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SPECTRUM AND EVOLUTION OF BOLTZMANN SYSTEMS

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The spectrum of the Boltzmann equation is discussed for the case of soft interaction potentials. The continuous spectrum of single-particle motions is important in this case. Quasi-collective excitations (modes and resonances) are studied. Relaxation towards an equilibrium state is considered for large time values on basis of the spectrum pattern so derived. It is demonstrated that the decays of the initial perturbation are qualitatively different for hard and soft potentials.

1. INTRODUCTION

MUCH progress has been made recently in kinetics based on the linear Boltzmann equation. At the same time, many questions calling for further investigation have been raised. Two of them, namely the determination of the spectrum of the Boltzmann equation in the case of soft interaction potentials and the analysis of relaxation of a specified initial distribution to the equilibrium value, are the subject of the present paper, which is a continuation of an article in which the first problem was investigated for hard potentials.[1]

The first to investigate the spectrum of the linearized collision operator were Wang-Chang and Uhlenbeck[2] (see[3]). They developed a proper collision-integral theory for the case of a gas of Maxwellian molecules (the spectrum turned out to be discrete). This theory was used in an analysis of sound propagation[2] as the basis for a discussion of one of the spectral branches of the Boltzmann equation, the sound trajectory.

The study of sound at ultrahigh frequencies[4,5] and of the thermalization of neutrons (see the reviews by Beckurts[6] and Nelkin[7]) called for a detailed investigation of the spectrum, concerning which little was generally known in the early 60's (see, in particular, the remarks of Wang-Chang and Uhlenbeck[3], p. 112). Recently, the eigenvalues (and trajectories) have been calculated for definite interaction models (see, for example,[8,9]) and the structure of the spectrum has been made clear for a wide class of potentials. Studies of neutron thermalization by the method of Van Kampen[10] and Case[11], reported in a number of papers (of which[12] is among the latest), have revealed the presence of a continuous spectrum. Grad[13], in an investigation of the asymptotic theory of the Boltzmann equation for small perturbations in a gas, established its existence rigorously in the absence of density gradients. The continuum picture has enabled him to separate the soft interactions. The present author[1], using the generalized Weyl theorem, determined the continuous spectrum in the inhomogeneous case and related it with the problem of high-frequency sound.

In Sec. 2, which proceeds along the lines of the earlier paper[1], we investigate the spectrum in the case of soft interaction potentials. It turns out here that an important role is played by the continuous spectrum (the continuum of single-particle motions). The observed collective excitations (modes and resonances) lie in the continuum. The question of sonic excitation in "soft" systems becomes complicated, for it no longer corresponds to a spectral branch. We note that the situation here (for arbitrary frequencies) is closer to that arising when an attempt is made to discuss high-frequency sound in a gas of particles with a hard potential[1].

A natural development of Boltzmann's general result concerning the establishment of the equilibrium state is a detailed study of the concluding stage of the evolution. From among the latest papers devoted to this question, we call attention to those of Sirovich[14] and by Kuscer and Corngold[15]. A known fact is the exponential relaxation in the homogeneous case. In the presence of density gradients, the relaxation depends on the choice of the initial distribution in space. The typical initial conditions with which Sirovich has dealt[14] lead to a $t^{-3/2}$ diffusion law. His analysis, however, is not fully satisfactory because of a number of limitations.

The question of the behavior of the system at large values of the time is considered in a very general manner in Sec. 3 of the present paper. In particular, we investigate homogeneous relaxation in the case of soft potentials. This relaxation depends on the initial conditions, but can be described essentially by an exponential function of a fractional power of t.

The general character of the analysis makes it possible to consider in the paper, besides small perturbations of a simple gas, also the evolution of a rarefied impurity in an equilibrium medium[1]. We shall henceforth assume that the medium is non-absorbing. The main formal difference between the indicated cases lies in the number of the collision invariants. In the latter case, obviously, only the number of particles is conserved. It is clear that in the absence of external forces, a Maxwellian distribution is established in the course of time in the here-considered systems "with conservation." under consideration,

2. SPECTRUM OF RELAXATION FREQUENCIES

The evolution of the systems under consideration is described by a linear Boltzmann equation

$$\frac{\partial\varphi}{\partial t} = -\mathbf{v}\frac{\partial\varphi}{\partial\mathbf{r}} + J\varphi, \tag{1}$$

where φ is the perturbation of the distribution function; the latter is written in the form[2]

$$f(\mathbf{v}, \mathbf{r}, t) = f_0 + f_0^{1/2}\varphi(\mathbf{v}, \mathbf{r}, t), \quad f_0 = (2\pi)^{-3/2}e^{-v^2/2}. \tag{2}$$

The sum of the flux and collision terms in the right side of (1) determines the evolution operator.

The following are the known properties of the collision operator J: a) it is self-adjoint and nonpositive, b) it has a zero eigenvalue, c) it is isotropic in velocity space. The multiplicity of the zero eigenvalue is obviously equal to the number of collision invariants. When the particle number, momentum, and energy are conserved the corresponding normal eigenfunctions are:

$$f_0^{1/2},\ \mathbf{v}f_0^{1/2},\ \sqrt{3/2}(1 - {}^1/{}_3v^2)f_0^{1/2}. \tag{3}$$

The collision term, as is well known [16], is

$$J\varphi = -\nu(v)\varphi + \int K(\mathbf{v}, \mathbf{v}_1)\varphi(\mathbf{v}_1)\,d\mathbf{v}_1. \tag{4}$$

Here $\nu(\mathrm{v})$ is the frequency of collisions of the particle with velocity v. An important factor for the subsequent analysis is that the kernel $K(\mathrm{v}, \mathrm{v}_1)$ is perfectly continuous for a broad class of interaction potentials [13]. These include repulsion potentials of the form α/r^s $(s > 2)$ with a finite effective radius[3]. In the case of these potentials, to which we shall henceforth confine ourselves, the collision frequency, using the normalization condition $\nu(0) = 1$, is equal to (see [1], formulas (8)—(10))

$$\nu(v) = \frac{\pi^{1/2}}{2^{(\gamma+1)/2}\Gamma\left(\frac{\gamma+3}{2}\right)}\int |\mathbf{v} - \mathbf{v}_1|^\gamma f_{0_1}d\mathbf{v}_1,\ \gamma = \frac{s-4}{s}. \tag{5}$$

It is easy to verify that the function $\nu(\mathrm{v})$ is monotonic. As $\mathrm{v} \to \infty$ we get

$$\nu(v) \approx \frac{\pi^{1/2}}{2^{(\gamma+1)/2}\Gamma\left(\frac{\gamma+3}{2}\right)}v^\gamma. \tag{6}$$

The frequency increases (or remains constant) in the case $s \geq 4$ (hard potentials), and decreases to zero if $s < 4$ (soft potentials). We shall henceforth discuss these cases separately, since there are qualitative differences between the properties of systems with soft and hard interactions.[4]

Thus, we proceed to study the spectrum of the Boltzmann equation. Going over to the Fourier representation

$$\varphi(\mathbf{v}, \mathbf{k}, t) = \int \varphi(\mathbf{v}, \mathbf{r}, t)e^{i\mathbf{k}\mathbf{r}}d\mathbf{r}, \tag{7}$$

we obtain

$$\frac{\partial\varphi(\mathbf{k})}{\partial t} = i\mathbf{k}\mathbf{v}\varphi(\mathbf{k}) + J\varphi(\mathbf{k}) = S(\mathbf{k})\varphi(\mathbf{k}). \tag{8}$$

We are interested in values of p and **k** for which we can solve the equation

$$p\varphi = (i\mathbf{k}\mathbf{v} - \nu(v))\varphi + K\varphi. \tag{9}$$

We first determine the spectrum of single-particle motions. As indicated, it plays an important role in the case of soft potentials. According to the invariance property of the continuous part of the spectrum relative to the fully continuous perturbation[5] it is sufficient to study the equation

[1]The so-called heat-bath (thermalizer) model. It is used for the study of a large number of questions. These include neutron transport, electron processes in a plasma, and Brownian motion.

[2]We use henceforth dimensionless quantities introduced in [1].

[3]We have in mind the ordinary regularization (inclusion of "glancing" collisions), which ensures a finite total cross section within the framework of the classical theory. It should be noted that quantum mechanics gives a finite total cross section for the indicated potentials.

[4]This was pointed out earlier by Grad [13].

[5]The generalized Weyl theorem which we used is given in the book by Glazman [17], p. 41.

$$[p+\nu(v)-i\mathbf{kv}]\varphi=0, \quad (10)$$

which is obtained when the operator K is crossed out from (9). In the case of the problem with initial conditions ($p = \lambda + i\omega$ and k real), the continuous-spectrum region is determined from the relations

$$\lambda=-\nu(v), \quad (11)$$

$$\omega=\mathbf{kv}. \quad (12)$$

Consequently, for soft potentials the continuum of the relaxation frequencies fills in the complex p-plane the strip

$$-1\leqslant \operatorname{Re}p<0. \quad (13)$$

It is clear that single-particle excitation will have arbitrarily long lifetimes and should greatly influence the evolution of the soft system. As follows from (13), the long-lived excitations having a collective character will be quasicollective (they lie in the continuum). For the stationary situation ($k = k_1 - ik_2$, $p = i\omega$), the region of the continuous spectrum is given by the condition (see [1], formulas (46)—(50))

$$k_1\geqslant \omega k_2, \quad (14)$$

i.e., it lies above the ray $k_1 = \omega k_2$ (see Fig. 2 of [1]). In this case, too, the collective excitations (if they exist) are in the continuum.

The question of collective excitations reduces to a study of the dispersion equation. We obtain the latter with the aid of a modified method of moments. Using the complete system of the functions $\psi_n(v)$, normalized with weight $\nu(v)$:

$$\int \nu(v)\psi_m\psi_n d\mathbf{v}=\delta_{mn}, \quad (15)$$

we represent the kernel $K(\mathbf{v}, \mathbf{v}_1)$ in the form

$$K(\mathbf{v},\mathbf{v}_1)=\nu(v)\nu(v_1)\sum_{m,n}K_{mn}\psi_m(\mathbf{v}_1)\psi_n(\mathbf{v}), \quad (16)$$

$$K_{mn}=\int\psi_m(\mathbf{v})K(\mathbf{v},\mathbf{v}_1)\psi_n(\mathbf{v}_1)d\mathbf{v}\,d\mathbf{v}_1. \quad (17)$$

The first five functions are chosen in the form of linear combinations of the conservations (3). The corresponding matrix elements are then

$$K_{mn}=\delta_{mn} \qquad (m\leqslant 5). \quad (18)$$

In this representation, the collision operator is[6)]

$$J\varphi=\nu(v)\Big[-\varphi+\sum_{m,n}C_mK_{mn}\psi_n\Big],$$

$$C_m=\int \nu\psi_n\varphi d\mathbf{v}. \quad (19)$$

Substituting (19) in (9), we get

$$\varphi(\mathbf{v},\mathbf{k},p)=\frac{\nu(v)}{[p+\nu(v)-i\mathbf{kv}]}\sum_{m,n}C_m(p,k)K_{mn}\psi_n. \quad (20)$$

Multiplying this expression by $\nu\psi_l$ and integrating[7)], we arrive at a system of equations and the existence conditions for their solutions yields the dispersion equation

$$D(p,\mathbf{k})=\|\delta_{lm}-\sum_n K_{mn}T_{ln}(p,\mathbf{k})\|=0, \quad (21)$$

$$T_{ln}(p,\mathbf{k})=\int\frac{\mathbf{v}^2\psi_l\psi_n d\mathbf{v}}{[p+\nu-i\mathbf{kv}]}. \quad (22)$$

In the case of hard potentials, the function $D(p, \mathbf{k})$, with $\operatorname{Re} p > -1$, is obviously analytic. Taking this circumstance into account, and also the properties of the evolution operator $S(\mathbf{k})$, we can represent the solutions of (21) for small values of k in the form of expansions

$$p_i(\mathbf{k})=\lambda_i-ikp_i^{(1)}+(ik)^2p_i^{(2)}+\ldots, \quad (23)$$

where λ_i are the eigenvalues of the collision operator, the coefficients $p_i^{(j)}$ are real, and $p_i^{(2)} > 0$ (see [1]). The longest lifetimes are possessed by the hydrodynamic modes (sonic, thermal, and transverse[8)]):

$$p_{1,2}=\pm i\sqrt{5/3}\,k-\alpha_1k^2+\ldots, \quad (24)$$

$$p_{3,4}=-\alpha_{3,4}k^2+\beta_{3,4}k^4+\ldots \quad (25)$$

It is clear that in the case of the thermalizer (heat-bath) model, one branch of the diffusion type (density) emerges from zero. Inversion of the series (24) and (25) with $p = i\omega$ yields the positions of the corresponding branches; all describe propagation. In particular, a density wave exists in the presence of a pulsating source of impurity particles. Its low-frequency propagation constant has in the diagonal approximation the form

$$k\approx\sqrt{\frac{\omega}{2}}\Big[\langle u^2/\nu\rangle+\frac{K_{22}}{\langle u^2\nu\rangle(1-K_{22})}\Big]^{-1/2}(1-i). \quad (26)$$

The angle brackets $\langle\ldots\rangle$ denote here averaging over the Maxwellian distribution, and the matrix element is taken with respect to the function

6)Confining ourselves to a finite number of terms, we obtain collision-term models that generalize the well known BGK model. All possess conservation. The representation used in [1] does not possess this property.

7)In the case when the denominator in (20) vanishes and φ becomes a singular function, it is necessary to use suitable regularization to determine the integral correctly.

8)Owing to the invariance of the operator $S(\mathbf{k})$ to rotations about the vector $\mathbf{k}$, the transverse mode is two-fold degenerate.

$\psi_2 = \langle u^2\nu\rangle^{-1/2} u f_0^{1/2}$. In the case of a Maxwellian background - impurity interaction, the coefficient in (26) becomes exact.

In the case of soft potentials, the function $D(p, \mathbf{k})$ is singular in the region $-1 \leq \mathrm{Re}\, p < 0$, of greatest interest[9)], and consequently the investigations of the quasicollective excitations should be carried out in a special manner (see footnote[7)]). Such excitations can be of two types: modes and resonances. In order to describe them, let us consider for simplicity a spatially-homogeneous case. Noting that the function $D(p) = D(p, 0)$ is analytic when p is complex, we investigate the real values of interest to us, resorting to one of the limiting transitions $p = \lambda \pm i0$. Calculating the upper and lower limits (+) and (−) and carrying out algebraic transformations, we get

$$D^{\pm}(\lambda) = \tilde{D}(\lambda) \mp i\pi\Gamma(\lambda), \tag{27}$$

$$\tilde{D}(\lambda) = \left\| \delta_{lm} - \sum_n K_{mn} \mathrm{P} T_{ln}(\lambda) \right\|, \tag{28}$$

$$\Gamma(\lambda) = \nu^2 \left| \frac{dv}{d\nu} \right| \sum_{lmn} \psi_m \tilde{D}^{ml}(\lambda) K_{ln} \psi_n \Big|_{v=\lambda} \tag{29}$$

where P is the symbol for the principal value of the integral, $\tilde{D}^{ml}(\lambda)$ is the co-factor of the element of the determinant (28), and v_λ is the solution of the equation $\nu(v_\lambda) + \lambda = 0$. The resonance is characterized by the condition

$$\tilde{D}(\lambda) = 0. \tag{30}$$

The value of $\Gamma(\lambda)$ determines in this case the width of the resonance. The quasicollective modes are resonances of zero width, i.e., it is necessary to satisfy in addition to (30) the equation

$$\Gamma(\lambda) = 0. \tag{31}$$

Equations (30) and (31) are independent, so that the existence of modes (which differ from the "conservations") has low probability. Resonances do exist. Unfortunately, those revealed by the preliminary calculations do not include small-width long-lived resonances that can determine the behavior of the system for medium values of t. It should be noted that in the study of quasi-collective excitations we are dealing as a rule with resonances, such as the Langmuir and undamped sonic branches obtained by Vlasov[18] [10)].

[9)]Outside this region, there is no spectrum for the thermalizer model.

[10)]This can be verified by using one of the limiting transitions $p = i\omega \pm 0$.

Interest attaches to the structure of the functions corresponding to the quasicollective excitations. They are written in the form

$$\varphi_\lambda = \mathrm{P}\frac{\nu \Sigma C_m(\lambda) K_{mn}\psi_n}{\lambda + \nu(v)} + \mu(\lambda)\delta(\lambda + \nu(v)), \tag{32}$$

where $C_m(\lambda)$ and $\mu(\lambda)$ are determined from the system of equations for their moments and from the normalization condition (for example, $C_0 = 1$). It is easy to establish that $\mu(\lambda)$ is proportional to $D(\lambda)$. In spite of the vanishing of the second term, resonance corresponds to a singular eigenfunction. It can be shown that condition (30) with allowance for (31) leads to vanishing of the numerator in the first term of (32). Consequently, φ_λ is a regular function and the mode corresponds to an eigenvalue on the continuous spectrum.

In the inhomogeneous case, we should expect the eigenvalues of the collision operator to generate diffusion trajectories. In this case, there exist thermal and transverse modes. As shown by preliminary calculations, the corresponding trajectories are "dissolved" in the continuum at certain values of the wave number ($k_{pr} \approx 0.9$, $p(k_{pr}) \approx -0.6$). The question of the sonic excitation becomes complicated, for it no longer corresponds to a spectral branch. One can hope that sonic excitation has the same character as a resonance. We note that formal application of perturbation theory makes it possible to obtain the adiabatic propagation velocity of the perturbation for vanishingly small k.

We were unable to reveal any modes in the stationary case. For all potentials, there exists at $\omega = 0$ a resonance $k_1 = 0$, $k_2 \approx 1$, which describes diffusion at the free-path wavelength. It is obvious that the third "trajectory" obtained in [19] comes from it. This fact sheds light on the nature of the "special" solutions of the Boltzmann equation, referred to in [1].

3. EVOLUTION TO THE EQUILIBRIUM STATE

Assume that a certain perturbation is specified at the initial instant

$$\varphi(\mathbf{v}, \mathbf{r}, t)_{t=0} = \varphi_0(\mathbf{v}, \mathbf{r}).$$

We investigate its behavior at larger values of the time.

In considering Eq. (1), it is convenient to use the Fourier representation (7), (8). To solve the initial-condition problem we apply to Eq. (8) the Laplace transformation

$$\varphi(\mathbf{v}, \mathbf{k}, p) = \int_0^\infty \varphi(\mathbf{v}, \mathbf{k}, t) e^{-pt} dt. \tag{33}$$

As a result we get

$$p\varphi(\mathbf{k},p)-\varphi_0(\mathbf{k})=S(\mathbf{k})\varphi(\mathbf{k},p). \qquad (34)$$

According to the inversion formula, we have

$$\varphi(\mathbf{k},t)=\frac{1}{2\pi i}\int_{-i\infty+\sigma}^{i\infty+\sigma}\frac{e^{pt}dp}{(p-S(\mathbf{k}))}\varphi_0(\mathbf{k}). \qquad (35)$$

The last expression is formal, and in order to calculate it we must find $(p - S(\mathbf{k}))^{-1}$, the resolvent of the operator $S(\mathbf{k})$. We use for this purpose the representation (19) for the collision operator. After simple manipulations we reduce (35) to the form

$$\varphi(\mathbf{v},\mathbf{k},t)=\frac{1}{2\pi i}\int_{-i\infty+\sigma}^{i\infty+\sigma}\left[\varphi_0(\mathbf{v},\mathbf{k})+\frac{\sum_m F_m(\mathbf{v})B_m(p,\mathbf{k})}{D(p,\mathbf{k})}\right]\frac{e^{pt}dp}{[p+\nu(v)-i\mathbf{kv}]}. \qquad (36)$$

Here

$$B_m(p,\mathbf{k})=\sum_l b_l(p,\mathbf{k})D^{lm}(p,\mathbf{k}), \qquad (37)$$

$$b_l(p,\mathbf{k})=\int\frac{\nu\psi_l\varphi_0 d\mathbf{v}}{[p+\nu-i\mathbf{kv}]}, \qquad (38)$$

$$F_m(\mathbf{v})=\nu(v)\sum_n K_{mn}\psi_n(\mathbf{v}). \qquad (39)$$

Expression (36) makes it possible to determine the form of $\varphi(\mathbf{v},\mathbf{k},t)$ at large values of t in the case of hard interaction potentials. As is well known, the asymptotic behavior of the function as $t\to\infty$ depends on the location and character of the singularities of its Laplace transformation. In this case, the integrand in (36) is, in accordance with (37) and (38), the quotient of two functions which are regular in the region $\mathrm{Re}\, p > -1$. It follows therefore that its singularities (poles) in the indicated region can be only zeroes of the denominator, i.e., the roots $p_i = p_i(\mathbf{k})$ of the dispersion equation (21). The picture of these trajectories [1] enables us to confine ourselves in the case of an asymptotic analysis ($t\to\infty$) to consideration of small values of k. Taking these considerations into account, we shift the Laplace contour to the left, as shown in Fig. 1 (we take into account the non-hydrodynamic mode, having the homogeneous case in mind). Calculating the contributions of the encountered poles, and also the integral of the first term, we represent the solution (36) in the form

$$\varphi(\mathbf{v},\mathbf{k},t)=\varphi_0 e^{(-\nu+i\mathbf{kv})t}+\sum_i R_i(\mathbf{v},\mathbf{k})e^{p_i(\mathbf{k})t}+\frac{1}{2\pi i}\int_L, \qquad (40)$$

where

$$R_i(\mathbf{v},\mathbf{k})=\sum_{m,l}\frac{F_m(\mathbf{v})b_l(p_i(\mathbf{k}),\mathbf{k})}{[p_i(\mathbf{k})+\nu-i\mathbf{kv}]_{p=p_i(\mathbf{k})}}\operatorname*{res}_{p_i(\mathbf{k})}\left[\frac{D^{lm}(p,\mathbf{k})}{D(p,\mathbf{k})}\right]. \qquad (41)$$

At large values of t, the principal term is the second one; the integral over the shifted contour and the first term (transient term) are exponentially small compared with the second term.

We shall now find it useful to expand $R_i(\mathbf{v}, \mathbf{k})$ in powers of $\mathbf{k}$. These quantities can be readily obtained by using the series (23), and also the expansions

$$\operatorname*{res}_{p_i(\mathbf{k})}\left[\frac{D^{lm}(p,\mathbf{k})}{D(p,\mathbf{k})}\right]=r_{i,lm}^{(0)}+ikr_{i,lm}^{(1)}+\ldots \qquad (42)$$

Greatest interest attaches to the hydrodynamic modes. For these modes, the lowest-order term in the expansion $R_i(\mathbf{v}, \mathbf{k})$ is:

$$R_i^{(0)}(\mathbf{v})=\sum_{l,m}^{5}r_{i,lm}^{(0)}\psi_m(\mathbf{v})\int\psi_l(\mathbf{v})\varphi_0(\mathbf{v},\mathbf{r})\,d\mathbf{v}d\mathbf{r}, \qquad (43)$$

where the coefficients $r_{i,lm}^{(0)}$ are calculated with the aid of the fifth-order determinant located in the upper left corner of $D(p, \mathbf{k})$.

The relations obtained above enable us to determine the asymptotic attenuation of any specified initial perturbation. We consider first the class of distributions whose Fourier components do not vanish when $k = 0$ (a simple example is a Gaussian density perturbation). In the asymptotic analysis, it is sufficient to take into account only the longest-lived (hydrodynamic) modes. Let us determine the contribution made to the evolution, say, of the thermal mode. According to (40), this contribution is given by the integral

$$\frac{1}{(2\pi)^3}\int R_3(\mathbf{v},\mathbf{k})e^{p_3(k)t-i\mathbf{kr}}d\mathbf{k}. \qquad (44)$$

It is clear that at large values of t (and fixed

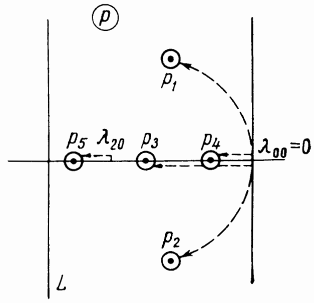

FIG. 1.

values of $\mathbf{r}$) the main contribution is made by the point where the function $p_3(k)$ has a maximum, $k = 0$. Using the expansion (25) of $p_3(k)$ and calculating the integral (44) by the saddle-point method, we obtain for it the asymptotic expression

$$(4\pi\alpha_3 t)^{-3/2} R_3(\mathbf{v}). \quad (45)$$

The transverse mode is considered similarly. An estimate of the contribution of the sonic mode at the fixed point $\mathbf{r}$ by the saddle-point method leads to exponential damping of the contribution with increasing time (in this case the point $k = 0$ is not the saddle point). In view of the propagation character of the mode, it is necessary to calculate its contribution in a reference frame tied to the wave. Appropriate manipulations show that the amplitude of the signal decreases like $t^{-5/2}$ (a "geometric" decrease like t^{-1} is added to the ordinary diffusion).

Thus, the relaxation of the perturbation class under consideration obeys the $t^{-3/2}$ law (we shall call it typical). Obviously, for one- and two-dimensional perturbations of this kind we have respectively the $t^{-1/2}$ and t^{-1} asymptotics. In the general case, the asymptotic behavior depends on the character of the initial distribution in coordinate space. This dependence can be readily traced by starting from the form (40) (see (41)). For example, if $\varphi_0(k) \sim k^n (k \to 0)$, the attenuation law will be

$$\varphi(t) \sim t^{-(n+3)/2}. \quad (46)$$

A similar relaxation is exhibited, in particular, by the initial distribution obtained by n-fold differentiation of the "typical" perturbation with respect to $\mathbf{r}$.

Using the foregoing analysis we can readily solve the problem of the asymptotic behavior of macroscopic quantities; in the calculation of the mean values it is sufficient to use the asymptotic expression for the distribution function. Obviously, the hydrodynamic variables (density, velocity, and temperature) attenuate in synchronism with the distribution function. We note that the Navier-Stokes approximation yields for them an analogous attenuation law[11], and in this sense the asymptotic behavior obtained above for the distribution function is hydrodynamic. From among the macroscopic non-hydrodynamic quantities we separate those whose molecular attributes are orthogonal to the conservations (3). These include, in particular, tangential stresses and heat flow. As follows from (42) and (43), the attenuation of such quantities is faster by at least $\sqrt{t}$. Thus, for example, the heat flux

$$\mathbf{q}(\mathbf{r}, t) = {}^1/_2 \int (v^2 - 5)\mathbf{v} f_0^{1/2} \varphi(\mathbf{v}, \mathbf{r})\, d\mathbf{v} \quad (47)$$

has an asymptotic form $\mathbf{q}(t) \sim t^{-2}$.

Let us consider briefly relaxation in the spatially-homogeneous case. Putting $k = 0$ in (40), we see that the asymptotic regime will describe the term with the smallest decrement[12]

$$R_1(\mathbf{v})\, e^{\lambda_1 t}. \quad (48)$$

The collision-operator eigenvalue closest to zero, λ_1, is best calculated by a variational method. In this case the problem is greatly simplified by the fact that the corresponding eigenfunction depends only on the energy variable. According to the available data [8], $\lambda_1 \approx 0.1$, so that the relaxation time is $t_0 \approx 10\tau$, where τ is the mean free-path time. Comparison of the results (46) and (48) point to a much larger rate of relaxation in velocity space. This leads to the well known motion of a local equilibrium established within a time on the order of t_0. However, as shown by the asymptotic behavior of the higher-order moments, this notion is meaningless in the study of the approach to equilibrium.

Let us now proceed to the evolution of soft systems. In order to reveal its characteristic features, it is sufficient to consider the spatially-homogeneous case. We represent (36) with $k = 0$ in a form that is convenient for analysis. As shown by a study of the spectrum, the integrand in (36) has the following singularities: a cut in the interval $(-1, 0)$, and possible poles other than $p = 0$. Contracting the Laplace contour to them (see Fig. 2) and carrying out the appropriate calculations, we obtain

$$\varphi(\mathbf{v}, t) = e^{-\nu t}\varphi_0 + \sum_m F_m(\mathbf{v}) \Big\{ e^{-\nu t} \frac{1}{2} \Big[\Big(\frac{B_m(\lambda)}{D(\lambda)} \Big)^+ + \Big(\frac{B_m(\lambda)}{D(\lambda)} \Big)^- \Big]_{\lambda = -\nu(v)} + \sum_i \operatorname*{res}_{p=\lambda_i} \Big[\frac{B_m(p)\, e^{pt}}{D(p)\,(p+\nu)} \Big] + \frac{1}{2\pi i} \mathrm{P} \int_{-1}^{0} \frac{e^{\lambda t} d\lambda}{\lambda + \nu} \Big[\Big(\frac{B_m(\lambda)}{D(\lambda)} \Big)^+ - \Big(\frac{B_m(\lambda)}{D(\lambda)} \Big)^- \Big] \Big\} \quad (49)$$

The first term is the already known transient term, the second results from the "moving" pole $p = -\nu(v)$ on the cut, the third is the contribution

[11] In the case of soft potentials, the result may differ from that of Navier-Stokes.

[12] The contribution of the conservations obviously is of no interest and can be eliminated in general.

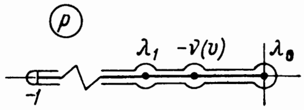

FIG. 2.

of the eigenvalues, and finally the fourth is the integral along the cut. The transient term, which reflects the influence of the initial state, cannot be discarded at $t \to \infty$ as before, for now it can turn out to be appreciable. The contribution from the quasicollective modes in the third term, after subtracting the conservations, obviously relaxes exponentially. As already noted, their existence is not very likely, and greater interest attaches to the resonances. They become manifest in the fourth term. The expression in the square brackets has, according to (27), the structure

$$\frac{B_m{}^+D^- - B_m{}^-D^+}{\tilde{D}^2 + \Gamma^2}, \tag{50}$$

i.e., the resonance ($\tilde{D} = 0$) makes a contribution which is inversely proportional to its width Γ. The presence of a small-width resonance with a characteristic velocity exceeding the average velocity can lead, at average values of t, to an exponential evolution.

Let us investigate the asymptotic behavior of the relaxing quantities. As follows from (49), it is impossible to indicate a single attenuation law for the particle distribution (the relaxation depends essentially on the velocity of the chosen component). In view of this, it is expedient to consider quantities that are averaged with respect to v (macroscopic). To exclude the conservation, we agree to consider "orthogonal" quantities. An analysis of (49) shows that the slowest to relax is the high-velocity component. (the "tail" of the particle distribution). This makes it difficult to obtain the asymptotic behavior of the macroscopic quantity as $t \to \infty$ directly with the aid of (49). The analysis that follows will enable us to "sum" the infinite series in (49).

We write the formal solution of the problem with the initial condition in the form

$$\varphi(\mathbf{v}, t) = e^{(-\nu(v)+K)t}\varphi_0(\mathbf{v}). \tag{51}$$

In this case the macroscopic quantity whose asymptotic behavior interests us will be

$$\bar{p}(t) = \int p(\mathbf{v}) f_0^{1/2} e^{(-\nu(v)+K)t}\varphi_0(\mathbf{v})\, d\mathbf{v}. \tag{52}$$

As indicated, as $t \to \infty$ the main contribution is made by the high-velocity component. This enables us to specify more concretely the integrand in (52). Let us first specify the behavior of the quantities contained in it when $v \to \infty$. Let $p(v) \sim v^n$ and let us also assume that

$$\varphi_0(\mathbf{v}) \sim 1/v^r.$$

The basis of the analysis that follows is the estimate[13)]

$$K\varphi_0 < C/v^{r+1}, \tag{53}$$

where C is a constant. Expanding in suitable manner the operator exponential, estimating the terms of the resultant series with the aid of (53), and summing the majorant series, we can verify that the main contribution to the asymptotic value is made by the integral

$$\int_0^\infty v^{n-r+2} \exp\left[-\nu(v)t - \frac{v^2}{4}\right] dv. \tag{54}$$

Calculating this integral by the saddle-point method, we obtain the following attenuation law

$$p(t) \sim t^{s(n-r+1)/(s+4)} \exp(-\delta t^{2s/(s+4)}), \tag{55}$$

where

$$\delta = \frac{s+4}{2s}\left(\frac{s}{2(4-s)}\right)^{(4-s)/(2+s)}$$

Thus, the relaxation is appreciably slower in the case of soft interaction potentials. An obvious consequence of (55) is the decrease of the relaxation rate for softer potentials.

4. CONCLUSION

1. A study of the spectrum of a system of particles interacting via a soft potential has shown that its properties are determined essentially by the continuous spectrum (single-particle motions). As is well known [1], the contribution of the collective modes predominates in the case of hard interactions at moderate gradients, while the single-particle motions predominate if the gradients are appreciable. These results are perfectly natural from the physical point of view.

An analysis of the dispersion equation has revealed the existence of excitations of collective character (modes and resonances) lying in the continuum. As a rule, quasicollective excitations are resonances. In particular, the branches of plasma oscillations without attentuation, considered by Vlasov [18], are resonances. Of course, under certain conditions it is possible to observe quasicollective excitations experimentally. A quantum-mechanical illustration of this are the brighter bands against the background of a continuous spectrum ("quasistationary" levels).

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13)Its derivation can be found in the paper of Grad [13] (see also the paper of Carleman [20]).

The obtained picture of the spectrum of Boltzmann systems has enabled us to investigate in detail the approach to the equilibrium state. As a result of an asymptotic analysis of the solution of the initial-condition problem, we established the law governing the attenuation of the perturbation as a function of its initial form in coordinate space. An examination of homogeneous relaxation has shown that in a system of particles with soft interaction this relaxation is much slower than in the case of hard particle interaction. The increase in the duration of the relaxation in the case of softer potentials is physically obvious. We note, however, that in the case of very soft Coulomb interaction the existing theory did not reveal a qualitatively different attenuation law.

2. The representation (19) used in this paper for the collision integral yields, if we confine ourselves in it to a finite number of terms, models of the collision term which are convenient for concrete considerations (and which are rigorously proved)[14)]. They are generalizations of the models obtained by Gross and Jackson [21] for a gas of Maxwellian molecules to include the case of an arbitrary interaction law. It should be noted that the eigenfunctions of the model collision operator constitute a complete set. This was proved by Koppel [12] for hard potentials and can be established without essential changes in the case of soft interactions. Of course, the completeness property enables us to solve the initial-condition problem for a homogeneous gas by expansion in eigenfunctions. Unfortunately, we have no such a possibility in the general case.

One of the obvious consequences of our analysis (see also [1]) is the need for using a modified method of moments to study excitations of collective character in the case when the continuum plays an appreciable role. In particular, the use of the standard method of moments would lead to qualitatively incorrect results (for example, to an attenuation in the form $\exp(-\alpha t)$). It is obvious that in the absence of a continuous spectrum (homogeneous gas of Maxwellian molecules) the modified method coincides with the standard one.

The results of our earlier paper [1] and of the present one reveal that the dispersion approach is not adequate for the description of sonic excitation of high frequency in the case of hard potentials and of any frequency for soft potentials. It should be noted that in the case of Maxwellian molecules it is possible, by analytically continuing the dispersion equation (20), to obtain its solution for arbitrary values of the defining parameter. The trajectories for the values of the parameter in excess of the limiting values [1] will lie in this case essentially on the "unphysical" sheet (it is used in quantum mechanics to study "quasistationary" levels) [15)]. Inasmuch as this case is exceptional, and the connection between the trajectories and real excitations is not yet clear, it is necessary to investigate quasicollective sound on the basis of a formulation of the problem corresponding to the experimental conditions.

[14)]We note that a model of any order leaves all the main properties of the collision operator unchanged.

[15)]The author takes this opportunity to thank B. S. Pavlov for a discussion of this question.

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