NONLINEAR SPIN WAVES IN FERROMAGNETS AND ANTIFERROMAGNETS

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Nonlinear spin waves in magnetically ordered crystals are investigated. Stationary-profile waves, propagated perpendicularly to the equilibrium direction of the magnetic moment in ferromagnetic crystals, are considered in the case of anisotropy of the easy-axis type and in the case of anisotropy of the easy-plane type; waves propagated perpendicularly to the anisotropy axis in antiferromagnetic crystals, with anisotropy of the easy-plane type, are also considered. It is shown that in such crystals, along with periodic waves, detached aperiodic waves are possible; in them, the magnetic moments of the atoms are in an equilibrium state both before the forward front of the wave (at $-\infty$) and behind its rear front (at $+\infty$). In the case of anisotropy of the easy-axis type, the magnetic-moment density vectors at $+\infty$ and at $-\infty$ can be directed along the anisotropy axis in opposite directions (magnetic-moment flip waves). In the case of anisotropy of the easy-plane type, the magnetic-moment density vectors at $+\infty$ and at $-\infty$ lie in the plane of easy magnetization; the angle between the corresponding directions of the magnetic moment is in general not a multiple of 2π (magnetic-moment rotation wave). Together with the solitary waves considered earlier (in which the directions of the magnetic-moment density vectors at $+\infty$ and at $-\infty$ coincide), the magnetic-moment flip waves and the magnetic-moment rotation waves form all the three types of detached aperiodic spin waves that are in principle possible. It is shown that velocities of all the types of aperiodic wave are bounded from above; in particular, in the case of an antiferromagnet the velocity of a magnetic-moment rotation wave of the sublattices cannot exceed the phase velocity of a small-amplitude spin wave.

INTRODUCTION

THIS paper investigates nonlinear spin waves in magnetically ordered crystals. It considers one of the classes of nonlinear spin waves: stationary-profile waves; that is, those motions in which the magnetic-moment density depends on the coordinates and the time only in the combination $\mathbf{r} - \mathbf{V}t$, where $\mathbf{V}$ is the constant velocity of the wave.

We show that along with periodic spin waves, aperiodic (detached) spin waves can be propagated in ferromagnets and antiferromagnets. Such a wave is a region of magnetic-moment excitation that moves along the crystal, and in which the magnetic moments of the atoms are in an equilibrium state both before the forward front of the wave (at $-\infty$) and behind its rear front (at $+\infty$).

If the crystal has anisotropy of the easy-plane type, then the vector magnetic-moment density after passage of the wave returns to the plane of easy magnetization; the angle between the directions of the magnetic moment at $+\infty$ and at $-\infty$ is, in general, not a multiple of 2π. Such a wave may be called a magnetic-moment rotation wave. If the crystal has anisotropy of the easy-axis type, then the magnetic-moment density vectors at $+\infty$ and at $-\infty$ can be antiparallel; such a wave may be called a magnetic-moment flip wave. Together with the solitary waves considered earlier [1], in which the values of the magnetic-moment vector at $+\infty$ and at $-\infty$ coincide, magnetic-moment rotation waves and magnetic-moment flip waves form all the three types of detached, aperiodic spin waves that are in principle possible.

We emphasize that in the case of magnetic-moment flip waves and of magnetic-moment rotation waves, the state of the crystal behind the rear front of the wave (at $+\infty$) differs from the state of the crystal before its forward front (at $-\infty$): the magnetic-moment density vector is oriented differently at $+\infty$ and at $-\infty$. In this sense, waves of these types are reminiscent of shock waves. In contrast to shock waves, however, the structure of these waves (as also the structure of the solitary waves) is determined by nondissipative properties of the crystal. (Dissipative processes occur, of course, also in the solitary waves and in magnetic-moment flip and rotation waves; these processes, however, do not play a determining role in them and lead only to a slow diminution of the amplitude of the wave during its propagation along the crystal.)

1. EQUATIONS DESCRIBING A STATIONARY-PROFILE SPIN WAVE IN A FERROMAGNET WITH ANISOTROPY OF THE EASY-PLANE TYPE

We first consider spin waves of finite amplitude in ferromagnets with anisotropy of the plane-of-easy-magnetization type. For this purpose we shall start from the equation of motion of the magnetic moment of a ferromagnet (see, for example,[2]),

$$\partial \mathbf{M}/\partial t = g[\mathbf{M}\mathbf{H}^e], \tag{1*}$$

where $\mathbf{M}$ is the magnetic-moment density, $\mathbf{H}^e$ is the effective field

$$\mathbf{H}^e = \mathbf{H} + \alpha\Delta\mathbf{M} + \beta\mathbf{n}(\mathbf{M}\mathbf{n}), \tag{2}$$

$\mathbf{H}$ is the magnetic field, which satisfies the equations of magnetostatics

$$\operatorname{div}(\mathbf{H} + 4\pi\mathbf{M}) = 0, \quad \operatorname{rot}\mathbf{H} = 0, \tag{3}$$

g is the gyromagnetic ratio, $\mathbf{n}$ is a unit vector in the direction of the anisotropy axis (the z axis), α is an exchange-interaction constant, and β is an anisotropy constant (in the case being considered, a crystal with anisotropy of the plane-of-easy-magnetization type, $\beta < 0$).

On specializing to stationary waves propagated along the anisotropy axis, and on setting the external magnetic field equal to zero, we reduce equations (1)—(3) to the form

$$V\mathbf{M}' + g[\mathbf{M}\mathbf{H}^e] = 0, \tag{4}$$

$$\mathbf{H}^e = \alpha\mathbf{M}'' + \mathbf{n}(\beta - 4\pi)(\mathbf{M}\mathbf{n}), \tag{5}$$

where V is the velocity of propagation of the wave (a prime on a quantity denotes differentiation with respect to the variable z).

On multiplying equation (4) scalarly by $\mathbf{M}$ and by $\mathbf{H}^e$, we get two integrals of the motion: $M^2 \equiv M_0^2 = \text{const}$ and

$$\alpha\{(\mathbf{M}')^2 - (\mathbf{M}_0')^2\} - (4\pi - \beta)(\mathbf{M}\mathbf{n})^2 = 0, \tag{6}$$

where M_0^2 is the value of the square of the magnetic-moment density and where $\mathbf{M}_0'$ is the value of the derivative $\mathbf{M}'$ at a point where the vector $\mathbf{M}$ is parallel to the x axis (the x axis lies in the plane of easy magnetization; in other respects the direction of this axis is arbitrary).

On multiplying equation (4) scalarly by $\mathbf{n}$ and on introducing the notation

$$M_x + iM_y = M_t e^{i\varphi}, \quad M_t = M_0 \sin\theta, \quad M_z = M_0\cos\theta, \tag{7}$$

we get, after simple transformation,

$$\varphi' = \sin^{-2}\theta\,\{\varphi_0' - \cos\theta\, V(\alpha g M_0)^{-1}\}, \tag{8}$$

where φ_0' is the value of the derivative φ' at a point where $\mathbf{M}$ is parallel to the x axis. We note that the equation obtained from (4) by vector multiplication by $\mathbf{n}$ is a consequence of the relations (6) and (8).

On taking into account that

$$(\mathbf{M}')^2 = M_0^2\{\theta'^2 + \varphi'^2\sin^2\theta\}$$

and on using (8), we can reduce the relation (6) to the form

$$(\theta'\sin\theta)^2 = \Psi(\cos\theta), \tag{9}$$

where

$$\begin{aligned}\Psi(\zeta) &= \theta_0'^2 + 2\varphi_0'\frac{V}{\alpha g M_0}\zeta \\ &+ \left[\frac{4\pi-\beta}{\alpha}\left(1 - \frac{V^2}{V_0^2}\right) - \theta_0'^2 - \varphi_0'^2\right]\zeta^2 \\ &- \frac{4\pi-\beta}{\alpha}\zeta^4, \quad V_0 = gM_0\sqrt{\alpha(4\pi-\beta)},\end{aligned} \tag{10}$$

and where θ_0' is the value of the derivative θ' at a point where the vector $\mathbf{M}$ is parallel to the x axis. On integrating this equation, we get

$$z - Vt = \int \{\Psi(\cos\theta)\}^{-1/2} d\cos\theta. \tag{11}$$

The relation (11), which determines the dependence of the angle θ on the coordinates and the time, together with the relations (7) and (8), completely describes the distribution of the magnetic-moment density of a ferromagnet in a nonlinear stationary-profile spin wave.

2. MAGNETIC-MOMENT ROTATION WAVES AND PERIODIC WAVES IN FERROMAGNETS

Equation (11), which determines $\cos\theta$ as a function of the variable $z - Vt$, has two types of solutions: aperiodic solutions (decreasing at infinity) and periodic solutions, corresponding to two types of stationary-profile waves—magnetic-moment rotation waves and periodic waves. We consider first the magnetic-moment rotation wave. In such a wave, obviously, the derivative $\mathbf{M}'$ and the projection $\mathbf{M}\cdot\mathbf{n}$ of the magnetic-moment density on the anisotropy axis approach

*$[\mathbf{M}\mathbf{H}^e] \equiv \mathbf{M}\times\mathbf{H}^e$.

zero as z approaches $\pm\infty$. On setting, therefore, $\theta_0' = \varphi_0' = 0$ in equation (11) and on performing the integration, we get

$$\cos\theta = \left(1 - \frac{V^2}{V_0^2}\right)^{1/2} \mathrm{ch}^{-1}\frac{z - Vt}{z_0},$$
$$z_0 = \left(\frac{\alpha}{4\pi - \beta}\right)^{1/2}\left(1 - \frac{V^2}{V_0^2}\right)^{-1/2}. \qquad (12)$$

According to this relation, the velocity of a magnetic-moment rotation wave cannot exceed a critical velocity V_0. If the velocity V is close to V_0, then the amplitude of the wave (that is, the largest angle of deviation of the vector **M** from the plane of easy magnetization) is small; the quantity z_0, which determines the breadth of the wave, is (for V close to V_0) large and proportional to $(1 - V^2/V_0^2)^{-1/2}$.

By use of the relations (12), (7), and (8) we can follow the changes of the vector magnetic-moment density in a magnetic-moment rotation wave. The component M_z of this vector along the anisotropy axis increases with increase of the variable $z - Vt$ from zero (at $z \to -\infty$) to its largest value, equal to $M_0(1 - V^2/V_0^2)^{1/2}$, and then becomes smaller again and vanishes at $z \to +\infty$. The projection $\mathbf{M}_t = \mathbf{M} - \mathbf{n}(\mathbf{M}\cdot\mathbf{n})$ of the magnetic-moment density on the plane of easy magnetization decreases in absolute value from M_0 (at $z \to -\infty$) to M_0V/V_0 and then increases again to M_0 (at $z \to +\infty$). (The dependence of the quantities M_z and M_t on the variable $z - Vt$ is shown schematically in Fig. 1.)

According to (8), the vector $\mathbf{M}_t$ rotates about the anisotropy axis, making with the x axis an angle

$$\varphi = -\left\{\mathrm{arctg}\left(\frac{V_0}{V}\mathrm{sh}\frac{z - Vt}{z_0}\right) + \frac{\pi}{2}\right\} \qquad (13)$$

(the x axis is chosen in the direction of the vector **M** at $z \to -\infty$). It is easy to see that if $z \to +\infty$, then $\varphi \to -\pi$; thus in a magnetic-moment rotation wave, the vectors **M** at $+\infty$ and at $-\infty$ are antiparallel.

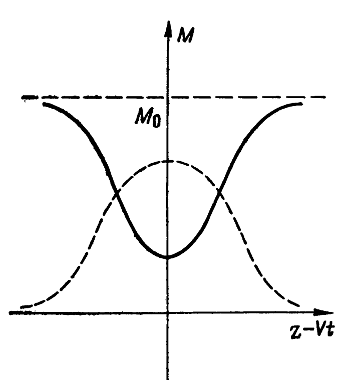

FIG. 1. Solid curve, M_t; dashed curve, M_z.

We now consider those oscillations of the magnetic moment of a ferromagnet for which the derivative of the magnetic moment and the component of the moment along the anisotropy axis do not vanish simultaneously; that is $M_0' \neq 0$. In this case, the relation (11) determines a periodic function $\theta(z - Vt)$, which runs through all values between $\theta = \arccos\zeta_+$ and $\theta = \arccos\zeta_-$, where ζ_+ (ζ_-) is the smallest, in absolute value, positive (negative) root of the equation $\Psi(\zeta) = 0$. Thus when $M_0' \neq 0$, there is propagated in the ferromagnet a periodic wave of finite amplitude, in which the angle θ performs oscillations between θ_+ and θ_-, where $\theta_\pm = \arccos\zeta_\pm$.

We remark that a periodic wave of finite amplitude is possible for arbitrary values of the characteristic quantities of the ferromagnet, if only $M_0' \neq 0$. In fact, for existence of such a wave it is sufficient that the equation $\Psi(\zeta) = 0$ shall have a root in the interval $(-1, 0)$ and a root in the interval $(0, 1)$; it is easy to see that two such roots actually do exist, if we take into account that

$$\Psi(0) = \theta_0'^2 > 0, \quad \Psi(\pm 1) = -(\varphi_0' \mp V/\alpha g M_0)^2 < 0.$$

The length λ of the periodic wave is obviously determined by the relation

$$\lambda = 2\int_{\zeta_-}^{\zeta_+} \{\Psi(\zeta)\}^{-1/2} d\zeta. \qquad (14)$$

It is easily seen that if $M_0' \to 0$, then $\zeta_- \to 0$ and $\zeta_+ \to (1 - V^2/V_0^2)^{1/2}$; at the same time $\lambda \to \infty$, and the periodic wave degenerates to the magnetic-moment rotation wave considered above.

In the periodic wave, the projection $\mathbf{M}_t$ of the magnetic-moment density on the plane of easy magnetization rotates about the axis of anisotropy. The angle φ between the vector $\mathbf{M}_t$ and the x axis (in contrast to the angle θ between the vector **M** and the anisotropy axis) is not a periodic function of the coordinates and the time. In particular, the values of the angle φ at two points the distance between which is the wavelength λ differ from each other by the amount

$$\Delta\varphi = 2\int_{\zeta_-}^{\zeta_+} (1 - \zeta^2)^{-1}\left(\varphi_0' - \frac{V}{\alpha g M_0}\zeta\right)\{\Psi(\zeta)\}^{-1/2} d\zeta, \qquad (15)$$

which in general is not a multiple of π.

We consider by way of example a harmonic spin wave in the simplest case, when $\theta_0' = 0$. On introducing the notation

$$\zeta_0 = 2\frac{V}{V_0}q\left(q^2 - 1 + \frac{V^2}{V_0^2}\right)^{-1}, \quad q^2 = \frac{\alpha}{4\pi - \beta}\varphi_0'^2$$

and on supposing that $\zeta_0 \ll 1$ and $\zeta_0^3 \ll Vq/V_0$, we get, according to (11),

$$\theta = \frac{\pi}{2} - \zeta_0 \sin^2 \frac{\pi(z - Vt)}{\lambda},$$

$$\lambda = 2\pi\left(\frac{\alpha}{4\pi - \beta}\right)^{1/2}\left(q^2 - 1 + \frac{V^2}{V_0^2}\right)^{-1/2}. \quad (16)$$

On substituting this relation into (8), we determine the dependence of the angle φ on the coordinates and the time:

$$\varphi = \varphi_0'\left(q^2 - 1 + \frac{V^2}{V_0^2}\right)^{-1} \times \left\{(q^2 - 1)(z - Vt) + \frac{V^2}{V_0^2}\frac{\lambda}{2\pi}\sin\frac{2\pi(z - Vt)}{\lambda}\right\}. \quad (17)$$

From this we get for the quantity $\Delta\varphi$, which describes the angle of rotation of the vector $\mathbf{M}_t$ in a wavelength,

$$\Delta\varphi = 2\pi q(q^2 - 1)(q^2 - 1 + V^2/V_0^2)^{-3/2}. \quad (18)$$

We see that the angle $\Delta\varphi$ is a multiple of 2π only for definite values of the derivative φ_0'; namely, for $\varphi_0' = 0$ and $\varphi_0' = \pm((4\pi - \beta)/\alpha)^{1/2}$, and also if $|\varphi_0'| \gg ((4\pi - \beta)/\alpha)^{1/2}$.

3. MAGNETIC-MOMENT FLIP WAVES IN A FERROMAGNET WITH ANISOTROPY OF THE EASY-AXIS TYPE

In the preceding section, we considered a ferromagnetic crystal with anisotropy of the easy-plane type, and we showed that in such a crystal two types of stationary spin waves are possible: aperiodic (in which the magnetic-moment excitation decreases at infinity) and periodic. We now consider stationary spin waves in a ferromagnetic crystal with anisotropy of the axis-of-easy-magnetization type. In such a crystal also, periodic and aperiodic waves can be propagated; in principle two types of aperiodic waves are possible, waves in which the vector magnetic-moment density before the forward front of the wave (at $-\infty$) and behind its rear front (at $+\infty$) are parallel and waves in which these vectors are antiparallel. Aperiodic waves of the first type, solitary waves, and also periodic waves were investigated in our previous paper [1]; we shall now turn to the study of the second type of waves, magnetic-moment flip waves.

We shall start from the equation of motion of the magnetic moment of a ferromagnet and the equations of magnetostatics, (1)—(3) (in the case being considered, a crystal with anisotropy of the axis-of-easy-magnetization type, $\beta > 0$). We restrict ourselves to the consideration of waves propagated perpendicularly to the anisotropy axis, and we suppose that the external magnetic field is equal to zero; we then get Eq. (4), in which the role of effective field is played by the quantity

$$\mathbf{H}^e = \alpha\mathbf{M}'' + \beta\mathbf{n}(\mathbf{Mn}) - 4\pi V^{-2}(\mathbf{MV})\mathbf{V} \quad (19)$$

and a prime denotes differentiation with respect to the variable x (the x axis is chosen in the direction of $\mathbf{V}$; the z axis as before is directed along the anisotropy axis).

On multiplying Eq. (4) scalarly by $\mathbf{M}$ and by $\mathbf{H}^e$, we get two integrals of the motion: $M^2 \equiv M_0^2 = \text{const}$ and

$$\alpha(\mathbf{M}')^2 + \beta\{(\mathbf{Mn})^2 - M_0^2\} - 4\pi V^{-2}(\mathbf{MV})^2 = 0 \quad (20)$$

(we have taken into account that in an aperiodic wave the quantities $\mathbf{M}'$ and $\mathbf{M}_t = \mathbf{M} - \mathbf{n}(\mathbf{M}\cdot\mathbf{n})$ vanish simultaneously). On further multiplying equation (4) scalarly by $\mathbf{n}$ and on using (7), we get

$$\frac{\partial}{\partial x}(V\cos\theta + \alpha g M_0\varphi'\sin^2\theta) + 2\pi g M_0\sin^2\theta\sin 2\varphi = 0. \quad (21)$$

We remark that the equation obtained from (4) by vector multiplication by $\mathbf{n}$ is a consequence of relations (20) and (21).

We shall be interested in those motions of the magnetic moment for which the vector $\mathbf{M}$ always lies in a plane that makes a constant angle φ with the plane $(\mathbf{n}, \mathbf{V})$; that is, in plane spin waves (see Fig. 2). By taking into account that in a plane wave $(\mathbf{M}')^2 = M_0^2\theta'^2$, relations (20) and (21) can be put into the form

$$\theta'^2 = \alpha^{-1}(4\pi\cos^2\varphi + \beta)\sin^2\theta, \quad (20')$$

$$\theta' = 2\pi g M_0 V^{-1}\sin 2\varphi\sin\theta. \quad (21')$$

On solving equation (20′), we get

$$\theta = 2\,\mathrm{arctg}\exp\frac{x - Vt}{x_0}, \quad x_0 = \alpha^{1/2}(4\pi\cos^2\varphi + \beta)^{-1/2}. \quad (22)$$

On substituting this expression into (7), we find the dependence on coordinates and time of the components of the vector $\mathbf{M}$ parallel (M_z) and perpendicular (M_t) to the anisotropy axis:

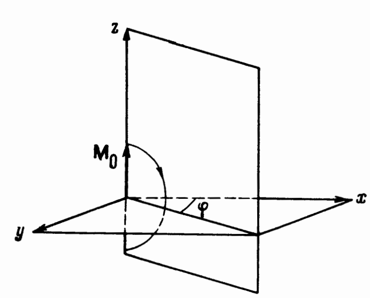

FIG. 2.

$$M_z = -M_0 \,\mathrm{th}\,\frac{x-Vt}{x_0}, \quad M_t = M_0\,\mathrm{ch}^{-2}\frac{x-Vt}{x_0}. \quad (23)$$

We see that the value of M_t rises from zero (at $x \to -\infty$) to its largest value M_0 and then decreases again to zero (at $x \to +\infty$). As for M_z, this quantity decreases monotonically from M_0 (at $x \to -\infty$) to $-M_0$ (at $x \to +\infty$); therefore a wave of the type considered can be called a magnetic-moment flip wave. (The dependence of the quantities M_z and M_t on the variable $x - Vt$ is shown schematically in Fig. 3.)

For the velocity of a magnetic-moment flip-wave—determined, obviously, from the condition for compatibility of equations (20′) and (21′)—we get

$$V = 2\pi g M_0 \alpha^{1/2} \sin 2\varphi (4\pi \cos^2\varphi + \beta)^{-1/2}. \quad (24)$$

We see that the value of V depends on the angle φ between the vector M_t and the direction along which the quantities that describe the ferromagnet change (the x axis). On increase of the angle φ, the velocity V rises from zero (at $\varphi = 0$) to a maximum value

$$V_c = gM_0(2\alpha)^{1/2}\{2\pi + \beta - \beta^{1/2}(4\pi+\beta)^{1/2}\}^{1/2} \quad (25)$$

and then decreases again to zero (at $\varphi = \pi/2$). We remark that the velocity of a magnetic-moment flip wave (like the velocity of a solitary wave or of a magnetic-moment rotation wave) is bounded from above: it cannot exceed the critical velocity V_c.

If the magnetic moment rotates in a plane perpendicular to the direction along which the quantities that describe the ferromagnet change (that is, if $\varphi = \pi/2$), then the velocity of the wave vanishes, and formulas (22) and (23) become the well-known relations that describe the structure of a domain wall in a ferromagnet (see Landau and Lifshitz[3,4]):

$$M_z = -M_0\,\mathrm{th}\,\frac{x}{x_0}, \quad M_t = M_0\,\mathrm{ch}^{-2}\frac{x}{x_0}, \quad x_0 = \left(\frac{\alpha}{\beta}\right)^{1/2}. \quad (26)$$

In the general case $V \neq 0$, the magnetic-moment flip wave can be interpreted as a uniformly moving domain wall[1)].

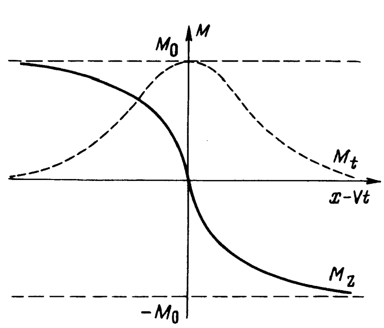

FIG. 3.

4. EQUATIONS DESCRIBING A STATIONARY-PROFILE SPIN WAVE IN AN ANTIFERROMAGNET WITH ANISOTROPY OF THE EASY-PLANE TYPE

In Sec. 2, on investigating the magnetic-moment rotation wave in a ferromagnet with anisotropy of the easy-plane type, we saw that on passage of such a wave, whatever its amplitude, the vector magnetic-moment density rotates through an angle π. It is of course obvious to begin with that the vectors M at $+\infty$ and at $-\infty$ lie in the plane of easy magnetization; the fact that these vectors are also antiparallel is in the case of anisotropy of the easy-plane type (in contrast to the case of anisotropy of the easy-axis type) in a certain sense accidental. In particular, there is no ground for expecting that a similar situation will persist in the case of other magnetically ordered crystals (for example, antiferromagnets) with anisotropy of the easy-plane type. In fact, we shall show that in anitferromagnetic crystals with anisotropy of the easy-plane type, the angle of rotation of the magnetic moments in an aperiodic wave depends on the amplitude of the wave and is not in general a multiple of π.

In investigating nonlinear spin waves in antiferromagnets, we shall start from the equations of motion of the magnetic moments of the sublattices (see, for example,[2])

$$\partial \mathbf{M}_i/\partial t = g[\mathbf{M}_i\mathbf{H}_i^e] \quad (i = 1, 2), \quad (27)$$

where M_i is magnetic-moment density of the i-th magnetic sublattice, H_i^e is the effective field

$$\mathbf{H}_1^e = \mathbf{H} + \alpha\Delta\mathbf{M}_1 + \beta\mathbf{n}(\mathbf{M}_1\mathbf{n}) + \alpha_{12}\Delta\mathbf{M}_2 - \delta\mathbf{M}_2,$$
$$\mathbf{H}_2^e = \mathbf{H} + \alpha\Delta\mathbf{M}_2 + \beta\mathbf{n}(\mathbf{M}_2\mathbf{n}) + \alpha_{12}\Delta\mathbf{M}_1 - \delta\mathbf{M}_1, \quad (28)$$

H is the magnetic field, related by equations (3) to the total magnetic-moment density $M = M_1 + M_2$, g is the gyromagnetic ratio, n is a unit vector in the direction of the anisotropy axis (the z axis), α, α_{12}, and δ are exchange-interaction constants, and β is a magnetic-anisotropy constant (in the case being considered, a crystal with

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[1)]We emphasize that a magnetic-moment flip wave is propagated in a crystal in the absence of a magnetic field and is caused by an initial excitation of the magnetic-moment density (and not by a constant field applied to the crystal). In this it differs significantly from the translation of a domain wall under the action of a magnetic field applied to the crystal, investigated by Landau and Lifshitz [3] anisotropy of the plane-of-easy-magnetization type, $\beta < 0$).

We shall be interested in those magnetic-moment excitations in which the vector $\mathbf{M}$ at the initial instant of time is parallel to the anisotropy axis, and the direction along which the quantities describing the crystal change is parallel to the plane of easy magnetization; we shall suppose that the external magnetic field is zero. It is easy to verify that the waves that occur in this case have a velocity parallel to the plane of easy magnetization and that the following relations are satisfied for all $\mathbf{r}$ and t:

$$\mathbf{M} = 2\mathbf{n}(\mathbf{M}_1\mathbf{n}),\ \mathbf{M}_2 = 2\mathbf{n}(\mathbf{M}_1\mathbf{n}) - \mathbf{M}_1,\ \mathbf{H} = 0. \quad (29)$$

On eliminating $\mathbf{M}_2$ from formula (28) by means of these relations, and on using the fact that in a stationary-profile wave all quantities depend on the coordinates and the time only in the combination $\mathbf{r} - \mathbf{V}t$, where $\mathbf{V}$ is the constant speed of the wave, we can reduce the equation for $\mathbf{M}_1$ to the form

$$V\mathbf{M}_1' + g[\mathbf{M}_1\mathbf{H}_1^e] = 0,$$
$$\mathbf{H}_1^e = (\alpha - \alpha_{12})\mathbf{M}_1'' + \delta\mathbf{M}_1 + 2\mathbf{n}(\alpha_{12}\mathbf{M}_1''\mathbf{n} - \delta\mathbf{M}_1\mathbf{n}), \quad (30)$$

where the prime denotes differentiation with respect to the variable $\xi = V^{-1}\mathbf{r}\cdot\mathbf{V} - Vt$ (we have taken into account that the magnetic anisotropy constant is much smaller than the constant of exchange interaction between sublattices, $|\beta| \ll \delta$).

Equations (30) are formally analogous to equations (4) and (5); therefore to integrate them, we apply the same method that was used in Sec. 1 to integrate the equations of motion of the magnetic moment in a ferromagnet. Thus on multiplying the first of equations (30) scalarly by $\mathbf{M}_1$ and by $\mathbf{H}_1^e$, we get two integrals of the motion: $M_1^2 \equiv M_0^2$ = const and

$$(\alpha - \alpha_{12})\{(\mathbf{M}_1')^2 - (\mathbf{M}_0')^2\} + 2\alpha_{12}\{(\mathbf{M}_1'\mathbf{n})^2 - (\mathbf{M}_0'\mathbf{n})^2\} - 2\delta(\mathbf{M}_1\mathbf{n})^2 = 0, \quad (31)$$

where M_0^2 is the value of the square of the magnetic-moment density of each of the sublattices, and where M_0' is the value of the derivative M_1' at a point where the vector $\mathbf{M}_1$ parallel to the x axis (the x axis lies in the plane of easy magnetization; in other respects the direction of this axis is arbitrary).

Further, on multiplying the first of Eqs. (30) scalarly by $\mathbf{n}$ and introducing the notation

$$M_{1x} + iM_{1y} = M_{1t}e^{i\varphi},\ M_{1t} = M_0\sin\theta,\ M_{1z} = M_0\cos\theta, \quad (32)$$

we get after simple transformations

$$\varphi' = \sin^{-2}\theta\,\{\varphi_0' - \cos\theta\,V(gM_0)^{-1}(\alpha - \alpha_{12})^{-1}\}, \quad (33)$$

where φ_0' is the value of the derivative φ' at a point where the vector $\mathbf{M}_1$ is parallel to the x axis. We note that the equation obtained from (30) by vector multiplication by $\mathbf{n}$ is a consequence of the relations (31) and (33).

On noting that, according to (32) and (33),

$$(\mathbf{M}_1')^2 = M_0^2\{\theta'^2 + \sin^{-2}\theta[\varphi_0' - \cos\theta\,V(gM_0)^{-1}(\alpha - \alpha_{12})^{-1}]^2\},$$
$$(\mathbf{M}'\mathbf{n})^2 = M_0^2\theta'^2\sin^2\theta,$$

we can reduce the relation (31) to the form

$$(\theta'\sin\theta)^2 = \Psi(\cos\theta), \quad (34)$$

where

$$\Psi(\zeta) = (\alpha + \alpha_{12} - 2\alpha_{12}\zeta^2)^{-1}\Big\{(\alpha + \alpha_{12})\theta_0'^2 + 2\varphi_0'V(gM_0)^{-1}\zeta + 2\delta\Big[1 - \frac{V^2}{V_0^2} - \frac{\alpha + \alpha_{12}}{2\delta}\theta_0'^2 - \frac{\alpha - \alpha_{12}}{2\delta}\varphi_0'^2\Big]\zeta^2 - 2\delta\zeta^4\Big\} \quad (35)$$

V_0 is the phase velocity of a small-amplitude spin wave

$$V_0 = gM_0\sqrt{2\delta(\alpha - \alpha_{12})} \quad (36)$$

and θ_0' is the value of the derivative θ' at a point where the vector $\mathbf{M}_1$ is parallel to the x axis. On integrating this equation, we get

$$\xi = \int\{\Psi(\cos\theta)\}^{-1/2}d\cos\theta. \quad (37)$$

The relation (37), which determines the dependence of the angle θ on the coordinates and the time, together with the relations (32), (33), and (29), completely describes the distribution of the total magnetic-moment density and of the magnetic-moment density of each of the sublattices of the antiferromagnet in a nonlinear stationary-profile spin wave.

In closing this section, we remark that if in the expression (35) for the function Ψ we set $\alpha_{12} = 0$ and carry out the substitution $2\delta \to 4\pi - \beta$, then Eqs. (34) and (35) become Eqs. (9) and (10), which describe nonlinear spin waves in a ferromagnet. For this reason, the formulas of Sec. 2 for the magnetic-moment density of a ferromagnet can be obtained from the corresponding formulas of Sec. 5 for the quantity $\mathbf{M}_1$, by setting $\alpha_{12} = 0$ in the latter and making the substitution $\xi \to z - Vt$, $2\delta \to 4\pi - \beta$.

5. MAGNETIC-MOMENT ROTATION WAVES AND PERIODIC WAVES IN ANTIFERROMAGNETS

Equation (37), which determines $\cos\theta$ as a function of the variable $\xi = V^{-1}\mathbf{r}\cdot\mathbf{V} - Vt$, has, as does equation (11), two types of solutions, aperiodic (decreasing at infinity) and periodic solutions, corresponding to two types of stationary-profile waves, magnetic-moment rotation waves and periodic waves. We consider first the magnetic-moment rotation wave. In such a wave, before the forward front of the wave and behind its rear front, the magnetic moments of the sublattices are antiparallel and lie in the plane of easy magnetization. If we choose the x axis along the direction of the vector $\mathbf{M}_1$ at $\xi \to -\infty$ and note that at infinity $\theta' = \varphi' = 0$, we can reduce the relation (37) to the form

$$|\xi| = \frac{1}{2}\sqrt{\frac{\alpha_{12}}{\delta}}\left\{W_0 \ln\frac{W+W_0}{W-W_0} - \ln\frac{W+1}{W-1}\right\}, \quad (38)$$

where

$$W = \left(\frac{\alpha+\alpha_{12}}{2\alpha_{12}} - \cos^2\theta\right)^{1/2}\left(1 - \frac{V^2}{V_0^2} - \cos^2\theta\right)^{-1/2},$$

$$W_0 = \left(\frac{\alpha+\alpha_{12}}{2\alpha_{12}}\right)^{1/2}\left(1 - \frac{V^2}{V_0^2}\right)^{-1/2}$$

We remark that according to this relation, the velocity of a magnetic-moment rotation wave cannot exceed the velocity V_0 of a small-amplitude spin wave.

By use of the relations (38), (32), (33), and (29), we can follow the changes of the quantities that describe the antiferromagnet in a magnetic-moment rotation wave. At $\xi \to -\infty$ the magnetic moments of the sublattices are directed, as has already been pointed out, along the x axis (in opposite directions), and the total magnetic moment is zero. With increase of ξ, there appear components of the magnetic moments along the anisotropy axis. The vector $\mathbf{M}$ is always parallel (or antiparallel) to the vector $\mathbf{n}$ and increases in absolute value from zero (at $\xi \to -\infty$) to a largest value M_{max} (at $\xi = 0$),

$$M_{max} = 2M_0(1 - V^2/V_0^2)^{1/2}, \quad (39)$$

and then again decreases and vanishes at $\xi \to +\infty$, with $M(\xi) = M(-\xi)$. The projections of the magnetic moments of the sublattices on the anisotropy axis behave similarly and are connected with the quantity M by the relation $\mathbf{M}_1 \cdot \mathbf{n} = \mathbf{M}_2 \cdot \mathbf{n} = \frac{1}{2}M$. (The dependence of the total magnetic-moment density M on the quantity ξ is shown schematically in Fig. 4.)

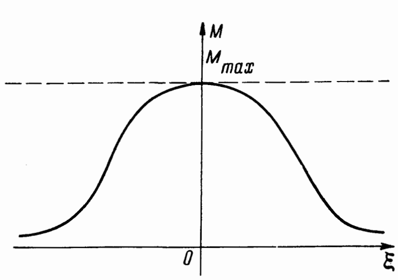

FIG. 4

The projections of the magnetic moments of the sublattices on the plane of easy magnetization, $\mathbf{M}_{1t} = \mathbf{M}_t - \mathbf{n}(\mathbf{M}_1 \cdot \mathbf{n})$ and $\mathbf{M}_{2t} = \mathbf{M}_2 - \mathbf{n}(\mathbf{M}_2 \cdot \mathbf{n})$, diminish in absolute value from M_0 (at $\xi \to -\infty$) to M_0V/V_0 (at $\xi \to 0$) and then increase again, reaching M_0 at $\xi \to +\infty$. The vectors $\mathbf{M}_{1t}$ and $\mathbf{M}_{2t}$ meanwhile rotate about the anisotropy axis. At $\xi \to +\infty$ these vectors are no longer directed along the x axis but make with this axis an angle

$$\varphi_0 = -2\left(\frac{\alpha+\alpha_{12}}{\alpha-\alpha_{12}}\right)^{1/2}\frac{V}{V_0}\int_0^{\zeta_0} d\zeta\left(1 - \frac{2\alpha_{12}}{\alpha+\alpha_{12}}\zeta^2\right)^{1/2}$$
$$\times (1-\zeta^2)^{-1}(\zeta_0^2-\zeta^2)^{-1/2}$$
$$\zeta_0^2 = 1 - V^2/V_0^2, \quad (40)$$

which depends on the wave velocity V and is in general not a multiple of π.

According to formulas (38) and (39), the breadth of a magnetic-moment rotation wave is equal in order of magnitude to $\xi_0 \sim (\alpha/\delta)^{1/2} \times M_0/M_{max}$. If $M_{max} \sim M_0$, then the value of ξ_0 has the same order of magnitude as the lattice constant; in this case the relations (38) to (40) describe the nonlinear motions of the magnetic moments, of course, only qualitatively. (We emphasize that in the case of a ferromagnet, the breadth z_0 of a magnetic-moment rotation wave, according to formula (12), always exceeds the lattice constant significantly.) Therefore the greatest interest attaches to the case when the angle between the vectors $\mathbf{M}_1$ and $\mathbf{M}_2$ deviates slightly from its equilibrium value (equal to π), and consequently the total magnetic moment that occurs is small, $M_{max} \ll M_0$. The velocity of such a wave is obviously close to the value of V_0. The total magnetic moment and the projections of the vectors $\mathbf{M}_{1,2}$ on the anisotropy axis are determined, according to (38), by the formulas

$$M = 2M_0\left(1 - \frac{V^2}{V_0^2}\right)^{1/2}\operatorname{ch}^{-1}\frac{\xi}{\xi_0}, \quad \mathbf{M}_1\mathbf{n} = \mathbf{M}_2\mathbf{n} = \frac{1}{2}M,$$
$$\xi_0 = \left(\frac{\alpha+\alpha_{12}}{2\delta}\right)^{1/2}\left(1 - \frac{V^2}{V_0^2}\right)^{-1/2}. \quad (41)$$

We emphasize that the value of ξ_0, which determines the breadth of a magnetic-moment rotation wave, is in this case large (proportional to M_0/M_{max}).

On integrating the relation (33), we get expressions for the components of the vectors $\mathbf{M}_{1,2}$ in the plane of easy magnetization:

$$M_{1x} = -M_{2x} = M_0\cos\varphi(\xi), \quad M_{1y} = -M_{2y} = M_0\sin\varphi(\xi),$$

$$\varphi(\xi) = -2\left(\frac{\alpha+\alpha_{12}}{\alpha-\alpha_{12}}\right)^{1/2}\operatorname{arctg}\exp\frac{\xi}{\xi_0}. \quad (42)$$

For the angle φ_0 through which the magnetic moments of the sublattices rotate on passage of the wave, we thus find

$$\varphi_0 = -\pi\left(\frac{\alpha+\alpha_{12}}{\alpha-\alpha_{12}}\right)^{1/2}. \tag{43}$$

We see that even if the velocity V of the wave is close to the velocity V_0 of a small-amplitude spin wave (and consequently the total magnetic moment in the wave is small), the vectors M_1 and M_2 rotate about the anisotropy axis through an appreciable angle (remaining almost exactly antiparallel to each other).

We now consider those oscillations of the magnetic moments of the antiferromagnet in which the derivative of the magnetic moments and the components of the moments along the anisotropy axis do not vanish simultaneously; that is, $M_0' \neq 0$. In this case the relation (37) determines a periodic function $\cos\theta = \zeta(\xi)$, which runs through all values between ζ_- and ζ_+, where $\zeta_+(\zeta_-)$ is the smallest, in absolute value, positive (negative) root of the equation

$$P(\zeta) \equiv \frac{\alpha+\alpha_{12}}{2\delta}(\theta_0')^2 + \zeta\frac{V\varphi_0'}{\delta g M_0} + \zeta^2\left[1 - \frac{V^2}{V_0^2} - \frac{\alpha+\alpha_{12}}{2\delta}(\theta_0')^2 - \frac{\alpha-\alpha_{12}}{2\delta}(\varphi_0')^2\right] - \zeta^4 = 0. \tag{44}$$

Thus when $M_0' \neq 0$, there is propagated in the antiferromagnet a periodic wave of finite amplitude, in which the angle θ performs oscillations between θ_+ and θ_-, where $\theta_\pm = \arccos\zeta_\pm$.

As in the case of a ferromagnet, a periodic wave in an antiferromagnet is possible for arbitrary values of the quantities characteristic of the crystal, if only $M_0' \neq 0$. In fact, for existence of such a wave it is sufficient that Eq. (44) have a root in the interval $(-1, 0)$ and a root in the interval $(0, 1)$; that two such roots actually exist is easily verified if one takes into account that

$$P(0) = \frac{\alpha+\alpha_{12}}{2\delta}(\theta_0')^2 > 0,$$

$$P(\pm 1) = -\frac{\alpha-\alpha_{12}}{2\delta}\left(\varphi_0' \mp \frac{V_0}{gM_0(\alpha-\alpha_{12})}\right)^2 < 0.$$

The length λ of a periodic wave in the case of an antiferromagnet is determined as usual by formula (14), in which the expression (35) must be substituted for the function Ψ. If $M_0' \to 0$, then, as in the case of a ferromagnet,

$$\zeta_- \to 0, \quad \zeta_+ \to (1 - V^2/V_0^2)^{1/2}, \quad \lambda \to \infty,$$

and the periodic wave degenerates to a magnetic-moment rotation wave.

The projections of the sublattice magnetic-moment densities on the plane of easy magnetization rotate, in a periodic wave, about the axis of anisotropy. The angle φ between the vector M_{1t} and the x axis (in contrast to the angle θ between the vectors $M_{1,2}$ and the anisotropy axis) is not a periodic function of the coordinates and the time. In particular, the values of the angle φ at two points the distance between which is a wavelength λ differ from one another by the amount

$$\Delta\varphi = 2\int_{\zeta_-}^{\zeta_+} (1-\zeta^2)^{-1}\left(\varphi_0' - \frac{2V\delta g M_0}{V_0^2}\right)\{\Psi(\zeta)\}^{-1/2}\,d\zeta, \tag{45}$$

which is in general not equal to π.

By way of example, we consider a harmonic spin wave in the simplest case, when $V = V_0$ and $\theta_0' = 0$. On introducing the notation $\zeta_0 = 2V_0(\alpha - \alpha_{12})^{-1}(gM_0)^{-1}\varphi_0'^{-1}$ and on supposing that $\zeta \ll 1$, we get according to (37)

$$\theta = \frac{\pi}{2} - \zeta_0 \sin^2\frac{\pi\xi}{\lambda}, \quad \lambda = 2\pi\left(\frac{\alpha+\alpha_{12}}{\alpha-\alpha_{12}}\right)^{1/2}\varphi_0'^{-1}. \tag{46}$$

On substituting this expression into (33), it is easy to verify that the angle φ in this case is a linear function of the coordinates and the time, $\varphi = \Delta\varphi\,\xi/\lambda$, where the quantity $\Delta\varphi$, which describes the angle of rotation of the vectors M_{1t} and M_{2t} in a wavelength, has the form

$$\Delta\varphi = 2\pi\left(\frac{\alpha+\alpha_{12}}{\alpha-\alpha_{12}}\right)^{1/2}. \tag{47}$$

We turn now to the problem of experimental observation of nonlinear spin waves. In the case of antiferromagnetic crystals it is possible, in principle, to detect these waves by using the phenomenon of magnetoacoustic resonance. If the phase velocity V_0 of a small-amplitude spin wave is greater than the velocity of sound s, then resonance between a small-amplitude sound wave and a spin wave is impossible. When $V_0 > s$, however, there should occur "nonlinear" magnetoacoustic resonance: a sound wave of finite amplitude should excite through resonance a nonlinear spin wave, propagated with velocity $V = s$, and the resonance should set in at a strictly determined value of the amplitude of the sound wave. Nonlinear magnetoacoustic resonance should be observed most easily in crystals with anisotropy of the easy-plane type, placed in a strong, constant, homogeneous magnetic field directed perpendicular to this plane. The advantage of such crystals consists in the facts that, first, the magnetoelastic coupling parameter is very large in them[5] and, second, by choice of the external magnetic field it is possible to produce a phase velocity V_0

sufficiently close to the velocity of sound s, thanks to the fact that observation of nonlinear magneto-acoustic resonance does not require too large a sound amplitude.

In the case of ferromagnetic crystals, the velocity of a detached spin wave amounts, according to (10), to a few tens of m/sec. For detection of such a wave, therefore, it is possible to use "magnetomechanical" resonance, by moving a crystal through a narrow region of constant magnetic field; resonance should occur when there is a definite relation between the velocity of motion of the crystal and the size of the magnetic field.

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[1] I. A. Akhiezer and A. E. Borovik, JETP **52**, 508 (1967), Soviet Phys. JETP **25**, 332 (1967).

[2] A. I. Akhiezer, V. G. Bar'yakhtar, and M. I. Kaganov, UFN **71**, 533 (1960), Soviet Phys. Uspekhi **3**, 567 (1961).

[3] L. D. Landau and E. M. Lifshitz, Physik. Z. Sowjetunion **8**, 153 (1935).

[4] L. D. Landau and E. M. Lifshitz, Electrodynamics of Continuous Media (Addison-Wesley Publishing Company, Inc., Reading, Mass., 1960).

[5] V. G. Bar'yakhtar, M. A. Savchenko, and V. V. Tarasenko, JETP **49**, 944 (1965), Soviet Phys. JETP **22**, 657 (1966).

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