what we have set forth above.

Phase stability is realized in the case discussed, as a result of the inequality

$$\frac{Y^2}{n^2-1} - \left(\mathbf{C} - \frac{e}{mc^2}\mathbf{A}\right)^2 - 1 \geqslant 0. \tag{11}$$

This inequality determines the boundaries of the region of phase variation θ_i and θ_{i+1}, and the equilibrium phase satisfies the equation

$$\left(\mathbf{C} - \frac{e}{mc^2}\mathbf{A}\right)\frac{\partial \mathbf{A}}{\partial \theta} = 0. \tag{12}$$

It is evident from Eq. (9) that for $n \to 1$ the region of phase variation collapses to a point and the oscillations become small. Here the expression under the integral in Eq. (9) approaches zero, and in this limit

$$Y = a\sqrt{n^2-1}, \tag{13}$$

where a is a constant which does not depend on n. Then from Eqs. (10) and (13) we can obtain

$$\beta = \frac{an}{\sqrt{n^2-1}}, \quad u_{\parallel} = \frac{1}{n}. \tag{14}$$

Equations (13) and (14) have been obtained on the assumption that the particle is held in the trap. However, for a change in the wave parameters, the conditions for containment can be destroyed. The containment of the particle in the trap can be investigated on the basis of Eqs. (6)—(8) in the K′ frame or Eqs. (9)—(11) in the laboratory system. Such an investigation is not possible in the general case for an arbitrary form of the function **A**.

We will consider as an example a particular case which is of interest.

Let the electromagnetic field be a plane transverse circularly polarized wave with a definite frequency ω, an amplitude $\mathscr{E}$, and a phase velocity 1/n which changes slowly along the direction of propagation:

$$A_x = \frac{c}{\omega}\mathscr{E}(z)\cos\omega\theta, \quad A_y = -\frac{c}{\omega}\mathscr{E}(z)\sin\omega\theta. \tag{15}$$

The class of fields represented by Eq. (15) includes, in particular, the well known approximation of geometrical optics, where $\mathscr{E}(z) = \mathscr{E}_0/\sqrt{n(z)}$, and the homogeneous wave $\mathscr{E}(z) = \mathscr{E}_0 = \text{const}$.

Containment of a particle in the trap can be realized in such a field. In this case the adiabatic invariant (9) is expressed in terms of well known functions. Aside from a constant factor it is equal to

$$I = \sqrt{\frac{\lambda(z)}{n^2-1}}\,\tau B(\tau). \tag{16}$$

Here

$$\lambda = \frac{e\mathscr{E}(z)}{\omega mc}, \quad B(\tau) = \frac{E(\tau) - (1-\tau)K(\tau)}{\tau},$$

where K(τ) and E(τ) are the complete elliptic integrals in the Legendre form of the first and second kind, respectively, and

$$\tau = \frac{Y^2 - [1 + (C-\lambda)^2](n^2-1)}{4\lambda C(n^2-1)} < 1, \tag{17}$$

where C is the length of the constant vector **C**.

Inequality (17) is the condition for containment of a particle in the trap. It follows from Eq. (16) that this condition can be destroyed as $n \to 1$ only in the case when the wave amplitude $\mathscr{E}(z)$, in this event, approaches zero faster than $n^2 - 1$.

If the condition for containment is satisfied, the expressions for the particle energy, longitudinal momentum, period of oscillation, equilibrium phase θ_0, and the extreme phase values θ_1 and θ_2 take the form

$$\beta = \frac{Yn}{n^2-1} + 2\sigma\left[\frac{\lambda C}{n^2-1}\left(\tau - \sin^2\omega\frac{\theta-\theta_0}{2}\right)\right]^{1/2}, \quad (18)$$

$$p_{\|} = \frac{Y}{n^2-1} + 2\sigma\left[\frac{\lambda C}{n^2-1}\left(\tau - \sin^2\omega\frac{\theta-\theta_0}{2}\right)\right]^{1/2}, \quad (19)$$

$$T = \frac{8nK(\tau)}{\omega(n^2-1)}\left[\tau + \frac{1+(C-\lambda)^2}{4\lambda C}\right]^{1/2}, \quad (20)$$

$$\mathrm{tg}\,\omega\theta_0 = -C_y/C_x, \quad (21)$$

$$\theta_{1,2} = \theta_0 \pm \frac{2}{\omega}\arcsin\sqrt{\tau}. \quad (22)$$

In these formulas the arithmetic values of the roots are used, as

$$\sigma = -(1-nu_{\|})\,/\,|1-nu_{\|}|.$$

Equations (18)—(22), together with the adiabatic invariant (16) and equality (17), describe the motion of the particle in the trap produced by the wave (15) when its parameters change slowly.

From these expressions it is evident that the particle energy, longitudinal momentum, and phase θ oscillate with a period T about their equilibrium values

$$\bar{\beta} = \frac{Yn}{n^2-1}, \quad \bar{p}_{\|} = \frac{Y}{n^2-1}, \quad \bar{\theta} = \theta_0,$$

where the energy and longitudinal momentum oscillate in the same phase, and the phase oscillations, which imply phase stability, are shifted with respect to the oscillations of energy by a quarter period.

The equilibrium phase value, as we can see from Eq. (21), is determined only by the initial conditions and does not depend on the variation of the wave parameters. From the expression for the extreme phase values, Eq. (22), it follows that the motion of the particle originates within a single period of the wave, i.e.,

$$\omega|\theta-\theta_0| < \pi.$$

If the wave amplitude $\mathscr{E}(z)$ for $n \to 1$ does not vanish or approach zero more slowly than $n^2 - 1$, the conditions for containment of the particle in the trap are always fulfilled, and, as follows from Eq. (16), τ approaches zero. Here the oscillations become small, i.e., the extreme phase values approach the equilibrium value, and the equilibrium energy value about which oscillations occur approaches infinity according to the law

$$\bar{\beta} = n\left[\frac{1+(C-\lambda)^2}{n^2-1}\right]^{1/2},$$

which follows from Eqs. (16)—(18). The amplitude of the oscillations in energy in this case assume the form

$$\Delta\beta = \sqrt{2\pi IC}\left(\frac{\lambda}{n^2-1}\right)^{1/4},$$

i.e., $\Delta\beta/\bar{\beta} \to 0$. In this limit the period of oscillation varies according to the law

$$T = \frac{2\pi}{\omega}\left[\frac{1+(C-\lambda)^2}{C\lambda}\right]^{1/2}\frac{n}{n^2-1}$$

and also approaches infinity for $n \to 1$. This means, as we would expect, that an infinitely large energy is not reached in finite time intervals.

The mechanism which we have discussed can be applied to the problem of the origin of cosmic rays. The existence of waves described by the vector potential (1) is extremely probable under cosmic conditions. In particular, this potential can be used to describe an electromagnetic impulse whose shape, amplitude, and group velocity 1/n vary slowly, for example as the result of dispersion, damping, and variation of density of the medium.

We note also that in the case of a circularly polarized wave propagated along the magnetic field, where the wave parameters and the longitudinal field vary slowly, it is also possible to construct an adiabatic invariant based on the exact solution obtained by Roberts and Buchsbaum[3] of the equations of motion of a particle for a wave with constant parameters propagated along a constant magnetic field.

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