"MOST DANGEROUS" INSTABILITIES OF A COLLISION-DOMINATED PLASMA IN A MAGNETIC FIELD

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The instabilities of a collision-dominated plasma that are the most difficult to stabilize by the use of shear are investigated. The analysis is based on the equations of two-fluid hydromagnetics in which the ion motion in the direction of the magnetic field and temperature perturbations are included. The instabilities of a collision-dominated plasma are classified. Estimates are given of the shear required to stabilize the various instabilities.

1. INTRODUCTION

IN the present work, by "most dangerous" we mean those plasma instabilities that are the most difficult to stabilize by the use of shear in a magnetic field. The role of shear in the stabilization of drift instabilities in a plasma was first studied in [1–3]. It was assumed in this work that collisions between the plasma particles are not important (collisionless-plasma approximation). Under these conditions primary attention was focused on the case in which the plasma density is inhomogeneous in space but the temperature is uniform. Later authors, in particular Kadomtsev and Pogutse, carried out a more complete analysis of the stabilizing role of shear taking account of temperature inhomogeneities and collisions. However, Kadomtsev and Pogutse [4] did not consider certain kinds of plasma instabilities that can be important when particle collisions are frequent (collision-dominated plasma). Until recently these instabilities have not been treated by most authors because, as a rule, the conventional system of hydromagnetic equations neglects the ion motion along the magnetic field and also neglects certain features of the temperature perturbations. This point has been made by Moiseev.[5]

It is the purpose of the present work to obtain a general pattern for the most dangerous instabilities of a collision-dominated plasma. Primary attention is given to the analysis of the shortwave (along the field) plasma perturbations in the approximation in which the magnetic field is taken to be uniform (no shear). The results of this analysis are then used to obtain estimates of the shear required to stabilize the plasma.

2. FORMULATION OF THE PROBLEM AND INITIAL ASSUMPTIONS

As in most present-day work on the stability of collision-dominated plasmas (cf. the survey in [6]) we assume that the plasma pressure p is small compared with the pressure of the magnetic field $B^2/8\pi$. The plasma perturbations are described in terms of a system of macroscopic equations which have been obtained in [7] 1) It is assumed that all the conditions listed in Secs. 1 and 2 of Part 2 of [7] are satisfied; in particular, we assume that the magnetic field is straight and uniform and that the perturbation electric field is potential.

Using the conventional procedure for linearizing the equations [6] and writing all perturbations in the form $\exp(-i\omega t + ik_y y + ik_z z) f(x)$ where x is the direction of the plasma inhomogeneity, using (3.22) of [7] one obtains the following relation between the perturbation frequency ω and the wave vector k (dispersion equation):

$$\begin{vmatrix} \omega-\omega_n & -1 & -\omega_n(1+s) & 0 \\ -2(k_z v_i)^2 & \omega+i\Delta_1 & -(k_z v_i)^2 & -(k_z v_i)^2 \\ -\omega_T & -{}^2/_3 & \omega-(1+s)\omega_T+i\Delta_2(1+2\nu) & -i\Delta_2 \\ -\omega_T & -{}^2/_3 & -i\Delta_2-\omega_T(1+s) & \omega+ic_1\Delta_1+i\Delta_2 \end{vmatrix} = 0. \qquad (2.1)$$

1)Some of the results seen in the present work can also be obtained through the use of the equations given by Braginskiĭ. [8]

Here

$$\omega_n = -k_y \frac{cT_0}{eB_0}\frac{\partial \ln n_0}{\partial x}, \quad \omega_T = -k_y \frac{c}{eB_0}\frac{\partial T_0}{\partial x},$$

$$\Delta_1 = {}^4/_3 \cdot 0.96 (k_z v_i)^2 \tau_i, \quad \Delta_2 = \frac{2m_e}{m_i \tau_e}, \quad \nu = \frac{(k_z v_i)^2}{2c_2 \Delta_2^2},$$

$$c_1 = \frac{3.9}{2 \cdot 0.96}, \quad c_2 = \frac{1}{2 \cdot 3.16}, \quad s = 0.71, \quad v_i^2 = \frac{T_0}{m_i}. \tag{2.2}$$

The zero subscript denotes unperturbed quantities. The remaining notation is the same as in [7].

In deriving (2.1) it is assumed that the electrons and ions have the same temperature T_0 in the unperturbed state and that the velocity of each plasma component along the magnetic field is zero, $V_{0ze} = V_{0zi} = 0$. In addition, we have neglected the transverse thermal conductivity and small deviations from plasma neutrality. It follows from (2.1) that a given wave vector $\mathbf{k}$ corresponds to four oscillation branches which differ from each other by the value of the frequency ω. These various oscillation branches are investigated in Secs. 3—6.

If the solution of (2.1) is found and the functions

$$\mathrm{Re}\,\omega_\alpha = \mathrm{Re}\,\omega_\alpha(\mathbf{k}), \quad \mathrm{Im}\,\omega_\alpha \equiv \gamma_\alpha = \gamma_\alpha(\mathbf{k}), \quad \alpha = 1, 2, 3, 4 \tag{2.3}$$

are known then, using the method developed in [4,9,10], one can obtain approximate values for the critical shear required for stabilization of the instabilities. Let us recall some of the basic features of the method.

First, the relations in (2.3) must be written in terms of a plasma in a magnetic field with shear. In this case the quantities k_y and k_z are replaced by k_b and $k_\parallel$ where k_b is the projection of the wave vector along the binormal to the line of force while $k_\parallel \equiv (\mathbf{k} \cdot \mathbf{B})\ \mathrm{B}$ is given by

$$k_\parallel \approx k_b \theta x_0 / a_\perp. \tag{2.4}$$

Here, θ is the change in direction of the line of force in a distance of the order of the dimensions of the plasma inhomogeneity $a_\perp$, while x_0 is a length of the order of the region of localization of the perturbation in the direction of the plasma inhomogeneity (that is to say, the distance along x over which f(x) is nonvanishing).

The procedure for determining x_0 is the following.[4,9,10] If $\gamma \gtrsim \mathrm{Re}\,\omega$ we can

$$x_0 \approx 1/k_b. \tag{2.5}$$

When $\gamma \ll \mathrm{Re}\,\omega$, the quantity x_0 is taken to be the smaller of the quantities

$$x_0 \simeq \min\left\{a_\perp, \frac{1}{\gamma}\frac{\partial\,\mathrm{Re}\,\omega}{\partial k_x}\right\} \tag{2.6}$$

(k_x is the wave vector in the direction of the inhomogeneity).

Knowing the functions in (2.3) we can use (2.5) and (2.6) to determine x_0. On the other hand, substituting in (2.4) the wave number $k_\parallel = k_{\parallel\mathrm{eff}}$ corresponding to these values of the frequency and growth rate we can find the value of the parameter θ at which an instability is just possible (critical θ):

$$\theta \approx \theta_0 \approx \frac{k_{\mathrm{eff}}}{k_b}\frac{a_\perp}{x_0}. \tag{2.7}$$

It follows from (2.5)—(2.7) that for the largest-scale perturbations, $k_b a_\perp \approx 1$, the larger the value of $k_{\parallel\mathrm{eff}}$ the larger the critical value of θ,

$$\theta_0 \approx k_{\parallel\,\mathrm{eff}} / a_\perp. \tag{2.8}$$

Hence, when $k_b a_\perp \approx 1$ the most dangerous instabilities are those with the minimum wavelength along the field (largest $k_\parallel$). It is also clear that when k_b, γ and $\mathrm{Re}\,\omega$ remain constant, the most dangerous instabilities are those for which $k_\parallel$ is largest. For these reasons these kinds of instabilities will receive primary attention.

3. HYDRODYNAMIC DRIFT-TEMPERATURE INSTABILITY

In the present section we assume as a rough approximation that

$$\omega \sim k_z v_i \sim \omega_T \tag{3.1}$$

and that each of the quantities in (3.1) is large compared with Δ_2. Furthermore, we neglect terms like $\omega\tau_i$. [The assumption that these terms are small is the basis of the derivation of (2.1)]. It then follows that

$$1 - \frac{\omega_n}{\omega} - \left(\frac{k_z v_i}{\omega}\right)^2 \left[\frac{8}{3} + \frac{\omega_n}{\omega}\left(\eta - \frac{2}{3}\right)\right] = 0; \quad \eta \equiv \frac{\partial \ln T_0}{\partial \ln n_0}. \tag{3.2}$$

1. Plasma Instability with $\eta \gg 1$; Growth Rate and Stability Limits

Baĭkov [11] has shown that among the solutions of (3.2) there are solutions for which $\mathrm{Im}\,\omega > 0$. These solutions correspond to unstable perturbations. We now find the maximum growth rate and the stability limits for this instability. Since the existence of a density gradient plays a stabilizing role (this has been shown in [11] and is also shown in a later part of the present work), it is sufficient to consider the case $\omega_n = 0$ in finding the maximum growth rate. Starting from (3.2) we find that an instability arises if

$$k_z v_i \leqslant k_{zg} v_i = \frac{27}{32\sqrt{2}}|\omega_T| \simeq 0.6\,\omega_T. \tag{3.3}$$

The growth rate γ as a function of k_z reaches a maximum value given by

$$\gamma_{max} \approx 0.21\omega_T \tag{3.4}$$

at a value $k_z = k_{z\,opt}$ given by the relation

$$k_{z\,\mathrm{opt}}\, v_i \approx 0.37|\omega_T|. \tag{3.5}$$

The real part of ω is very small for $k_z = k_{opt}$ and $\mathrm{Re}\,\omega \ll \gamma$, so that the instability is almost "aperiodic."

2. Instability Limits for Arbitrary η

It follows from (3.2) that the instability condition for arbitrary η is given by

$$\frac{1}{4}\left[\frac{2}{27} + \alpha\eta^2\left(\frac{2}{9}+\eta\right)^2\right]^2 - \frac{1}{(27)^2}(8\alpha\eta^2+1)^3 \geqslant 0, \tag{3.6}$$
$$\alpha = (k_z v_i/\omega_T)^2.$$

Since a is a positive quantity by definition, the condition in (3.6) can only be satisfied if

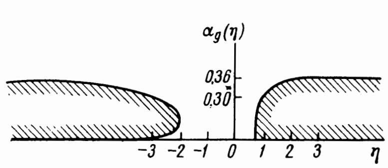
$$\eta \geqslant 2/3, \qquad \text{or} \qquad \eta \leqslant -2. \tag{3.7}$$

This means that the instability is possible only in a plasma in which the temperature gradient is nonzero and not too small. [Only the first inequality in (3.7) was obtained by Baĭkov.[11] According to a private communication from V. D. Shafranov, the instability criterion $\eta > 2/3$ has also been given in[12] but the details are not known to the present author.]

If the condition in (3.7) is satisfied the dependence of the limiting value $\alpha_g = \alpha_g(\eta)$ is given by

$$\alpha_g = \frac{1}{27\eta^2}\left\{-\frac{183}{8} + \frac{9^3\eta}{32}\left(\frac{4}{9}+\eta\right) \pm \left[\left[\frac{183}{8} - \frac{9^3\eta}{32}\left(\frac{4}{9}+\eta\right)\right]^2 - 64\cdot 9\left(1-\frac{3}{2}\eta\right)\right]^{1/2}\right\}. \tag{3.8}$$

This functional dependence is shown in the figure (the cross hatched regions correspond to instability).



4. DISSIPATIVE DRIFT-TEMPERATURE INSTABILITY

Let us now assume that the condition in (3.3) is not satisfied. There will be no instability for the large values of k_z considered in the preceding section. However, it still does not follow that the plasma is stable against these perturbations, because we have neglected dissipative effects (viscosity etc.) in the dispersion relation (3.2); as a result, the analysis given above is not complete. Taking account of these effects can mean that the perturbation frequency will exhibit a small positive imaginary part when $k_z > k_{zg}$. Let us consider the situation in which this occurs.

We assume that the original assumptions in Sec. 3 still hold. However, in addition to the principal terms in the dispersion relation (2.1) we now include small corrections. As a result we find

$$D = D_0 + \frac{i\Delta_1}{\omega_T}a_1 + \frac{i\Delta_2}{\omega_T}a_2 = 0, \tag{4.1}$$

where

$$D_0 = \alpha\left[x^2\left(x-\frac{1}{\eta}\right) - \alpha\left(\frac{8}{3}x + 1 - \frac{2}{3\eta}\right)\right],$$
$$a_1 = \alpha\left[(c_1+1)\left(x-\frac{1}{\eta}\right)x - 2c_1\alpha\right],$$
$$a_2 = \alpha\left[x\left(x-\frac{1}{\eta}\right) - 2\alpha\right] - c_2 x\left\{x^3 - x^2\left(1+s+\frac{1}{\eta}\right) - 10/3\,\alpha x + 2\alpha(1-2/3\eta)\right\},$$
$$x = \omega/\omega_T, \qquad \alpha = (k_z v_i/\omega_T)^2. \tag{4.2}$$

Let us now consider the following limiting cases.

1. Viscosity Instability

We assume that $\Delta_1 \gg \Delta_2$. This means that the most important dissipative effects are the longitudinal viscosity and the longitudinal thermal conductivity. The condition $\Delta_1 \gg \Delta_2$ can be written in the following more transparent form:

$$k_z v_i \gg \frac{1}{\tau_i}\left(\frac{m_e}{m_i}\right)^{1/4}. \tag{4.3}$$

Neglecting terms containing Δ_2 in (4.1), we find from the resulting equation that if (3.3) is not satisfied the perturbations are still unstable (dissipative instability). The limits for this instability are given by

$$\alpha_g = \frac{x^2(x-1/\eta)}{8/3x + 1 - 2/3\eta}, \tag{4.4}$$

where

$$x = \frac{c_1+1}{c_1+4}\left(\frac{1}{\eta} - \frac{3}{2}\right). \tag{4.5}$$

Substituting the numerical value of c_1 from (2.2), we have as an approximation

$$\alpha_g = \frac{27}{64}\left(1 - \frac{4}{9\eta^2}\right). \tag{4.6}$$

The maximum value of α_g is reached when $|\eta| \to \infty$. This value of α_g corresponds to a limiting value of k_{zg} given by

$$k_{zg}v_i = 3\sqrt{3}|\omega_T|/8 \approx 0.64|\omega_T|. \tag{4.7}$$

The growth rate and frequency of this perturbation are of order (numerical coefficients are omitted)

$$\gamma \sim \tau_i\omega_T^2, \quad \mathrm{Re}\,\omega \sim \omega_T. \tag{4.8}$$

Comparing (4.7) and (3.3) we conclude that allowance for the longitudinal ion viscosity and the longitudinal ion thermal conductivity expands the instability region. However, if the inequality in (3.7) holds, the numerical value of this expansion is small. Hence, in practice, the resulting stability limit in a plasma with $\eta > 2/3$ and $\eta < -2$ remains essentially unchanged and is given by the relations of Sec. 3. As far as the case $-2 < \eta < 2/3$ is concerned, however, here the most dangerous instability is the dissipative instability treated in the present section.

The viscosity instability cannot be treated with the equations derived by Braginskiĭ [8] because these equations, as noted in, [7] contain "superfluous terms" in the viscosity tensor which cancel the second-order terms that are neglected in [8]. In particular, these equations could lead to the erroneous conclusion that there is a viscosity instability in a plasma in which the temperature is spatially uniform.

2. Heat-exchange Instability

In the present case we assume that the dominant effects are heat exchange between the ions and the electrons and perturbations of the electron temperature due to the fact that the thermal conductivity is not high enough. In accordance with (4.1), the initial dispersion equation for $\Delta_2 \gg \Delta_1$ is

$$D_0 + i\Delta_2 a_2 = 0. \tag{4.9}$$

It is easy to show that all perturbation described by (4.9) are characterized by $\mathrm{Im}\,\omega \le 0$ if $\nabla T_0 = 0$ and $\nabla n_0 \ne 0$. (This statement can be proved simply by assuming that it does not hold: let us assume that the instability occurs and write the equation for the stability limits. It will then be evident that this equation cannot be satisfied under any conditions.)

Let us now consider the other limiting case $\nabla T_0 \ne 0$ and $\nabla n_0 = 0$. As follows from (4.2) and (4.9) the stability limit for $\nabla n_0 = 0$ is given by the relations

$$x^3 - {}^8/_3\alpha_g x - \alpha_g = 0,$$

$$\alpha_g(x^2 - 2\alpha_g) - c_2 x\left[x^3 - x^2\left(1 + s + \frac{1}{\eta}\right) - \frac{10}{3}\alpha_g x + 2\alpha_g\left(1 - \frac{2}{3\eta}\right)\right] = 0. \tag{4.10}$$

Since the factor c_2 is numerically small ($c_2 \approx 0.16$), as an approximation we can take this quantity equal to zero. Physically this means that the basic dissipative effect is heat exchange and not the effect of the finite electron thermal conductivity. As a result we get from (4.10)

$$\alpha_g \approx {}^9/_8, \tag{4.11}$$

and the value of α_g corresponding to this value of k_{zg} is

$$k_{zg}v_i = \frac{3}{2\sqrt{2}}|\omega_T| \approx 1.1|\omega_T|. \tag{4.12}$$

A comparison of (4.12) with (3.3) and (4.7) shows that heat exchange between the ions and electrons in a plasma with $|\eta| \gg 1$ can expand the instability region to a somewhat greater degree than the ion viscosity. The limiting value of k_z for the heat-exchange instability exceeds the value of k_{zg} for the hydrodynamic instability by approximately a factor of 2. The growth rate associated with the heat-exchange instability is still rather small so that

$$\gamma \sim \Delta_2, \quad \mathrm{Re}\,\omega \sim \omega_T. \tag{4.13}$$

An analysis of intermediate values of η, the details of which are not given here, shows that the heat-exchange dissipative instability occurs when

$$\eta > {}^2/_3 \quad \text{and} \quad \eta < 0. \tag{4.14}$$

3. Instability due to Finite Electron Thermal Conductivity

It follows from (4.9) that the growth rate for the instability considered in Sec. 2 increases as k_z is reduced. Under these conditions, however, the relative roles of heat exchange and the finite electron thermal conductivity become different than they are when $k_z v_i \approx \omega^*$. If we assume $\omega \approx \omega^*$, then heat exchange can be neglected in (4.9) if $(k_z v_i/\omega^*)^2$ is smaller than the numerical value of the small parameter c_2. Omitting terms containing the heat exchange in (4.9) and writing $\omega \approx \omega^*$, we obtain an equation that has already been obtained by Galeev, Oraevskiĭ, and Sagdeev [13]

$$\omega^2 - \omega\left[\omega_n + (1+s)\omega_T - \frac{i(k_z v_i)^2}{c_2\Delta_2}\right] - \omega_n \frac{i(k_z v_i)^2}{c_2\Delta_2} = 0. \quad (4.15)$$

It follows from (4.15) that an instability arises if

$$\eta < 0. \quad (4.16)$$

The growth rate reaches a maximum of order

$$\gamma_{max} \sim |\omega_T \omega_n|^{1/2} \quad (4.17)$$

where

$$k_z v_i \sim \omega^* (\Delta_2/\omega^*)^{1/2}, \quad \omega^* = \max(\omega_T, \omega_n). \quad (4.18)$$

5. UNSTABLE PERTURBATIONS WITH $\omega \lesssim \Delta_2$

In Secs. 3 and 4 we have assumed that heat exchange between the ions and electrons is small but that the electron thermal conductivity is high. In terms of the notation being used here this corresponds to the approximation (cf. beginning of Sec. 3):

$$\min\{\omega, k_z v_i, \omega^*\} \gg \Delta_2, \quad (5.1)$$

In some sense the ratio ω^*/Δ_2 characterizes the degree to which the plasma is collision-dominated. We will call the plasma weakly collision-dominated or highly collision-dominated depending on whether this ratio is large or small. From this point of view the criterion in (5.1) means that in Secs. 3 and 4 we have been considering instabilities of a weakly collision-dominated plasma, but only those for which $\omega \gg \Delta_2$. Thus, in order to obtain the complete behavior picture that follows from the dispersion equation (2.1) we must also consider the following limiting cases: perturbations of a weakly-collision dominated plasma with $\omega \lesssim \Delta_2$ and perturbations of a highly collision-dominated plasma.

1. Weakly Collision Dominated Plasma

The inequalities $\omega \lesssim \Delta_2$ and $\omega^* \gg \Delta_2$ imply that $\omega \ll \omega^*$. In (2.1) we now omit terms of order ω/ω^* and terms containing Δ_1 (the fact that the latter is small will be shown later); in this way we obtain the following cubic equation for the perturbation frequency $\omega = 2i\Gamma\Delta_2$:

$$\Gamma(\Gamma+1)(\Gamma+\nu) + 2\nu c_2 s(\eta - 2/3)(\Gamma + 1 - \nu/2) = 0, \quad (5.2)$$

where

$$\nu = (k_z v_i)^2 / 2c_2\Delta_2^2.$$

Using the fact that the parameter $2c_2 s \approx 0.22$ is small, we can solve this equation in terms of Γ. Assuming that ν is of order unity we have

$$\Gamma_1 = -1, \quad \Gamma_2 = -\nu, \quad \Gamma_3 = 2c_2 s(2/3 - \eta)(1 - \nu/2). \quad (5.3)$$

The first two roots correspond to damped perturbations characterized by $\mathrm{Re}\,\omega = 0$, while the third corresponds to a growing perturbation when

$$(2/3 - \eta)(1 - \nu/2) > 0. \quad (5.4)$$

and a damped perturbation in all other cases. It follows from (5.4) that when $\eta > 2/3$ the perturbation is unstable so long as k_z is not too small

$$k_z v_i > 4c_2 m_e / m_i \tau_e. \quad (5.5)$$

This is, in fact, the lower limit for the drift-temperature instabilities considered in Secs. 3 and 4.

It also follows from (5.4) that an instability is possible when $\eta < 2/3$ if the inverse condition to (5.5) is satisfied:

$$k_z v_i < 4c_2 m_e / m_i \tau_e. \quad (5.6)$$

This is a new kind of plasma instability, the possibility of which was pointed out recently by Moiseev.[5] It follows from (5.3) that for unstable perturbations

$$\gamma \approx 0.22\,\Delta_2\left(\frac{2}{3} - \eta\right)\left[1 - \frac{(k_z v_i)^2}{4c_2\Delta_2^2}\right], \quad \mathrm{Re}\,\omega \ll \gamma. \quad (5.7)$$

We now wish to consider the change in the solution of (5.2) when the parameter ν is reduced (i.e., reduction of k_z). In this case (5.2) has one large root, $\Gamma = -1$, and two small roots which satisfy the approximate equation

$$\Gamma^2 + \nu\Gamma + 2c_2 s\left(\frac{2}{3} - \eta\right) = 0. \quad (5.8)$$

It then follows that when $\nu \ll 8c_2 s(2/3 - \eta)$

$$\omega^2 = -4/3(1 - 3/2\,\eta)s(k_z v_i)^2. \quad (5.9)$$

This result was obtained by Moiseev[5] for $\eta = 0$ and by Baĭkov for $\eta \neq 0$.[11] From the analysis given above it follows that the growth rates found in these two papers through the use of (5.9) do not represent the maximum growth rate; the latter is actually determined by the approximate relation (5.7).

2. Highly Collision Dominated Plasma, $\sigma \equiv \Delta_2/\omega_n \gg 1$.

In (2.1) we now take $\sigma \gg 1$ and omit terms of order $1/\sigma$, thereby obtaining a cubic equation for $x \equiv \omega/\omega_n$:

$$x^3 - x^2[1 + \eta(1+s)] - 10/3\,x + 2\xi s(\eta - 2/3) = 0,$$

$$\xi = (k_z v_i/\omega_n)^2. \quad (5.10)$$

	k_z	γ	Conditions for onset of instability	Type of instability	Formulas used
I	$\approx k_{z\,max}$	$\approx \gamma_{max}$	a) $\omega^*\tau_i \gtrsim \mu$, $\eta > 2/3$, $\eta < -2$	Drift-temperature (hydrodynamic)	(3.3) — (3.7)
			b) $\omega^*\tau_i \lesssim \mu$, $-0.6 < \eta < 2/3$	Strong thermal	(5.11) — (5.14)
II	$\approx k_{z\,max}$	$\ll \gamma_{max}$	a) $\omega^*\tau_i > \mu^{1/2}$, $-2 < \eta < -2/3$	Viscosity	(4.6) — (4.8)
			b) $\mu^{1/2} > \omega^*\tau_i > \mu$, $-2 < \eta < 0$	Heat exchange	(4.12) — (4.14)
			c) $\omega^*\tau_i \ll \mu$, $\eta > 2/3$, $\eta < -0.6$	Dissipative instability of strong collision dominated plasma	(5.17) — (5.19)
III	$\ll k_{z\,max}$	$\approx \gamma_{max}$	$\omega^*\tau_i > \mu$, $\eta < 0$	Thermal-conductivity	(4.15) — (4.18)
IV	$\ll k_{z\,max}$	$\ll \gamma_{max}$	$\omega^*\tau_i > \mu$, $0 \leqslant \eta < 2/3$	Strong thermal instability	(5.6), (5.7)
V	$\lll k_{z\,max}$	$\approx \gamma_{max}$	Arbitrary η and $\omega^*\tau_i$	Drift-dissipative	(6.3), (6,4)

The solution of this equation for $\xi \ll 1$ and $\eta = 0$ has been obtained by Moiseev[5] and by Baĭkov[11] for $\xi \ll 1$ and $\eta \neq 0$. These authors have shown that there are unstable perturbations for which a necessary condition is[11]

$$-\frac{1}{1+s} < \eta < \frac{2}{3}. \tag{5.11}$$

We now consider the most dangerous perturbations corresponding to $\xi \sim 1$, limiting ourselves to $\eta = 0$. It follows from (5.10) that when $\eta = 0$ the instability limit is given by the relation $\xi_g = 1.6$ or, in more explicit form,

$$k_{zg} v_i \approx 1.3 |\omega_n|, \tag{5.12}$$

where the instability corresponds to $k_z < k_{zg}$. The maximum growth rate is reached for

$$k_{z\,\mathrm{opt}} v_i \simeq 0.45 |\omega_n|, \tag{5.13}$$

where

$$\gamma(k_{z\,\mathrm{opt}}) = \frac{\sqrt{3}}{2} \cdot 0.27\omega_n, \quad \mathrm{Re}\,\omega(k_{z\,\mathrm{opt}}) = -0.25\omega_n. \tag{5.14}$$

Let us now consider the stability of a highly collision-dominated plasma in the case in which (5.11) is not satisfied. Under these conditions we must retain terms of order $1/\sigma$ in the dispersion equation which have omitted in (5.10) (cf. Sec. 4). Then (5.10) is replaced by

$$x^3 - x^2[1 + (1+s)\eta] - {}^{10}/_3\xi x + 2\xi s(\eta - 2/3) + i\xi(2c_2\sigma)^{-1}(x^2 - x - 2\xi) = 0. \tag{5.15}$$

Using (5.15) we find that the limit for the dissipative instability is given by the relation

$$\frac{16}{9}\xi_g^2 + \xi_g\left[2\left(\frac{2}{3} - \eta\right)\eta - \frac{8}{9}s(s+1)\right] + s(s+1)\eta\left(\eta - \frac{2}{3}\right) = 0. \tag{5.16}$$

It follows, in particular, that when $|\eta| \gg 1$ the plasma is unstable if

$$k_z v_i \ll k_{zg} v_i = {}^3/_4 |\omega_T|. \tag{5.17}$$

When $\xi/\eta^2 \ll 1$ the real and imaginary parts of the frequency are given by

$$\mathrm{Re}\,\omega = -{}^{10}/_3 \frac{\omega_T}{1+s}, \quad \mathrm{Im}\,\omega \equiv \gamma = \frac{3}{20c_2} \frac{(k_z v_i)^2}{\Delta_2}. \tag{5.18}$$

The maximum growth rate for the dissipative instability obtains when $\xi/\eta^2 \sim 1$ and is of the following order:

$$\gamma_{max} \approx \omega_T^2/\Delta_2. \tag{5.19}$$

6. GENERAL BEHAVIOR OF THE MOST DANGEROUS INSTABILITIES IN A COLLISION-DOMINATED PLASMA

1. It follows from the analysis given above that, in particular, an inhomogeneous collision-dominated plasma located in a uniform magnetic field can be subject to time-growing perturbations only if those perturbations satisfy the approximate relation

$$k_z v_i \lesssim \omega^*. \tag{6.1}$$

It is interesting to note that this same criterion holds for a collisionless plasma, as indicated in[14].[2)]

2. The present analysis allows us to answer the question of whether there exist in a collision-dominated plasma unstable perturbations with the largest possible k_z such that

2)The criterion in (6.1) has been given earlier by Kadomtsev and Pogutse [4] as an upper estimate for drift instabilities for both collision-dominated and collisionless plasmas; however, no proof of this relation is given in [4]

$$k_{z\,max} v_i \approx \omega^*. \qquad (6.2)$$

The notion of $k_{z\,max}$ as given by (6.2) has been introduced by Kadomtsev and Pogutse.[4] However the question has, up to the present time, been only examined for the case of a collisionless plasma (unstable perturbations characterized by $k_z \sim k_{z\,max}$ in such a plasma correspond to the drift-temperature instability considered by Rudakov and Sagdeev).[15]

It follows from the present analysis that in a weakly collision-dominated plasma $\omega^* > \Delta_2$ the approximate relation in (6.2) holds for instabilities considered in Secs. 3 and 4 and for certain of the instabilities considered in Sec. 5 for a highly collision dominated plasma.

3. The largest possible k_z is not achieved for arbitrary values of η and ω^*/Δ_z. Not all perturbations characterized by $k_z \sim k_{z\,max}$ correspond to the largest growth rate $\gamma_{max} \approx \omega^*$. All the instabilities considered above can be classified in accordance with the scheme in items I-IV of the table [in the table $\mu \equiv (m_e/m_i)^{1/2}$].

4. The relation in (3.22) of[7] does not take account of the transverse inertia or the transverse viscosity of the ions [8], both of which are important for perturbations characterized by very small values of k_z. When these effects are taken into account, the dispersion equation for small values of k_z assumes the form [6,16]

$$\omega^2 + \omega\left(\omega_n + \omega_T + \frac{2i(k_z v_i)^2}{\Delta_2 (k_\perp \rho_i)^2}\right) - \frac{2i}{\Delta_2}[\omega_n + (1+s)\omega_T]\left(\frac{k_z v_i}{k\rho_1}\right)^2 = 0 \qquad (6.3)$$

Perturbations described by these equations are unstable for arbitrary values of η and ω^*/Δ_2. These perturbations are characterized by the following values of growth rate and longitudinal wave number:

$$\gamma \approx \mathrm{Re}\,\omega \approx \gamma_{max}, \quad k_z \approx k_{z\,max}\frac{\rho_i}{a_\perp}\left(\frac{a_2}{\omega^*}\right)^{1/2}. \qquad (6.4)$$

The results of this section are entered in item V of the table.

7. ESTIMATES OF REQUIRED SHEAR

The effect of shear on plasma stability becomes manifest at values

$$\theta \gtrsim \theta^{\mathrm{V}}_{min} \approx \frac{\rho_i}{a_\perp}\left(\frac{\Delta_2}{\omega_{Bi}}\right)^{1/2}. \qquad (7.1)$$

At this level perturbations given in V of the table characterized by $k_b a_\perp \sim 1$ and $\gamma \approx \gamma_{max}$ are stabilized. The next in ascending value of θ (for a weakly collision dominated plasma $\rho_i v_i/a_\perp^2 > \Delta_2$) is given by

$$\theta^{\mathrm{IV}} \simeq \frac{1}{S}\left(\frac{m_e}{m_i}\right)^{1/2}\frac{\rho_i}{a_\perp} \qquad (7.2)$$

where $S \equiv \rho_i l/a_\perp^2$ and $l \equiv v_i \tau_i$ is the particle mean free path.[3)]

Perturbations of type IV are stabilized when $\theta \gtrsim \theta^{\mathrm{IV}}$. For somewhat larger values of θ the effect of shear is manifest on stabilities of type III

$$\theta^{\mathrm{III}} \approx \frac{1}{S^{1/2}}\left(\frac{m_e}{m_i}\right)^{1/4}\frac{\rho_i}{a_\perp}. \qquad (7.3)$$

Perturbations of type I and II are stabilized at still higher values of θ

$$\theta \gtrsim \theta^{\mathrm{I}}_{min} \approx \theta^{\mathrm{II}}_{min} \approx \frac{\rho_i}{a_\perp}. \qquad (7.4)$$

We may assume here that at least those perturbations characterized by $k_b a \approx 1$ are stable.

Using the results of the preceding sections we can obtain estimates of θ for perturbations characterized by $k_b a \gg 1$. In general, stabilization of these perturbations requires larger values of θ than does the case of $k_b a \approx 1$. In particular, the criterion for stabilization of perturbations of type I for $k_b a \gg 1$ is of the form

$$\theta \gtrsim k_b a_\perp \theta^{\mathrm{I}}_{min} \approx k_b \rho_i. \qquad (7.5)$$

It should be remembered that the estimates obtained this way do not go beyond the range of applicability of (2.1) so long as (when $a_\perp \ll l$)

$$\theta \lesssim a_\perp/l. \qquad (7.6)$$

When $\theta \gtrsim a_\perp/l$ investigation of instabilities in the plasma is to be carried out by means of other starting equations (this point is discussed in greater detail by Kadomtsev and Pogutse [4]).

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