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GEOMETRIZATION OF THE ELECTROMAGNETIC FIELD AND VIOLATION OF INVARIANCE

B. A. ARBUZOV and A. T. FILIPPOV

Joint Institute for Nuclear Research

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A geometrical interpretation is given for an interaction violating CP invariance which had been proposed earlier. A space is considered which possesses absolute parallelism with the Minkowski metric but which also has torsion. It is shown that such a space is determined by an antisymmetric tensor of second rank, which we connect with the electromagnetic field tensor. Simple geometric considerations lead to equations which are a generalization of the Maxwell equations. It is shown that in such a space a spinor particle necessarily has an interaction which violates CP invariance.

1. INTRODUCTION

THE discovery of parity nonconservation in weak interactions in 1956[1] revealed a serious defect in theoretical ideas about the symmetries of space-time. In fact, up to 1956 invariance of the interactions of elementary particles under discrete transformations of space-time (space inversion P and time reversal T) was regarded as an obvious consequence of the most general properties of space-time, described by the postulates of the special theory of relativity (cf., e.g.,[2]). Therefore the observed nonconservation of P seemed incompatible with fundamental and well established properties of space.

A beautiful way out of this difficulty was found, however, Wigner,[3] Landau,[4] and Lee and Yang[5] suggested that the true operation of space inversion is not P, but the combined inversion CP, and that accordingly all interactions are CP invariant. Owing to CPT invariance, violation of which would lead to a far-reaching reexamination of the simplest principles of relativistic quantum mechanics (cf., e.g.,[6]), all interactions would be T invariant and the symmetry of space-time would in this way be preserved.

The hypothesis of CP invariance was confirmed by rather numerous experiments, and it gradually came to be regarded as one of the fundamental laws of nature. Therefore the discovery in 1964 of the decay $K_2^0 \to \pi^+\pi^-$, which is forbidden by CP conservation, was entirely unexpected. Simple ways to save CP were refuted after a more detailed experimental investigation of the decays of K mesons (see the reviews[8-12]), and it became clear that ideas about discrete symmetries of space-time are in need of a new and serious reexamination (on this point see[10,12]).

It must, by the way, be pointed out that in principle there are some possibilities of evading this difficulty. One such hypothesis about CP invariance was indeed indicated by Lee and Yang[12] as early as 1957 in a discussion of the possible violation of T invariance. They suggested that our world is "doubled" in terms of a new quantum number characterizing a new degree of freedom of particles; to each particle there then corresponds a "mirror" particle which differs from the first particle only in this new quantum number. Then the symmetry of the world is restored if we assume that the true operation of space inversion (or of time reversal) is the product of CP (or of T) and the operation of change from ordinary particles to "mirror" particles. A recent detailed discussion of this hypothesis[14] led to the not very consoling conclusion that the interaction of ordinary particles with the "mirror" particles must be extremely small, and therefore this simple way of rescuing customary ideas may be purely illusory. Another possibility of a new interpretation of discrete symmetries is considered by Lee and Wick,[15] but they introduce different definitions of the discrete symmetry transformations in different interactions, and essentially renounce the connection of the discrete symmetries with the properties of space-time. Moreover, there is a great deal of arbitrariness in the definition of the new operations, and no general principles can as yet been discerned which could enable us to eliminate this.

Accordingly, even if we suppose that these ways around the difficulties are still open, it is not much of an exaggeration to admit that as a whole the question of the discrete symmetries is now in about the same state as it was in 1957. Therefore all sorts of attempts at a geometrical interpretation of the discrete symmetries seem to be quite in order. There have so far been very few such attempts (we have been able to find only some papers[16] devoted to a possible geometric interpretation of the violation of P invariance; see also a paper by Smorodinskiĭ[45] which presents a mechanical model of nonconservation of P in a space with a torsion).

One of the possibilities has been pointed out in[17] and subsequently discussed in[18,19]. In these papers a geometric approach to the theory of weak interactions is developed, in which the weak interactions are explained in terms of local distortions of space-time at small distances "inside" particles. With this approach nonconservation of P arises as a consequence of simple geometric propositions. Thereafter[20-22] we made an attempt to give a similar geometric interpretation of nonconservation of CP. The basis of this interpretation is the setting up of a connection between the electromagnetic field and the curvature of space-time. Some additional physical assumptions about the nature of this connection allowed us to predict a number of effects in weak-electromagnetic interactions (whose coupling constant is $\sim Ge$, where G is the constant of the weak interactions and e is the charge of the electron). These papers gave a detailed discussion of the possibility of observing such effects, and also of the difference between the predictions of the geometric model and those of other models (cf., e.g.,[23-26]) of the violation of CP invariance, but the geometric interpretation itself was only indicated.

In the present paper we expound an attempt at a consistent construction of a geometric theory of the electromagnetic field, based on interpreting it as a torsion of space-time. Although in the final analysis we are trying to understand the connection between the weak and electromagnetic interactions (cf.[18,19] and[22]), we shall here not take into account the curvature of space-time, and accordingly shall not try to construct a unified theory of the weak interactions and electromagnetism.[1] We note that we shall here make wide use of the methods applied by Einstein in his attempts to formulate a unified theory of gravitation and electromagnetism,[2] but we shall refrain altogether from attempting to connect the electromagnetic field with the gravitational field.

It must be pointed out that the mathematical formalism used in this paper differs from that which we used originally for heuristic purposes.[20-22] In fact, the use of a nonsymmetric metric tensor is a purely formal procedure and does not throw much light on the geometry of space-time. The geometry is uniquely determined by giving the tensors of curvature and torsion (cf., e.g.,[29-35]), which can be expressed in well known ways in terms of the affine connection. In the general theory of relativity Euclidean space is generalized to Riemannian space with a symmetric connection, which defines zero torsion. It seems to us, however, that the simplest generalization of the pseudoeuclidean space is a space with zero curvature, the pseudoeuclidean metric, and a nonzero torsion (a nonsymmetric connection). We shall show that the study of much spaces naturally leads to a geometric interpretation of the free electromagnetic field. The simplest geometric restrictions that can be imposed on the torsion give generalized nonlinear Maxwell's equations, which in the weak-field approximation reduce to the ordinary Maxwell's equations. We shall then consider the Dirac equation in this space and show that it automatically contains a CP-odd interaction of the form that we postulated earlier,[20-22] but the detailed form of this interaction for various particles can evidently be established only in a unified theory of the weak interactions and electromagnetism.

2. FUNDAMENTAL PROPERTIES OF SPACES WITH ABSOLUTE PARALLELISM

In this section we shall briefly expound the theory of spaces with absolute parallelism, mainly following Cartan,[29] who first studied such spaces, and Einstein,[27,28] who applied these spaces in one version of unified field theory. We shall concentrate on the facts which are essential for the following sections. The mathematical details of the theory can be found in the papers of Cartan which we have cited, and also in books by Eisenhart[30] and by Schrödinger.[31] [3] As for the physical interpretation, as we have already stated, the model we have developed has no relation to unified field theories of gravitation and electromagnetism, and is rather based on an attempt to combine on a sin-

[1]The existence of a definite connection between these phenomena will, however, be at all times assumed.

[2]See papers by Einstein [27]; they are translated into Russian in the Collected Works of Einstein, [28] which also contains other papers devoted to the same problems.

[3]The simplest facts about spaces with torsion, and in particular about spaces with absolute parallelism, can be found in standard introductions to differential geometry (cf., e.g., [32-35]).

gle geometric foundation the phenomena of electromagnetism and of the weak interactions.

A space with absolute parallelism is defined locally by the condition that the result of parallel transfer of a vector from any point x to any point y does not depend on the path on which the transfer is carried out. This is equivalent to the condition that at every point x of any small region of space[4)] one can construct a system of linearly independent vectors $h^i_{(a)}(x)$ [a is the number of the vector, and $h^i_{(a)}$ is the projection of the a-th vector along the i-th axis of some given coordinate system at the point x; a, i = 0, 1, 2, 3], and the system of vectors at the point y is obtained from that at x by parallel transfer (see[27-34]), which is expressed in the usual way in terms of the coefficients of the affine connection $\Gamma^{j\cdots}_{k}$. In parallel transfer from the point x^k to the infinitesimally close point $x^k + \delta^k$ the contravariant components $A^i(x)$ of an arbitrary vector receive the increments

$$\delta A^i(x) = -\Gamma^i_{jk}(x) A^j(x)\,\delta x^k, \tag{2.1}$$

and the covariant components $A_j(x)$ receive the increments

$$\delta A_j(x) = \Gamma^i_{jk}(x) A_i(x)\,\delta x^k \tag{2.2}$$

(summation over repeated indices is always understood).

Accordingly we get for the frame vectors the equations

$$h^i_{(a),\,k} \equiv \partial_k h^i_{(a)} \equiv \frac{\partial}{\partial x^k} h^i_{(a)} = -h^j_{(a)}\Gamma^i_{jk}. \tag{2.3}$$

Introducing the normalized minors $h_{(a)i}$ of the determinant of the matrix $h^i_{(a)}$, defined by the equations[5)]

$$h^i_{(a)}(x)\, h^{(a)}_j(x) = \delta^i_j, \tag{2.4}$$

we get from (2.3) the following expression for Γ:

$$\Gamma^i_{jk} = h^i_{(a)} h^{(a)}_{j,\,k} = -h^{(a)}_j h^i_{(a),\,k} \tag{2.5}$$

[where the last equation follows from (2.4)]. For a space with absolute parallelism the equations (2.3) must be integrable, and from this it follows that

$$0 = h^i_{(a),\,jk} - h^i_{(a),kj} = \partial_k(\Gamma^i_{lj} h^l_{(a)}) - \partial_j(\Gamma^i_{lk} h^l_{(a)}). \tag{2.6}$$

Using (2.3) again, we now find that the Riemann curvature tensor R^i_{jkl} is zero:

$$R^i_{jkl} \equiv -\Gamma^i_{jk,\,l} + \Gamma^i_{jl,\,k} - \Gamma^s_{jk}\Gamma^i_{sl} + \Gamma^s_{jl}\Gamma^i_{sk} = 0. \tag{2.7}$$

It can be shown (cf., e.g.,[30,31]) that this last condition is also sufficient for the integrability of the equations (2.3).

Accordingly, an affine space possesses absolute parallelism if and only if its curvature tensor is identically zero (Schrödinger[31] calls such spaces integrable spaces). This condition is equivalent to the existence of frame vectors $h^i_{(a)}$ in terms of which the affine connection is expressed by the relation (2.5). Without loss of generality we can hereafter regard all sets of frame vectors as pseudo-orthogonal and normalized

$$h^i_{(a)} h^{(b)}_i = \delta^b_a, \quad h^i_{(a)} h_{(b)i} = \delta_{ab}, \quad h^{(a)i} h^{(b)}_i = \delta^{ab}. \tag{2.8}$$

Then from the geometric meaning of the quantities $h^i_{(a)}$ and $h_{(a)i}$ there follows their connection with the metric tensor:

$$g_{ij} = h_{(a)i} h^{(a)}_j, \qquad g^{ij} = h^i_{(a)} h^{(a)j}. \tag{2.9}$$

It is necessary to emphasize that the quantities $h^i_{(a)}$ and $h_{(a)i}$ are in general not uniquely defined. In fact, neither the relation (2.5) nor the relation (2.9) is altered by the transformation

$$h^i_{(a)} \to L^{\cdot(b)}_{(a)} h^i_{(b)}, \quad h^{(a)}_i \to L^{(a)}_{\cdot(b)} h^{(b)}_i, \tag{2.10}$$

where $L^{\cdot(b)}_{(a)}$ is any pseudoorthogonal matrix[6)] which does not depend on x.

We could free ourselves from this ambiguity by means of the following physical requirement. Let us consider coordinates x^k such that when we go over to a flat space (turn off the interactions) they go over continuously, in any finite region, into Cartesian coordinates. Since in this passage to the limit the axes of all frames h remain parallel at all times (in the sense of absolute parallelism), we find that in the Cartesian limit they will all be parallel to each other (in the usual sense), but in general will not be parallel to the axes of the Cartesian coordinate system. By using a transformation (2.10) we can always get all of the frames

4)In all that follows we shall be considering a four-dimensional space-time, although the methods we use can also be applied in the case of a space of any number of dimensions and with any metric.

5)Here and in what follows δ^i_j denotes the usual Kronecker symbol, whereas $\delta^{ij} = \delta_{ij}$ is the diagonal matrix with the elements $\delta_{00} = 1$, $\delta_{11} = \delta_{22} = \delta_{33} = -1$. The indices (a) are raised and lowered in the usual way by means of the metric tensor δ_{ab}. For example, $h^{(a)}_i = \delta^{ab} h_{(b)i}$, and so on.

6)A pseudoorthogonal matrix satisfies the conditions

$$L^{\cdot(c)}_a \delta_{cd} L^{\cdot(d)}_{(b)} = \delta_{ab}, \quad L^{\cdot(a)}_{(c)} \delta^{cd} L^{\cdot(b)}_{(d)} = \delta^{ab},$$

where δ_{ab} is the metric tensor of a pseudoeuclidean space (see above).

oriented the same as the fundamental Cartesian coordinate system—that is, in the limit we will have

$$h^i_{(a)} = \delta^i_a \qquad h_{(a)i} = \delta_{ai}. \tag{2.11}$$

These conditions could be adopted in the general case to eliminate the arbitrariness in the choice of the frame vectors. In the present paper we shall use a simpler formal approach, which we describe in the next section.

It is useful to point out that the use of orthogonal frame vectors $h^i_{(a)}$ and $h_{(a)i}$ is not at all necessary for the description of spaces with absolute parallelism. The whole theory could also be developed without introducing these objects (cf., e.g.,[32]). We use frame vectors, first, because spaces with absolute parallelism can be most simply and naturally described in terms of them, and second, because by means of orthogonal frame vectors we can most simply introduce spinors in noneuclidean spaces.[7] By the way, there are also other rather convenient ways to introduce spinors in noneuclidean spaces (see, in particular,[40,41]), which we propose to consider in another place.

In concluding this section we consider the conditions which relate the affine connection and the metric. This condition can be obtained by requiring that the metric structure given by the affine connection Γ be consistent with the metric defined by the metric tensor g_{ij}. In other words, the distance determined by the metric tensor, $ds^2 = g_{ij}dx^i dx^j$, must be the same as the distance which can be defined along any geodesic by means of the affine connection alone. As is shown, for example, in Schrödinger's book,[31] the necessary and sufficient condition for this requirement to be satisfied is that the symmetric part of the affine connection can be written in the form

$$\bar{\Gamma}^i_{jk} = \frac{\Gamma^i_{jk} + \Gamma^i_{kj}}{2} = \left\{ {i \atop jk} \right\} + g^{il} T_{ljk},$$

where $\left\{ {i \atop jk} \right\}$ is the Christoffel symbol, and the tensor T_{ljk}, symmetric in j and k, satisfies the condition

$$T_{ljk} + T_{jkl} + T_{klj} = 0,$$

but is otherwise arbitrary.

These conditions, however, impose no restrictions whatever on the antisymmetric part of the affine connection

$$\Omega^k_{ij} = \frac{\Gamma^k_{ij} - \Gamma^k_{jl}}{2}, \quad \Omega^k_{ij} = -\Omega^k_{jl}, \tag{2.12}$$

which is a tensor, and according to Cartan[29] (cf. also[30-35]) determines the torsion of the space.[8]

It would therefore be natural to take as the basis of our further constructions the strongest requirement: that the metric tensor at the point x must be obtained from that at any other point y by parallel transfer, i.e.,[9]

$$g_{ij|l} = g_{ij,l} - \Gamma^s_{il} g_{sj} - \Gamma^s_{jl} g_{is} = 0. \tag{2.13}$$

It can be verified that the general conditions for consistency of the metric with the affine connection follow from this requirement, but the converse is in general not true. Accordingly, in the general case the requirement (2.13) imposes important restrictions on the geometry of the space.

It is, however, not hard to verify that for a space with absolute parallelism the condition (2.13) is satisfied automatically. In fact, the metric tensor g_{ij} is determined in terms of the frame vectors $h^i_{(a)}$ by the relation (2.9), and frame vectors at different points go over into each other through parallel transfer, i.e.,

$$h^i_{(a)|l} = 0, \quad h_{(a)i|l} = 0. \tag{2.14}$$

This last condition can also be obtained purely formally, by using (2.4) and (2.5) and the definition of the covariant derivative of a vector

$$A^i_{|l} \equiv A^i_{|l} + \Gamma^i_{jl} A^j, \qquad A_{i|l} \equiv A_{i,l} - \Gamma^j_{il} A_j. \tag{2.15}$$

Thus we have shown that in any space with absolute parallelism the metric is consistent with the affine connection, and moreover the change of the metric tensor as we go from point to point can be obtained by applying parallel transfer to it.

3. PSEUDOEUCLIDEAN SPACE WITH TORSION AND THE FREE ELECTROMAGNETIC FIELD

Let us now consider the simplest of the spaces with absolute parallelism, i.e., a space in which the ordinary pseudoeuclidean metric

$$g_{ij} = \delta_{ij}, \quad g^{ij} = \delta^{ij}. \tag{3.1}$$

is preserved. We shall call such a space a pseudoeuclidean space with torsion. The metric relations in this space are the same as in the ordinary Minkowski geometry, but the parallel transfer is

[7]See [36-39]. The most complete results have been obtained by Fock. [39]

[8]The geometric interpretation of the tensor Ω can be found, for example, in the books [30,32-34].

[9]We use the symbol $|l$ to denote the covariant derivative with respect to the l-th coordinate.

decidedly different because of the presence of torsion. We shall not rewrite all of the formulas of the preceding section, but simply stipulate that in all of them g^{ij} and g_{ij} are to be replaced by δ_{ij}. We write out some of the most important notations:

$$\Gamma_{ijk} = \delta_{il}\Gamma^{l}_{jk}, \quad \Omega_{ijk} = \Omega^{l}_{ij}\delta_{lk}. \tag{3.2}$$

The relations (2.4) and (2.8) lead to the condition that the matrix $h_{(a)i}$ be pseudoorthogonal:

$$h^{(a)}_{i} h_{(a)j} \equiv h_{(a)i}\,\delta^{ab} h_{(b)j} = \delta_{ij}. \tag{3.3}$$

From (2.5) we can now find that

$$\Gamma_{ijk} = -\Gamma_{jik}. \tag{3.4}$$

This same symmetry condition can be derived from the condition (2.13) if we use the fact that $\delta_{ij,l} = 0$.

From the definition (2.12) of the torsion tensor Ω and the condition (3.4) it is not hard to find the useful relation

$$\Gamma_{ijk} = -\Omega_{ijk} + \Omega_{jki} + \Omega_{kij}, \tag{3.5}$$

from which it follows in particular that the affine connection Γ_{ijk} behaves like a tensor under transformations that preserve the metric (3.1).

Let us count the number of independent functions by which the geometry of a pseudoeuclidean space with torsion is determined. Owing to the orthogonality conditions (3.3) the matrix $h_{(a)i}$ has only six independent elements, in terms of which all geometric quantities can be expressed. We could take as these independent quantities the antisymmetric part of the matrix $h_{(a)i}$, but in the case of a pseudoeuclidean space with torsion we can proceed in a different way. Let us consider any Lorentz coordinate system and in it set

$$h_{(a)i}(x) = l_{(a)i} + F_{ij}(x)\, l^{j}_{(a)}, \tag{3.6}$$

where $l_{(a)i}$ is an arbitrary constant pseudoorthogonal matrix. It is obvious that the matrix F_{ij} is a tensor with respect to arbitrary Lorentz transformations of the coordinates and does not change under transformations (2.10) of the frame vectors.[10)] In the limiting case of an infinitely small torsion the frame vectors h do not depend on x, and therefore we shall assume that the tensor F_{ij} becomes infinitely small. In order for the set of vectors $h_{(a)i}$ to coincide with the set which is uniquely defined (in accordance with the requirement of Sec. 2) it is sufficient, by (2.10), to set $l_{(a)i} = \delta_{ai}$.

From the conditions (2.4) and the arbitrariness $l_{(a)i}$ of the frame vectors it follows that the tensor F_{ij} satisfies the condition

$$F_{ij} + F_{ji} + F_{ik}\delta^{kl}F_{jl} = 0. \tag{3.7}$$

Accordingly the matrix $\delta_{ij} + F_{ij}$ is pseudoorthogonal. As is shown in the Appendix, the symmetric part of such a matrix

$$\delta_{ij} + \bar{F}_{ij} = {}^1\!/_2(F_{ij} + F_{ji}) + \delta_{ij}$$

can be expressed in terms of

$$f_{ij} = {}^1\!/_2(F_{ij} - F_{ji}), \tag{3.8}$$

namely we have

$$\bar{F}_{ij} = \frac{1}{4}\left[\bar{s} - \frac{2s}{\bar{s}+4}\right]\delta_{ij} + \frac{2}{\bar{s}+4}\,(f_{ik}\delta^{kl}f_{lj}),$$

$$\bar{s} = \bar{F}^{\cdot k}_{k}, \quad s = f^{ij}f_{ji}. \tag{3.9}$$

Here $\bar{s}$ can be expressed in terms of s and d (see Appendix), but the formulas in question are rather cumbersome and we shall not need them at present. In the case of an infinitely small torsion we obviously get

$$\bar{F}^{\cdot k}_{i} = {}^1\!/_2 f^{\cdot l}_{i} f^{\cdot k}_{l} + o(f^2). \tag{3.10}$$

It is useful to give the explicit expression for the Γ_{ijk} in terms of F_{ij}:

$$\Gamma_{ijk} = F_{ji,k} + F^{\cdot s}_{i} F_{js,k} = -f_{ij,k} + {}^1\!/_2(F^{\cdot s}_{i} F_{js\,k} - F^{\cdot s}_{j} F_{is,k}), \tag{3.11}$$

where in the last expression the antisymmetry condition (3.4) is made explicit. Substituting (3.9) in (3.11), we can express the connection Γ_{ijk}, and thus also the torsion Ω_{ijk}, in terms of six independent and as yet entirely arbitrary functions f_{ij}. In this way the geometry of our space-time is so far extremely arbitrary.

To remove this arbitrariness and put some restriction on the choice of the space, we proceed in a way that has justified itself well in the construction of Einstein's general theory of relativity.[42] Roughly speaking, Einstein's basic argument was as follows. We find the irreducible tensors which can be constructed from the tensor that characterizes the geometry (in the case of Einstein's theory this is the Riemann curvature tensor R_{iklm}). We take the simplest irreducible tensor and equate it to zero. The simplest nontrivial equation that actually restricts the geometry of space-time is indeed the Einstein equation: $R_{ij} - {}^1\!/_2 g_{ij}R = 0$.

In our case the geometry is completely deter-

10)The expression (3.6) can be written in a form valid for arbitrary curvilinear coordinates, if instead of $l_{(a)i}$ we take the Lamé coefficients for the corresponding coordinates. F_{ij} will then retain its tensor character for curvilinear transformations of the coordinates.

mined by giving the tensor Ω_{ijk}. Let us break it up into irreducible tensors.[11] This is not hard to do by using the operations of symmetrization, alternation, contraction, and multiplication by the metric tensor δ_{ij} and the Levi-Civita tensor density ϵ_{ijkl}. It is obvious that in this way we cannot construct any irreducible scalars or second-rank tensors, but we can easily construct an irreducible vector V_i and pseudovector A_i:

$$V_i = \Omega_{ik}^{\cdot\,\cdot k}, \quad A_i = \varepsilon_{ijkl}\Omega^{jkl}. \tag{3.12}$$

In the absence of matter we do not have any other vectors and axial vectors at our disposal, and therefore it is natural to assume that the equations for the torsion field are of the form[12]

$$V_i = 0, \quad A_i = 0. \tag{3.13}$$

We shall verify later that these equations reduce to nonlinear equations for the tensor f_{ij} which are generalizations of the Maxwell equations and reduce to them for small values of $|f_{ij}|$. This solves the problem of the consistency of the equations (3.13), at least in the case of small torsion.[13]

From the equations (3.13) and the condition (3.5) it follows that

$$\Omega_{ijk} = -{}^1/_2\Gamma_{ijk}. \tag{3.14}$$

Therefore, using the representation (3.11), we get

$$\Omega_{ijk} = {}^1/_2 f_{ij,\,k} - {}^1/_4[F_i^{\cdot s}F_{js,\,k} - F_j^{\cdot s}F_{is,\,k}], \tag{3.15}$$

and the equations (3.13) can be written in the form

$$f^{ij}_{\ldots,j} = {}^1/_2[F^{is}F^{\;j}_{.s,\,j} - F^{js}F^{\;i}_{.s,\,j}]. \tag{3.16a}$$

$$f_{ij,\,k} + f_{jk,\,i} + f_{ki,\,j} = {}^1/_2[F_i^{\cdot s}F_{js,\,k} + F_j^{\cdot s}F_{ks,\,i} + F_k^{\cdot s}F_{is,\,j} - F_j^{\cdot s}F_{is,\,k} - F_k^{\cdot s}F_{js,\,i} - F_i^{\cdot s}F_{ks,\,j}]. \tag{3.16b}$$

In the approximation of weak torsion the nonlinear terms in (3.16a) and (3.16b) can be neglected, and we find that in this approximation the antisymmetric tensor f_{ij} satisfies the Maxwell equations

$$f^{ij}_{\ldots,j} = 0, \quad f_{ij,\,k} + f_{jk,\,i} + f_{ki,\,j} = 0. \tag{3.17}$$

This gives us grounds for supposing that the tensor f_{ij} is proportional to the electromagnetic field tensor H_{ij}.

To determine the proportionality constant we note that the electromagnetic field tensor H_{ij} has the dimensions of mass squared (in the system $\hbar = c = 1$), whereas the tensor f_{ij} proportional to it is dimensionless. Since, as was explained in detail in the Introduction, it seems to us natural to look for a unified theory of the weak and electromagnetic interactions, to find the size of the constant relating the tensors f and H we can use the universal constant G of the weak interaction,

$$G = 10^{-5} / m_p^2.$$

Therefore on dimensional grounds we set

$$f_{ij} = \lambda(Ge)H_{ij}, \tag{3.18}$$

where λ is a numerical constant and the factor e is displayed separately in order to indicate that effects of torsion of space (nonconservation of CP, as we shall see later) show up only in weak-electromagnetic interactions (cf.[20,22]). Of course we can hope to determine the exact form of the relation (3.18), i.e., the value of the constant λ, only from a more complete theory, which takes into account in a consistent way both the curvature and the torsion of space.

The smallness of the constant G allows us to justify the neglect of the nonlinear terms in (3.16). In fact, the necessary conditions for this neglect to be legitimate are

$$|f_{ij}| = \lambda Ge|H_{ij}| \ll 1; \quad \lambda Ge|\bar{E}| \ll 1, \quad \lambda Ge|\bar{H}| \ll 1,$$

where E and H are the electric and magnetic field strengths. These conditions can be written in the form

$$\lambda|E| \ll 5\cdot 10^{27}\ \mathrm{V/cm}, \quad \varepsilon \ll \lambda^2\cdot 10^{55}\ m_p/\mathrm{cm}^3,$$

where ϵ is the density of electromagnetic energy. Accordingly it is clear that in all ordinary situations the nonlinear terms in (3.16) can be neglected, and in any case there is a clear quantitative criterion for neglecting them.

It is useful to discuss the question of the uniqueness of our choice of the equations. Since in the theory of a pseudoeuclidean space with torsion the connection Γ_{ijk} is also a tensor, it would seem that instead of (3.13) one could get a different system of equations by replacing Ω by Γ in (3.12). It is not hard to verify, however, [see (3.5)] that

$$\Gamma_{ijk} + \Gamma_{jki} + \Gamma_{kij} = \Omega_{ijk} + \Omega_{jki} + \Omega_{kij}, \tag{3.19}$$

and therefore this second equation turns out to be precisely the same. Since (3.14) follows from this second equation, we get

[11] A general procedure for constructing irreducible tensors has been developed by Cartan. [29] Here we can confine ourselves to the use of simpler arguments.

[12] The attempt to equate the third-rank irreducible tensor to zero leads to indefinite equations, and the adoption of only one of the equations (3.13) as the fundamental equations gives indefinite equations.

[13] For a complete solution of this problem it would be desirable to derive these equations from a Lagrangian of some kind.

$$\Gamma^{ik}_{..k} = -2\Omega^{ik}_{..k} \qquad (3.20)$$

and the first equation is thus preserved. When we go on to spaces of more general types only Ω_{ijk} remains a tensor, and this question does not even arise.

4. A SPINOR FIELD IN A PSEUDOEUCLIDEAN SPACE WITH TORSION. VIOLATION OF CP INVARIANCE IN INTERACTIONS OF SPINOR PARTICLES WITH THE ELECTROMAGNETIC FIELD

In spaces with absolute parallelism the equations for spinor particles are determined in an extremely natural way. They can be constructed especially simply if we use the formalism of absolutely parallel frame vectors expounded above. In the construction of the equations for spinors we shall mainly follow Fock,[39] who has worked out the frame-vector method in detail in the case of Riemannian spaces without torsion. As will be seen in what follows, an additional simplification arises for pseudo-euclidean spaces with curvature, and there is no difficulty in the introduction of spinors.[14)]

We define a system of ordinary Dirac matrices $\gamma^{(a)}$ satisfying the anticommutation relations

$$\{\gamma^{(a)}, \gamma^{(b)}\} \equiv \gamma^{(a)}\gamma^{(b)} + \gamma^{(b)}\gamma^{(a)} = 2\delta^{ab}. \qquad (4.1)$$

In our space with torsion these matrices are not objects of vector nature, since parallel transfer of the vector $\gamma_{(a)}$ from one point to another would have to lead to different matrices $\gamma'_{(a)}$. It is not hard, however, to construct vector objects from the matrices $\gamma_{(a)}$ by using the frame-vector coefficients $h^i_{(a)}$. We define matrices β_i, depending on the point x, by the relation

$$\beta_i = \gamma^{(a)} h_{(a)i}. \qquad (4.2)$$

It follows from (3.3) that these matrices satisfy the anticommutation relations

$$\{\beta_i, \beta_j\} = \beta_i\beta_j + \beta_j\beta_i = 2\delta_{ij}, \qquad (4.3)$$

and their vector character is obvious.

Let us now consider bilinear spinor combinations $\bar\psi B\psi$, where B is a matrix, dependent on x, from the algebra of the matrices β_i, and $\bar\psi = \psi^+\beta_0$. To define the transformation of the spinors ψ and $\bar\psi$ under parallel transfer, we require (cf.[39]) that the quantity $\bar\psi\psi$ be a scalar, and that $\bar\psi\beta_i\psi$ be a vector. Then under parallel transfer from the point x^k to the point $x^k + \delta x^k$ these bilinear combinations must acquire the increments

$$\delta\overline{\psi}(x)\psi(x) = 0, \quad \delta\overline{\psi}(x)\beta_j\psi(x) = \Gamma^i_{jk}(\overline{\psi}\beta_i\psi)\delta x^k. \qquad (4.4)$$

When we define the increment $\delta\psi$ of the spinor as

$$\delta\psi(x) = C_k(x)\psi(x)\delta x^k, \qquad (4.5)$$

we find from (4.4)

$$\delta\overline{\psi}(x) = -\overline{\psi}(x)C_k(x)\delta x^k, \qquad (4.6)$$

$$\overline{\psi}[\beta_j, C_k]\psi\delta x^k + \overline{\psi}\delta\beta_j\psi = \Gamma^i_{jk}(\overline{\psi}\beta_i\psi)\delta x^k, \qquad (4.7)$$

where (cf. (2.3))

$$\delta\beta_j = \gamma^{(a)}\delta h_{(a)j} = \Gamma^i_{jk}\gamma^{(a)}h_{(a)j}\delta x^k = \Gamma^i_{jk}\beta_i\delta x^k. \qquad (4.8)$$

Substituting (4.8) in (4.7) we get the condition

$$[\beta_j, C_k] = 0, \qquad (4.9)$$

from which it follows that the spinor connection C_k is given by[15)]

$$C_k = ieIA_k, \qquad (4.10)$$

where I is the unit matrix and A_k is an arbitrary real vector, which has often been identified with the potential of the electromagnetic field. In order to get the usual interaction of a charged particle with the electromagnetic field, we adopt for the present this interpretation of the vector A_k, although it is not obligatory (cf.[18,19]).[16)] Accordingly, we have obtained an exceptionally simple expression for the spinor connection C_k. If we neglect the vector A_k spinors will not change on parallel transfer. The simplicity of this result is explained by the fact that

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[14)]Strictly speaking this statement applies only to the local aspect of the problem—the construction of a spinor field in only a finite part of the space. The possibility of uniquely defining spinors in the whole space depends on the topology of the space. We can explain this remark with the example of two-dimensional Euclidean spaces. A two-dimensional Euclidean space is topologically equivalent (homeomorphic) to one of five objects: a plane, a cylinder, a torus, a twisted-over cylinder (infinitely wide Möbius strip), or a twisted-over torus (Klein bottle). Whereas on a surface homeomorphic to the Euclidean plane there is an obvious definition of spinors by means of frame vectors, it is impossible on a Möbius strip, for example, to define even a continuous family of frame vectors, and it turns out to be impossible to define a spinor field uniquely. To avoid such questions, we adopt for the present the natural assumption that our pseudo-euclidean space with torsion is homeomorphic to the Minkowski space.

[15)]In fact, any four-rowed matrix C_k which commutes with all of the four-rowed matrices β_j satisfying the conditions (4.3) is proportional to the unit matrix (cf., e.g., [43]).

[16)]The problem of the interpretation of the arbitrariness in (4.10) can evidently be solved only in a more complete unified theory of the weak and electromagnetic interactions.

we are considering actually the simplest generalization of the pseudoeuclidean space.

It is now not hard to define the covariant derivative of a spinor

$$\psi_{|k} = \psi_{,k} - C_k\psi = (\partial_k - ieA_k)\psi, \qquad (4.11)$$

$$\bar{\psi}_{|k} = \bar{\psi}_{,k} + \bar{\psi}C_k = (\partial_k + ieA_k)\bar{\psi}. \qquad (4.12)$$

If we neglect the term ieA_k, the covariant derivative of a spinor is the same as the ordinary derivative.[17)] The Dirac equation in the pseudoeuclidean space with torsion can be written in the form

$$i\beta^k\psi_{|k} - m\psi = 0, \quad i\bar{\psi}_{|k}\beta^k + m\bar{\psi} = 0. \qquad (4.13)$$

From (4.13) and the condition $\beta_{i|j} = 0$ there follows the generalized condition of conservation of the current $\bar{\psi}\beta^i\psi$ of the spinor particles

$$(\bar{\psi}\beta^i\psi)_{|i} = 0. \qquad (4.14)$$

The relations (4.11), (4.12), and (4.9) also enable us to find that

$$(\bar{\psi}\beta^i\psi)_{,i} = (\bar{\psi}\beta^i_{,i}\psi) = h^i_{(a),i}(\bar{\psi}\gamma^{(a)}\psi). \qquad (4.15)$$

When we use the Maxwell equations (3.17) we find from this that in first approximation

$$(\bar{\psi}\beta^i\psi)_{,i} = 0. \qquad (4.16)$$

The question of current conservation in higher approximations requires that the current in the right members of the Maxwell equations be taken into account, and is beyond the scope of the present paper.

Let us now go to the limit of small torsion in (4.13). Confining ourselves to first-order quantities, we get

$$i\gamma^k(\partial_k - ieA_k)\psi - m\psi + if^{kl}(\partial_k - ieA_k)\gamma_l\psi = 0, \quad (4.17)$$

$$i(\partial_k + ieA_k)\bar{\psi}\gamma^k + m\bar{\psi} + if^{kl}(\partial_k + ieA_k)\bar{\psi}\gamma_l = 0, \quad (4.18)$$

where $\gamma^k = l^k_{(a)}\gamma^{(a)}$. The last terms in these equations correspond to the interaction Lagrangian

$$\mathcal{L} = \frac{i}{2}\lambda Ge[\bar{\psi}(\partial_k - ieA_k)\gamma_l\psi - (\partial_k + ieA_k)\bar{\psi}\gamma_l\psi]H^{kl}, (4.19)$$

which we had constructed earlier[20-22] on the basis of intuitive considerations about the connection of the electromagnetic field with a torsion of space-time. This Lagrangian is CP-odd and C-odd.

Such a simple Lagrangian, however, can still not explain the observed nonconservation of CP, and it must be extended somewhat to the case of interaction of different spinor particles (in particular, with change of strangeness). Besides this, to include terms which do not conserve parity it is necessary to make the interaction (4.19) γ_5 invariant. We have done such work earlier,[20-22] and the derivation of these hypotheses is obviously beyond the scope of the simplified model considered here, since it requires the essential unification of the weak and electromagnetic interactions in a single theory.

5. CONCLUSION

The main results of this paper are a proof that it is possible to interpret the electromagnetic field as a torsion of space-time, and a derivation of the equations for the electromagnetic field from simple geometric considerations. It is extremely important that this geometric theory of electromagnetism does not contradict the usual Maxwell equations, but on the contrary allows us to derive them and give an estimate of their range of applicability. Another important result is the derivation of a CP-odd interaction of spinor particles with the electromagnetic field, which arises quite automatically in the geometric theory, without any additional assumptions.

The main unsolved problem remaining is the construction of a unified theory of the weak and electromagnetic interactions. To solve it it will evidently be necessary to try to combine the ideas of the geometrization of electromagnetism with the ideas of the geometrization of the weak interactions.[17,19]

Even within the range of ideas of the present paper, however, there are some interesting problems. A very interesting one is the study of the nonlinear equations which are a generalization of the Maxwell equations. It would be useful to write them in Lagrangian form and try to construct a vector potential useful also in the general nonlinear case. A very interesting and difficult problem is that of the global structure of the pseudoeuclidean space-time and the possibility of constructing a continuous spinor field in the entire space-time.

APPENDIX

We shall derive the relation between the symmetric and antisymmetric parts of a pseudoorthogonal matrix. With a transformation SLS^{-1} (where S is a pseudoorthogonal matrix) any pseudoorthogonal matrix L_{ik} can be brought into one of the forms (cf., e.g.,[44])

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17)In calculations it is helpful to note the fact that the covariant derivatives of the β matrices are zero, by virtue of the definition (4.2) and the relations (2.14).

$$L_i^{(1)j} = \left(\begin{array}{cc|cc} \operatorname{ch}\chi & \operatorname{sh}\chi & \multicolumn{2}{c}{0} \\ \operatorname{sh}\chi & \operatorname{ch}\chi & & \\ \hline \multicolumn{2}{c|}{0} & \cos\varphi & \sin\varphi \\ & & -\sin\varphi & \cos\varphi \end{array}\right),$$

$$L_i^{(2)j} = \left(\begin{array}{ccc|c} 1+\frac{t^2}{2} & -\frac{t^2}{2} & t & 0 \\ t^2/2 & 1-t^2/2 & t & \\ t & -t & 1 & \\ \hline \multicolumn{3}{c|}{0} & 1 \end{array}\right). \qquad (A.1)$$

Let us set

$$L_{\ i}^{(1,2)j} = \delta_i{}^j + G_{\ i}^{(1,2)j}. \qquad (A.2)$$

For the matrix $G^{(1)}$ it is easy to find the relation

$$\bar{G}_{ij}^{(1)} = \frac{1}{4}\left(\bar{s} - \frac{2s}{\bar{s}+4}\right)\delta_{ij} + \frac{2}{\bar{s}+4}\hat{G}_{ik}^{(1)}\delta^{kl}\hat{G}_{lj}^{(1)}, \qquad (A.3)$$

where

$$\bar{s} = \bar{G}_i{}^i = G_i{}^i, \qquad s = \hat{G}_i{}^k G_k{}^i, \qquad (A.4)$$

$$\bar{G}_{ij}^{(1)} = {}^1/_2(G_{ij}^{(1)} + G_{ji}^{(1)}), \quad \hat{G}_{ij}^{(1)} = {}^1/_2(G_{ij}^{(1)} - G_{ji}^{(1)}). \qquad (A.5)$$

From the equations

$$d = -uv, \quad s = 2(u-v), \qquad \bar{s} = 2\sqrt{1+u} + 2\sqrt{1-v}, \qquad (A.6)$$

where

$$d = \det(\hat{G}), \quad u = \operatorname{sh}^2\chi, \quad v = \sin^2\varphi, \qquad (A.7)$$

we can find the expression for $\bar{s}$ in terms of s and d.

It is easily verified that the matrix $G^{(2)}$ satisfies the relation

$$\bar{G}_{ij}^{(2)} = {}^1/_2\hat{G}_i^{(2)k}\hat{G}_{kj}^{(2)}, \qquad (A.8)$$

which is of the same form as (A.3) if we set

$$\bar{s}_2 = s_2 = d_2 = 0. \qquad (A.9)$$

Since the relation (A.3) is invariant under pseudoorthogonal transformations, it has been proved that the relation (A.3) holds for any pseudoorthogonal matrix.

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