SOME FEATURES OF THE JOSEPHSON RADIATION

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Broadening of lines emitted from a superconducting tunnel structure is considered. The analysis is carried out under the assumption that the electrons interact with the electromagnetic field in quantum fashion. It is shown that the amplitude and frequency of the Josephson alternating current cannot be specified simultaneously. The dependence of the amplitude on the spectrum width of the emitted frequencies is estimated.

YANSON, Svistunov, and Dmitrenko were the first[1] to observe the radiation, predicted by Josephson,[2] from a superconducting tunnel structure. According to the theory,[2] a single spectral line should be observed with frequency $\omega = 2eU/\hbar$ (U is the voltage on the barrier). Account of electrodynamic effects (see[3-6]) leads to the appearance of a spectrum with frequencies $\omega_n = n\omega$. Experimentally, however, radiation is observed in a rather wide band of frequencies;[7] on the other hand, multiple frequencies, in correspondence with the theory, are evidently very weak and are not recorded. The band of radiated frequencies varies over rather wide limits, from 0.8×10^7 Hz[7] to 10^5 Hz[8] for the same central frequency $\sim 10^{10}$ Hz.

We consider the broadening of the Josephson radiation, assuming the interaction of the electrons with the electromagnetic field to be a quantum one. For this purpose, the scheme with the tunnel Hamiltonian[9] will be generalized to the case in which there are electric and magnetic fields that vary in time and space.[1)] We then separate the field canonical variables which will be subjected to the usual commutation relations. Since the amplitude and frequency are connected with the mutually complementary canonically-conjugate quantities, they cannot be specified simultaneously. For the homogeneous case (small voltages on the barrier), the dependence of the amplitude on the band of frequencies emitted will be estimated.

1. In the absence of a field, the tunnel structure is described by the Hamiltonian

$$H_I = H_1 + H_2 + T. \quad (1)$$

Here H_1 and H_2 are the Hamiltonians of the left- and right-side metals, respectively, and the interaction Hamiltonian T has the form

$$T = \sum_{\mathbf{p},\mathbf{q},\sigma} T_{\mathbf{pq}} a^+_{\mathbf{p}\sigma} b_{\mathbf{q}\sigma} + \text{herm. conj.} \quad (2)$$

where $a^+_{p\sigma}$, $a_{p\sigma}$, $b^+_{q\sigma}$, and $b_{q\sigma}$ are the particle creation and annihilation operators in states with quasimomentum and spin $\mathbf{p}$ and σ in the right metal and $\mathbf{q}$ and σ in the left metal; T_{pq} is the matrix element of the effective interaction. Turning the field on adds to the Hamiltonian (1) the term

$$H' = \int_{-\infty}^{t} d\tau \int d^3r\, \mathbf{jE} + \frac{1}{8\pi} \int d^3r (\varepsilon \mathbf{E}^2 + \mathbf{H}^2), \quad (3)$$

where $\mathbf{E}$ and $\mathbf{H}$ are the vectors of the electric and magnetic fields, ϵ is the dielectric constant of the material, and j the current-density operator, the specific form of which we shall determine by a self-consistent method.

Using the fact that wavelengths λ customarily used in the experiment always satisfy the inequality $\lambda \gg \lambda_L$, where λ_L is the London penetration depth, we rewrite the first term in (3) in the following way:

$$\int d^3r\, \mathbf{jE} = \int d^2r\, \hat{j}_x(0, \mathbf{r}) \int_1^2 dx\, E_x(x, \mathbf{r}) + \int d^3r\, \mathbf{j}_\perp \mathbf{E}_\perp. \quad (4)$$

Here x is the coordinate in the direction perpendicular to the plane of the barrier (x = 0 at the center of the barrier), $\mathbf{r}$ is the radius vector in the plane of the barrier, and $j_\perp$ and $\mathbf{E}_\perp$ are the components of the current and field parallel to the surface of the superconductors. The points 1 and

1)Recently Larkin, Ovchinnikov, and Fedorov [10] considered tunneling in superconductors, using the Gor'kov equations. We think that the method proposed in [10] can be considered to be the basis for a model scheme as applied to superconductors.

2 lie in the interior of the first and second superconductors, while the result of integration in (4) does not depend on the specific location of these points at the accuracy acceptable to us. See [6,11].) Inasmuch as we are interested only in how the interaction Hamiltonian changes in the presence of external fields, we shall use the London equations below and express $j_\perp$ in terms of $E_\perp$.

Integrating the operator equation of continuity from 0 to ∞, we obtain the following expression for the operator $\hat{j}_x(0, \mathbf{r})$:

$$\hat{j}_x(0, \mathbf{r}) = e\frac{\partial A(\mathbf{r})}{dt}, \quad A(\mathbf{r}) = \int_0^\infty dx\, \rho(x, \mathbf{r}), \tag{5}$$

ρ is the electron density operator. In (5), we have neglected the term $\int_0^\infty dx\, \nabla_\perp \cdot \mathbf{j}_\perp$, which makes a contribution to the average value of the order of λ_L/λ. Using (1), (3), and (5) we get an integral equation that determines the operator $j_x(0, \mathbf{r})$:

$$\hat{j}_x(0, \mathbf{r}, t) = -\frac{ie}{\hbar}[A(\mathbf{r})H_I]_- - \frac{ie}{\hbar}\int_{-\infty}^{t} d\tau \int d^2r' \int_1^2 dx\, E_x(x, \mathbf{r}') \times [A(\mathbf{r})\hat{j}_x(0\ \mathbf{r}', \tau)]_- \cdot \tag{6}$$

Solving (6), we get

$$\hat{j}_x(0, \mathbf{r}, t) = -\frac{ie}{\hbar}e^{-iC}[A(\mathbf{r})H_I]_- e^{iC}, \qquad C = \int d^2r A(\mathbf{r})\varphi(\mathbf{r}, t), \tag{7}$$

where $\varphi(\mathbf{r}, t)$ satisfies the relations (see [12])

$$\frac{\partial\varphi}{\partial t} = \frac{e}{\hbar}\int_1^2 dx\, E_x(x, \mathbf{r}, t); \quad \frac{\partial\varphi}{\partial \mathbf{r}} = \frac{8\pi e}{\hbar c^2}\lambda_L^2 \mathbf{j}_\perp; \quad \varphi(\mathbf{r}, -\infty) = 0. \tag{8}$$

To determine the canonical variables, we note that in the classical case of two interacting systems with generalized coordinates q (for the particles) and Q (for the field) the interaction potential V can be represented in the form

$$V(q, t) = \int_{-\infty}^{t} d\tau \int d^3r \frac{\delta V(q, Q)}{\delta Q}\dot{Q}, \tag{9}$$

where Q and $\dot{Q}$ are assumed to be given functions of time and the coordinates r, inasmuch as we consider motion in an external field. Comparing (3) with (9), we find that

$$\frac{\delta V(q, Q)}{\delta Q} = \frac{\hbar}{e}\hat{j}_x; \qquad \dot{Q} = \dot{\varphi}. \tag{10}$$

It is seen from (9) and (10) that the term representing the interaction with the field can be written as follows:

$$V(q, \varphi) = \frac{\hbar}{e}\int d^2r \int_0^\varphi \delta\varphi\, \hat{j}_x(q, \varphi). \tag{11}$$

Here we mean by q the dependence on the electron operators. By direct computation by means of Eq. (7), it can also be established that the interaction term has the form (11). Making use of the relation

$$e^{-iC}a^+_{\mathbf{p}\sigma}e^{iC} = \sum_{\mathbf{n}} \int d^2r\, e^{-i\varphi}\int_0^\infty dx\, \psi_{\mathbf{p}}(x, \mathbf{r})\psi_{\mathbf{n}}^*(x, \mathbf{r})a_{\mathbf{n}\sigma}, \tag{12}$$

we transform (11) to the form

$$V(q, \varphi) = \int d^2r B(\mathbf{r})\exp[-i\varphi(\mathbf{r})] + \text{herm. conj.} - T, \tag{13}$$

where

$$B(\mathbf{r}) = \sum_{\mathbf{n}, \mathbf{p}, \mathbf{q}, \sigma}\int_0^\infty dx\, \psi_{\mathbf{p}}(x, \mathbf{r})\psi_{\mathbf{n}}^*(x, \mathbf{r})a^+_{\mathbf{n}\sigma}b_{\mathbf{q}\sigma}T_{\mathbf{p}\mathbf{q}},$$

$\psi_p(x, \mathbf{r})$ are the single-particle states with infinitely high barrier. Thus we obtain a Hamiltonian which takes into account not only the time (see [12]) but also the space lag of the current.

We now express the second term in (3) and the contribution from the second term in (4) in terms of φ and the momentum θ canonically conjugate to it. With the assumed degree of accuracy (we neglect terms having the order λ_L/λ), we obtain the field Hamiltonian H_{II} in the form

$$H_{II} = \int d^2r\left[\frac{\theta^2}{2M} + \frac{M}{2}\bar{c}^2(\nabla\varphi)^2\right], \tag{14}$$

where

$$M = \varepsilon\hbar^2/4\pi de^2, \quad \theta = M\dot{\varphi}, \quad \bar{c}^2 = c^2 d/2\lambda_L\varepsilon.$$

2. We shall now assume that not only the electron system, but also the field system is described in quantum fashion. The total Hamiltonian H will be

$$H = H_1 + H_2 + \int d^2r B(\mathbf{r}) \times \exp[-i\varphi(\mathbf{r})] + \text{herm. conj.} + H_{II}, \tag{15}$$

and φ and θ satisfy the usual commutation relations:

$$[\varphi(\mathbf{r})\theta(\mathbf{r}')]_- = i\hbar\delta(\mathbf{r} - \mathbf{r}'), \quad [\varphi(\mathbf{r})\varphi(\mathbf{r}')]_- = 0, \quad [\theta(\mathbf{r})\theta(\mathbf{r}')]_- = 0. \tag{16}$$

For the calculation, it is convenient to assume that we turn on not the field but the interaction with the electron system (transparency of the barrier), but then the state of the field before turning on the interaction will be described, in accord with (8), by the specified mean values

$$\langle\theta\rangle = MeU/\hbar, \quad \langle\nabla\varphi\rangle = 2e\lambda_L[\mathbf{nH}]/\hbar c^2 = \mathbf{k}, \tag{17*}$$

where U is the voltage on the barrier, H the constant magnetic field, and n a unit vector perpendicular to the surface of the barrier.

It would appear that the transition to excitation annihilation and creation operators in orthonor-

*$[\mathbf{nH}] \equiv \mathbf{n} \times \mathbf{H}$.

malized states, which are the solutions of the corresponding field equation for the Hamiltonian (14), leads to a simplification of the problem. However, it is not difficult to see that such an approach leads to the appearance of divergences. In fact, by using (15) and (16) it is easy to obtain the operator equation

$$\nabla^2\varphi - \frac{1}{\bar{c}^2}\frac{\partial^2\varphi}{\partial t^2} = \frac{-i}{M\bar{c}^2}[B(\mathbf{r})\exp[-i\varphi(\mathbf{r})] - \text{herm. conj.}]. \tag{18}$$

This equation is the quantum analog of the equations used in[3,5,6]. If we consider the calculation of the mean voltage on the barrier as an example, using second-quantization operators creation and annihilation in orthogonal states, then we discover that the divergence appears as the result of the vanishing of the term $\int d^2r \langle \nabla^2\varphi \rangle$. By using (18) we can easily see that $\int d^2r \langle \nabla^2\varphi \rangle \neq 0$. Use of states with real boundary conditions would be natural in this case,[6] but such states are not orthogonal, inasmuch as the operator ∇^2 is not Hermitian here.

By using (18), it is possible to construct a perturbation theory in T which leads to the usual description of quasiparticle tunneling; however, the Josephson current proper is equal to zero. In order to understand the reason for this, we consider the homogeneous case, in which the voltage at the barrier can be regarded as constant. Actually, this corresponds to the region of voltages $eU \ll 2\Delta$ (Δ is the gap in the elementary-excitation spectrum in superconductors), i.e., those usually realized in experiments on Josephson tunneling.

3. In the homogeneous case, the expression (14) is greatly simplified and takes the form

$$H_{\text{II}} = \frac{\lambda_L s\mathbf{H}^2}{4\pi} + \frac{\hat{p}^2}{2M}, \qquad \hat{p} = \frac{1}{\sqrt{s}}\int d^2r\,\theta(\mathbf{r}), \tag{19}$$

where s is the area of the junction. The coordinate canonically conjugate to $\hat{p}$ is

$$\hat{Q} = \frac{1}{\sqrt{s}}\int d^2r\,\varphi(\mathbf{r}) = Q_0 + \sqrt{s}\,\mathbf{kr} + \hat{\gamma}. \tag{20}$$

From (16), we get

$$[\hat{Q}\hat{p}]_- = [\hat{\gamma}\hat{p}]_- = i\hbar. \tag{21}$$

Using (7), (15) and (19) in the first non-vanishing order of perturbation theory, we get the following expression for the current which flows through the barrier:

$$I = \frac{2e}{\hbar}\operatorname{Re}\int_{-\infty}^{t} d\tau \int d^2r \int d^2r' \,\{\langle [B(\mathbf{r}',\tau)\exp[-i\varphi(\mathbf{r}',\tau)] | B(\mathbf{r},t) \times \exp[-i\varphi(\mathbf{r},t)]]_-\rangle_0 + \langle [B^+(\mathbf{r}',\tau)\exp[i\varphi(\mathbf{r}',\tau)] | B(\mathbf{r},t) \times \exp[-i\varphi(\mathbf{r},t)]]_-\rangle_0\}. \tag{22}$$

Here we denote $\langle \ldots \rangle_0$ the averaging over the equilibrium ensemble of electron states, when $T_{pq} = 0$, and over the states of the Hamiltonian H_{II}, while

$$\varphi(\mathbf{r},t) = \hat{Q}(\mathbf{r},t)/\sqrt{s};$$

$\hat{Q}(t)$ and $B(t)$ are operators in the interaction representation,

$$\hat{Q}(t) = \exp\left[\frac{iH_{\text{II}}t}{\hbar}\right]\hat{Q}\exp\left[-i\frac{H_{\text{II}}t}{\hbar}\right] = Q_0 + \mathbf{kr}\sqrt{s} + \hat{\gamma} + t\hat{p}/M. \tag{23}$$

The second term in (22) leads to the usual quasiparticle current, and we shall not consider it below. The first term for $U \ll 2\Delta$ (see[12]) can be simplified and rewritten in the form

$$I_s(t) = I_0\int\frac{d^2r}{s}\operatorname{Im}\exp\left[i\frac{2Q_0}{\sqrt{s}} + 2\mathbf{kr}\right]\left\langle \exp\left[\frac{i2\hat{p}t}{\sqrt{s}M} + \frac{2i\hat{\gamma}}{\sqrt{s}}\right]\right\rangle_0, \tag{24}$$

where I_0 is the Josephson current.[2]

If it is assumed that, prior to turning on the interaction, we are given $\langle \hat{p} \rangle_0$, i.e., the voltage on the barrier is such that the variance $\langle(\Delta\hat{p})^2\rangle_0 = 0$, then we get $I_S \equiv 0$. The latter is a consequence of the fact that the variance $\langle(\Delta\hat{\gamma})^2\rangle_0 \to \infty$, in correspondence with the uncertainty relation $\langle(\Delta\hat{p})^2\rangle_0 \langle(\Delta\hat{\gamma})^2\rangle_0 \sim \hbar^2/4$. Actually, however, we always have a noise voltage on the barrier, due at finite temperatures both to the flow of quasiparticle current and to noise introduced from the external source, which as a rule is at room temperature and possesses a finite internal resistance. At temperatures close to zero, the fundamental contribution is made by the external noise. Taking it into account that the noise incident on the junction passes through a natural low frequency filter, we assume that the upper limiting frequency of the noise Ω is such that the inequality $\Omega l/\bar{c} \ll 1$ is satisfied (l is the characteristic transverse dimension of the junction). Satisfaction of this inequality is necessary since otherwise it would be necessary to take retardation into account and the homogeneous approximation would be invalid.

The existence of noise can be described by introducing in the Hamiltonian a potential that depends on the time in random fashion. We cannot solve the problem with a random potential. However, it is easy to obtain an estimate of Eq. (24) by starting out from the fact that the noise leads to the appearance of a variance of $\hat{p}$, $\langle(\Delta\hat{p})^2\rangle_0 = sM^2U_n^2/\hbar^2$ (U_n is the noise voltage), and consequently, a variance of $\hat{\gamma}$ is $\langle(\Delta\hat{\gamma})^2\rangle_0$

$\sim \hbar^2/\langle(\Delta\hat{p})^2\rangle_0$, i.e., to the existence of a rather strong cutoff in γ. Taking the foregoing into account, we get

$$I_s(t) \sim I_0 \exp\left[-\frac{\alpha^2\hbar^2}{2s^2M^2(\Delta\omega)^2}\right] \times \frac{1}{s}\int d^2r \sin\left[\frac{2eUt}{\hbar} + 2\mathbf{kr} + \varphi_0 + \chi(t)\right]. \quad (25)$$

Here φ_0 is the initial phase of the oscillation ($\varphi_0 = Q_0/\sqrt{s}$), $\chi(t)$ is the random phase, the energy spectrum $S(\Omega)$ of which is connected with the noise spectrum $W(\Omega)$ by the relation $S(\Omega) = W(\Omega)/\Omega$, $\Delta\omega$ is the width of the band of emitted frequencies, and α is a dimensionless parameter ($\alpha \sim 1$).

To get (25), we have assumed that the index of noise modulation can reach large values,[13] and therefore assumed that $(\Delta\omega)^2 \sim e^2U_n^2/\hbar^2$. The functional dependence of the current amplitude on $(\Delta\omega)^2$ will naturally depend strongly on the method of cutting off of the integrals in the calculation of the mean values. In the given case, we used a cutoff of the Gaussian type, i.e., $\sim \exp[-\hat{\gamma}/2 < (\Delta\hat{\gamma})^2\rangle_0]$. Thus the result is purely an estimate. We note that, regardless of the manner of the cut, the factor f which suppresses the amplitude when the band width of the emitted frequencies decreases has the form

$$f \approx 1 - \frac{\alpha^2\hbar^2}{2s^2M^2(\Delta\omega)^2} \quad (26)$$

for large $\Delta\omega$.

4. In conclusion, we estimate when a significant decrease in the amplitude of the Josephson current sets in. For the junctions employed in the experiment of Dmitrenko and Yanson,[7] we have $M \sim 10^{-29}$ g and $s \sim 10^{-2}$ cm^2. Using (25), it is easy to see that a decrease in the amplitude by a factor e will take place for a bandwidth $\Delta\omega \sim 10^4$ Hz. In the work of Dmitrenko and Yanson,[7] a bandwidth $\Delta\omega \sim 0.8 \times 10^7$ Hz was observed, i.e., the damping of the current amplitude was virtually nonexistent. In the experiments of Taylor et al.,[8] the bandwidth reached $\Delta\omega \sim 10^5$ Hz, which is rather close to the estimate obtained by us. We note that to obtain very high stability it is advantageous to use high frequencies, since the bandwidth of emitted frequencies in the approximation considered here ($\omega \ll 2\Delta/\hbar$) does not depend on the central frequency ω. For high frequencies $\omega \lesssim 2\Delta/\hbar$, however, an additional broadening will appear, connected with the excitation of a quasiparticle current due to the effects of retardation[12] even at zero temperature.

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[1] I. K. Yanson, V. M. Svistunov and I. M. Dmitrenko, JETP **48**, 976 (1965), Soviet Phys. JETP **21**, 650 (1965).

[2] B. P. Josephson, Phys. Lett. **1**, 251 (1962).

[3] R. Eck, D. Scalapino and B. Taylor, Phys. Rev. Lett. **13**, 15 (1964).

[4] I. M. Dmitrenko, I. K. Yanson and V. M. Svistunov, JETP Pis'ma **2**, 17 (1965), JETP Lett. **2**, 10 (1965).

[5] I. O. Kulik, JETP Pis'ma **2**, 134 (1965), JETP Lett. 2, 84 (1965).

[6] Yu. M. Ivanchenko, A. V. Svidzinskiĭ and V. A. Slyusarev, JETP **51**, 194 (1966), Soviet Phys. JETP **24**, 131 (1967).

[7] I. M. Dmitrenko and I. K. Yanson, JETP Pis'ma **2**, 242 (1966), JETP Lett. **2**, 154 (1966).

[8] B. Taylor, W. Parker and D. Landenberg, Proc. LT-10, Abstracts of LT-10 Papers, C-30, 117, C-31, 118, 1966.

[9] M. H. Cohen, L. M. Falicov and J. C. Phillips, Phys. Rev. Lett. **8**, 316 (1962).

[10] A. I. Larkin, Yu. N. Ovchinnikov and M. A. Fedorov, JETP **51**, 683 (1966), Soviet Phys. JETP **24**, 452 (1967).

[11] I. Swihart, J. Appl. Phys. **32**, 461 (1961).

[12] Yu. M. Ivanchenko, JETP **51**, 337 (1966), Soviet Phys. JETP **24**, 225 (1967).

[13] D. Middleton, An Introduction to Statistical Communication Theory (Russian transl., Sov. Radio, 1962).

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