DIELECTRIC CONSTANT OF A GAS OF RESONANT ATOMS

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The dielectric constant of a gas of resonant atoms is found for frequencies corresponding to the transition between a zero-momentum ground state and a unity-momentum excited state; pair collisions involving resonance energy transfer are assumed to make the main contribution. The interaction between the atoms is assumed to be of a dipole-dipole nature. It is shown that the shape of the spectral line is only approximately of the Lorentz type; broadening as well as a line shift occur. The line wings are also considered. The case of a transition between two excited states of which the lower is resonantly coupled to the ground state is investigated. The inverted population of the levels is considered and the frequency band in which generation occurs is determined.

1. INTRODUCTION

We determine in this paper the dielectric constant of a gas of resonant atoms at frequencies corresponding in the main to a transition between the excited and the ground states. We consider a physical picture in which a dielectric constant is determined completely by pair collisions of excited and unexcited atoms, with resonant energy transfer. The role of these collisions in the determination of the line shape was first noted by Vlasov and Fursov.[1] We shall show that the line shape is only approximately Lorentzian, and the line not only broadens but is also shifted. No such shift occurs for lines corresponding to a transition between two excited states. We assume that the angular momentum of the atom is equal to zero in the ground state and to unity in the excited state. The ground and excited levels are connected by an optical dipole transition, and consequently the interaction leading to the resonant energy transfer is dipole-dipole. In addition, in the greater part of the article we assume that the number of atoms in the excited state is small.

The region of applicability of the results is determined by a number of inequalities that follow both from the physical picture of the processes and from the analysis methods. The most stringent of these inequalities is the requirement that the Doppler width be small compared with the collision width, and that only pair collisions be taken into account. As is well known,[1,2] an important role in collisions with resonant energy transfer is played by impact parameters ρ of the order of[1)]

$$\rho_0 = g/\sqrt{v}, \tag{1.1}$$

where v is the relative velocity of the colliding atoms and g the reduced matrix element of the dipole transition of the individual atom. Thus, the condition for the collisions to be paired is

$$n\rho_0^3 = ng^3/v^{3/2} \ll 1, \tag{1.2}$$

where n is the density of the gas. The condition for the Doppler width to be small compared with the collision width is

$$n\rho_0^2 v = ng^2 \gg \omega v/c. \tag{1.3}$$

From inequalities (1.2) and (1.3) it follows that our analysis is valid in the following range of variation of the gas density n:

$$\omega_0\sqrt{T/Mc^2} \ll n \ll (T/M)^{3/4} \tag{1.4}$$

(T is the temperature and M the mass of atom, ω_0 is the transition frequency). This region exists for temperatures

$$T \gg M(\omega/c)^4. \tag{1.5}$$

In those cases when the number of atoms in the excited state is assumed to be small, the temperature is also bounded from above by the inequality

$$T \ll \omega_0. \tag{1.6}$$

We note also that conditions (1.2) and (1.3) lead

[1)] We use an atomic system of units, $\hbar = m = e = 1$.

to the inequality $n\lambda\!\!\!^{-3} \gg 1$, denoting that the collision width ng^2 is much larger than the natural width $\gamma = {}^4/_3 g^2/\lambda\!\!\!^{-3}$ ($\lambda\!\!\!^{-} = c/\omega_0$). This leads to the inequality

$$\lambda\!\!\!^{-} \gg \rho_0. \tag{1.7}$$

2. FUNDAMENTAL RELATIONS

The determination of the dielectric constant $\epsilon(\omega)$ reduces, as is well known,[3] to a determination of the polarization operator Π. In a transverse gauge (div $\mathbf{A} = 0$) the dielectric constant is connected with the scalar part of the polarizability tensor Π_{00} by the equation

$$\Pi_{00}(\mathbf{k}, \omega_k) = \frac{\mathbf{k}^2}{4\pi}[1 - \varepsilon(i|\omega_k|)] \tag{2.1}$$

(we shall henceforth omit the subscripts of Π).

The values of ϵ on the real axis obtained by analytically continuing (2.1) from the discrete set of points $i\omega_k = i2\pi nT$ (n—integer) on the positive imaginary axis to the entire upper half-plane. The dielectric constant can be expressed also in terms of the transverse part of the polarizability tensor. To find the polarizability tensor we shall use the temperature Green's function technique.

The Hamiltonian of the system of resonant atoms with dipole-dipole interaction in an external electric field with a scalar potential

$$\varphi = \varphi_{\mathbf{k},\omega} e^{i(\mathbf{kR}-\omega t)} + \text{c.c.}$$

is

$$H = \sum_{\mathbf{p}} \frac{\mathbf{p}^2}{2M} a_{\mathbf{p}}^{+} a_{\mathbf{p}} + \sum_{\mathbf{p}} \left(\frac{\mathbf{p}^2}{2M} + \omega_0\right) b_{\mathbf{p},\alpha}^{+} b_{\mathbf{p},\alpha}$$
$$+ \frac{1}{\Omega} \sum_{\mathbf{p}_1, \mathbf{p}_2, \mathbf{q}} V_{\alpha\beta}(\mathbf{q})\, a_{\mathbf{p}_1}^{+} b_{\mathbf{p}_2,\alpha}^{+} a_{\mathbf{p}_2+\mathbf{q}} b_{\mathbf{p}_2-\mathbf{q},\beta}$$
$$+ gk_\alpha \sum_{\mathbf{p}} (i b_{\mathbf{p},\alpha}^{+} a_{\mathbf{p}-\mathbf{k}} \varphi_{\mathbf{k},\omega} e^{-i\omega t} + \text{c.c.}), \tag{2.2}$$

where $a_{\mathbf{p}}$ and $b_{\mathbf{p}}$ are the annihilation operators of the normal and excited atoms, α is a vector index ($\alpha = 1, 2, 3$) characterizing the state with angular momentum $j = 1$, Ω is the volume of the system, M is the mass of the atom, $V_{\alpha\beta}(\mathbf{q})$ is the matrix element of the dipole-dipole interaction operator, and g is the reduced matrix element of the dipole transition between the excited and ground states:

$$\sum_{i=1}^{N} \int \psi^* r_{\alpha i} \psi_\beta \, d\mathbf{r}_1 \, d\mathbf{r}_2 \ldots d\mathbf{r}_N = g\delta_{\alpha\beta}. \tag{2.3}$$

For small $|\mathbf{q}|$ ($|\mathbf{q}| \ll 1$), the interaction potential $V_{\alpha\beta}(\mathbf{q})$ takes the form

$$V_{\alpha\beta}(\mathbf{q}) = 4\pi g^2 q_\alpha q_\beta / \mathbf{q}^2. \tag{2.4}$$

In the coordinate representation, the region of small $|\mathbf{q}|$ corresponds to distances much larger than the dimension of the atom, where the potential is equal to

$$V_{\alpha\beta}(\mathbf{R}) = -g^2 \frac{\partial^2}{\partial R_\alpha \partial R_\beta} \frac{1}{R} = g^2 \frac{R^2\delta_{\alpha\beta} - 3R_\alpha R_\beta}{R^5}. \tag{2.5}$$

The temperature Green's functions of the noninteracting particles of the ground and excited states are respectively

$$G^{(0)}(p) = \frac{1}{i\omega_p - \mathbf{p}^2/2M + \mu}, \tag{2.6}$$

$$G_{\alpha\beta}^{(0)}(p) = \frac{\delta_{\alpha\beta}}{i\omega_p - \mathbf{p}^2/2M + \mu - \omega_0}, \tag{2.7}$$

where μ is the chemical potential and $\omega_p = 2\pi nT$. (We use the formulas for Bose particles. The gas is henceforth assumed to be nondegenerate.)

The diagrams are constructed in the usual fashion, with the aid of the vertices of interaction with the external field and resonant interaction of the atoms with one another.

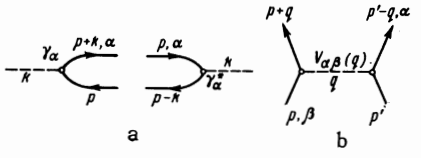

FIG. 1.

The diagrams of Fig. 1a correspond to interaction with the external field $\gamma_\alpha = ik_\alpha g$. Lines without an index correspond to the Green's function of the ground state, and the line with index to the excited state. Repeated indices in the diagrams imply summation. The diagram of Fig. 1b corresponds to resonant transfer of excitation from an atom with momentum $\mathbf{p}'$ to an atom with momentum p. For small transfers $\mathbf{q}$, the potential $V_{\alpha\beta}(\mathbf{q})$ takes the form (2.4). In this case the diagram of Fig. 1b can be regarded as made up of the vertices γ_α^* and γ_β, as well as the dashed line corresponding to the usual Coulomb interaction, with the D-function equal to

$$D(q) = 4\pi/\mathbf{q}^2. \tag{2.8}$$

We introduce a two-particle Green's function which is irreducible with respect to the Coulomb interaction: $K_{\alpha\beta}(\mathbf{p}, \mathbf{k}, \mathbf{p}')$ (Fig. 2), where p etc.

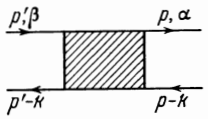

FIG. 2.

denote the 4-momentum $p(\mathbf{p}, \omega_p)$. The function $K_{\alpha\beta}$ is determined by a set of diagrams which cannot be broken into two parts joined by a single dashed line. The polarization operator $\Pi(k, \omega_k)$ is connected with the function $K_{\alpha\beta}$ by the relation

$$\Pi(k) = -k_\alpha k_\beta g^2 T \sum_{\omega_p, \omega_{p'}} \int d\mathbf{p}\, d\mathbf{p}'\, K_{\alpha\beta}(p, k; p'). \quad (2.9)$$

We assume $dp = dp_x dp_y dp_z/(2\pi)^3$ throughout. The summation is over the fourth-components of the momenta p and p′. Thus, the problem reduces to the determination of the two-particle Green's function $K_{\alpha\beta}$.

3. THE FOUR-POLE Γ

Inequality (1.2) allows us to use the gas approximation for the determination of the two-particle Green's function (the approximation of paired collisions). In this approximation, the diagrams are constructed with the aid of a reducible four-pole Γ, which is calculated in the ladder approximation.[4]

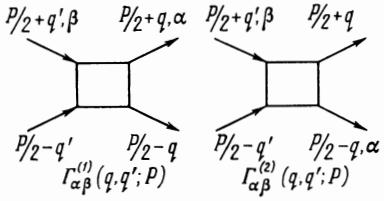

FIG. 3.

In our case there are two such four-poles, $\Gamma^{(1)}$ and $\Gamma^{(2)}$ (see Fig. 3), describing collisions without and with transfer of excitation, respectively. Since the elementary interaction (Fig. 1b) leads to transfer of excitation, the four-poles $\Gamma^{(1)}$ and $\Gamma^{(2)}$ satisfy the following system of equations:

$$\Gamma^{(2)}_{\alpha\beta}(q, q'; P) = V_{\alpha\beta}(\mathbf{q} - \mathbf{q}') - T \sum_{\omega_\varkappa} \int d\varkappa V_{\alpha\gamma}(\varkappa) \times G^{(0)}_{\gamma\sigma}\left(\frac{P}{2} + q - \varkappa\right) G^{(0)}\left(\frac{P}{2} - q + \varkappa\right) \Gamma^{(1)}_{\delta\beta}(q - \varkappa, q'; P),$$

$$\Gamma^{(1)}_{\alpha\beta}(q, q'; P) = -T \sum_{\omega_\varkappa} \int d\varkappa\, V_{\alpha\gamma}(\varkappa) G^{(0)}_{\gamma\delta}\left(\frac{P}{2} + q - \varkappa\right) \times G^{(0)}\left(\frac{P}{2} - q + \varkappa\right) \Gamma^{(2)}_{\delta\beta}(q - \varkappa, q'; P). \quad (3.1)$$

Adding and subtracting these equations, we obtain independent expressions for the quantities $\Gamma^{(\pm)}$:

$$\Gamma^{(\pm)} = \Gamma^{(1)} \pm \Gamma^{(2)}. \quad (3.2)$$

These equations have the usual form corresponding to non-exchange interaction with potential $+V_{\alpha\beta}$ for $\Gamma^{(+)}$ and $-V_{\alpha\beta}$ for $\Gamma^{(-)}$. The four-poles $\Gamma^{(\pm)}$ can be expressed directly in terms of the amplitudes $f^{(\pm)}$ for the scattering by these potentials:[4] [2)]

$$\Gamma_{\alpha\beta}(\mathbf{q}, \mathbf{q}'; P) = \frac{1}{M} f_{\alpha\beta}(\mathbf{q}, \mathbf{q}') + \frac{1}{M^2} \int d\varkappa\, f_{\alpha\gamma}(\mathbf{q}, \varkappa) \times f_{\beta\gamma}^*(\mathbf{q}', \varkappa) \left[\left(E - \frac{\varkappa^2}{M}\right)^{-1} + \left(\frac{\varkappa^2}{M} - \frac{\mathbf{q}'^2}{M} - i\delta\right)^{-1}\right], \quad (3.3)$$

where $E = i\omega_P + 2\mu - \omega_0 - \mathbf{P}^2/4M$ and ω_P is the fourth component corresponding to the 4-vector P.

The amplitude $f_{\alpha\beta}$ is defined as follows (this definition differs from the customary one by a factor $-1/4\pi$)

$$f_{\alpha\beta}(\mathbf{q}, \varkappa) = M \int d\mathbf{R}\, e^{-i\mathbf{q}\mathbf{R}} V_{\alpha\gamma}(\mathbf{R})\, \psi_{\varkappa,\beta}(\mathbf{R}, \gamma). \quad (3.4)$$

Here $\psi_{\kappa,\beta}$ is the solution of the Schrödinger equation for the relative motion with potential $V_{\alpha\gamma}$, corresponding to a momentum κ and a polarization β prior to scattering. To calculate the amplitude $f_{\alpha\beta}$, we shall use a method previously developed by Vaĭnshteĭn and one of the authors.[5] We seek the function $\psi_{\kappa,\beta}$ in the form

$$\psi_{\varkappa,\beta}(\mathbf{R}, \alpha) = e^{i\varkappa\mathbf{R}} S_{\alpha\beta}(\mathbf{R}, \varkappa). \quad (3.5)$$

Substituting (3.5) in the Schrödinger equation and neglecting the second derivatives of the function S, we obtain

$$2i\frac{\varkappa}{M} \nabla S_{\alpha\beta}(\mathbf{R}, \varkappa) = V_{\alpha\gamma}(\mathbf{R}) S_{\gamma\beta}(\mathbf{R}, \varkappa) \quad (3.6)$$

with boundary condition $S_{\alpha\beta}(-\infty) = \delta_{\alpha\beta}$.

The foregoing approximation is valid for energies much larger than the potential V.[6] Taking the potential at the most important distances $\rho \sim \rho_0$, we obtain the condition

$$v \gg 1/M^2, \quad (3.7)$$

which is satisfied in all cases of practical interest. The value of the amplitude f is determined by the region of the impact parameters $\rho \sim \rho_0$ which are much larger than the dimensions of the atom. We can therefore use the approximate expression (2.5) for the potential in (3.6).

It is convenient to seek the solution of (3.6) in a coordinate frame in which the x axis is directed along the relative-velocity vector $\mathbf{v} = 2\boldsymbol{\kappa}/M$, and the xy plane coincides with the collision plane.

Introducing the variable $\xi = x/R$, we rewrite (3.6) in the form

[2)] We shall henceforth present all the relations for $\Gamma^{(+)}$ and omit the superior index. The relations for $\Gamma^{(-)}$ can be obtained by reversing the sign of the potential.

$$iu\frac{\partial S_{x\beta}}{\partial\xi}=\left[-\frac{1}{2}+\frac{3}{2}(1-2\xi^2)\right]S_{x\beta}-3\xi(1-\xi^2)^{1/2}S_{y\beta},$$

$$iu\frac{\partial S_{y\beta}}{\partial\xi}=-3\xi(1-\xi^2)^{1/2}S_{x\beta}+\left[-\frac{1}{2}-\frac{3}{2}(1-2\xi^2)\right]S_{y\beta},$$

$$iu\frac{\partial S_{z\beta}}{\partial\xi}=S_{z\beta}. \qquad (3.8)$$

These equations contain a single parameter $u = v\rho^2/g^2$, and therefore appreciable changes in the wave function S take place when $u \sim 1$. This confirms the foregoing assumption regarding the existence of impact parameters $\rho \sim \rho_0 = g/\sqrt{v}$. The essential region of the variable ξ is also of the order of unity and consequently values $x \sim \rho \sim \rho_0$ are important.

Substituting (3.5) in the determination of the amplitude (3.4) and going over to integration with respect to x and with respect to the impact parameter ρ, we write (3.4) in the form

$$f_{\alpha\beta}(\mathbf{q}\varkappa)=M\int d\boldsymbol{\rho}\int dx\, e^{-i(\mathbf{q}-\varkappa)\mathbf{R}}V_{\alpha\gamma}(\mathbf{R})\,S_{\gamma\beta}(\mathbf{R},\varkappa). \qquad (3.9)$$

We consider first the amplitude for zero-angle scattering, $\varkappa = \mathbf{q}$. Using (3.6) and the boundary condition $S_{\alpha\beta}(-\infty) = \delta_{\alpha\beta}$, we obtain

$$f_{\alpha\beta}(\mathbf{q},\mathbf{q})=2i|\mathbf{q}|\int\rho\,d\rho\,d\varphi\,[S_{\alpha\beta}(\infty)-\delta_{\alpha\beta}] \qquad (3.10)$$

($S_{\alpha\beta}(\infty) \equiv S_{\alpha\beta}(\mathbf{R}, \mathbf{q})$ as $x \to \infty$).

Thus, to determine the amplitude it is sufficient to know the solution of the system (3.7) only for $\xi = +1$. The value of S at this point depends on the parameter u, and after averaging over the azimuthal angle φ contains a single a single vector—the particle relative-motion momentum $\mathbf{q}$.

Going over to integration with respect to the parameter u, we obtain the final expressions for the zero-angle scattering amplitude:

$$\mathrm{Im}\,f_{\alpha\beta}(\mathbf{q},\mathbf{q})=\pi g^2M[\mu_1\delta_{\alpha\beta}+\mu_2q_\alpha q_\beta/\mathbf{q}^2], \qquad (3.11)$$

$$\mathrm{Re}\,f_{\alpha\beta}(\mathbf{q},\mathbf{q})=-\pi g^2M[\mu_3'\delta_{\alpha\beta}+\mu_4'q_\alpha q_\beta/\mathbf{q}^2], \qquad (3.12)$$

where $\mu_1 \cdot \mu_2$, μ_3' are numerical coefficients.

The solution of Eqs. (3.8) and the subsequent integration with respect to the parameter u were performed with an electronic computer. The following values were obtained for the coefficients:

$$\mu_1=-2.62,\quad \mu_3'=-0.543,$$
$$\mu_2=0.875,\quad \mu_4'=0.967. \qquad (A)$$

We shall henceforth be interested also in the amplitude for scattering with momentum transfer $\mathbf{k}$ ($|\mathbf{k}| = 1/\lambda$). In this case the integral in (3.9) can be broken up into two regions

$$f_{\alpha\beta}(\mathbf{q}+\mathbf{k},\mathbf{q})=M\int_0^{\rho_1}d\rho\int_{-\infty}^{\infty}dx\,e^{-i\mathbf{kR}}V_{\alpha\gamma}(\mathbf{R})S_{\gamma\beta}(\mathbf{R},\mathbf{q})$$
$$+M\int_{\rho_1}^{\infty}d\rho\int_{-\infty}^{\infty}dx\,e^{-i\mathbf{kR}}V_{\alpha\gamma}(\mathbf{R})S_{\gamma\beta}(\mathbf{R},\mathbf{q}), \qquad (3.13)$$

where

$$\rho_0 \ll \rho_1 \ll \lambda. \qquad (3.14)$$

In the first of these integrals we can, by using the right side of the inequality (3.14), neglect the argument $\mathbf{k}\cdot\mathbf{R}$ of the exponential. The resultant integral can be extended to infinity, since the region of significance in the integration is $\rho \lesssim \rho_0$. The integral obtained in this fashion represents the zero-angle scattering amplitude. For the same reasons, we can replace $S_{\alpha\beta}$ in the second integral by its unperturbed value $\delta_{\alpha\beta}$:

$$f_{\alpha\beta}(\mathbf{q}+\mathbf{k},\mathbf{q})=f_{\alpha\beta}(\mathbf{q},\mathbf{q})+M\int_{\rho>\rho_1}d\mathbf{R}\,e^{-i\mathbf{kR}}V_{\alpha\beta}(\mathbf{R}). \qquad (3.15)$$

The latter integral can be easily calculated by using (2.5). It turns out to be insensitive to the lower limit and is equal to

$$M\int_{\rho>\rho_1}d\mathbf{R}\,e^{-i\mathbf{kR}}V_{\alpha\beta}(\mathbf{R})=-2\pi g^2M\left(\delta_{\alpha\beta}-\frac{q_\alpha q_\beta}{\mathbf{q}^2}-2\frac{k_\alpha k_\beta}{\mathbf{k}^2}\right). \qquad (3.16)$$

The last term in this expression is proportional to the Fourier component of the exact potential (see (2.4)) and coincides with the scattering amplitude in the Born approximation. The first two terms are equal to the integral of the exact potential over the internal region of the cylinder $\rho < \rho_1$, as can be verified by direct calculation.

Thus, the amplitude for scattering with transfer of small momentum $\mathbf{k}$ is equal to

$$\mathrm{Im}\,f_{\alpha\beta}(\mathbf{q}+\mathbf{k},\mathbf{q})=\mathrm{Im}\,f_{\alpha\beta}(\mathbf{q},\mathbf{q}),$$
$$\mathrm{Re}\,f_{\alpha\beta}(\mathbf{q}+\mathbf{k},\mathbf{q})=-\pi g^2M\left(\mu_3\delta_{\alpha\beta}+\mu_4\frac{q_\alpha q_\beta}{\mathbf{q}^2}-4\frac{k_\alpha k_\beta}{\mathbf{k}^2}\right), \qquad (3.17)$$

where

$$\mu_3=\mu_3'+2=1.457,\quad \mu_4=\mu_4'-2=-1{,}033. \qquad (B)$$

We see that the real part of the amplitude is not an analytic function of the scattering angle. We note that the forward scattering amplitude is infinite for the potential $V(\mathbf{R}) = \alpha/R^3$ which is independent of the angles.[7]

Let us estimate now the angle interval within which the amplitude differs noticeably from zero. This estimate can be readily obtained by comparing the total cross section $\sigma \sim \rho_0^2$ with the forward-

scattering amplitude. We have

$$\vartheta_{\text{eff}} \sim \frac{1}{q\rho_0} \sim \frac{1}{gM\gamma v} \ll 1. \tag{3.18}$$

The smallness of the effective angles follows from the inequality (3.7). We obtain an upper estimate for the amplitude for scattering through angles exceeding δ_{eff}, particularly for the scattering through an angle π:

$$|f(\mathbf{q}, -\mathbf{q})| \lesssim \sqrt{\sigma} \sim |f(\mathbf{q}, \mathbf{q})|\,\vartheta_{\text{eff}}. \tag{3.19}$$

Thus, the amplitude for large-angle scattering is small compared with the forward-scattering amplitude.

Let us consider now the role of the quadratic term (with respect to the amplitude) in the expression for the four-pole Γ (3.3). We are interested in its behavior in the case of analytic continuation to the real axis with respect to the variable $i\omega_P$. On approaching the real axis from above we have on the mass shell

$$i\omega_P = -2\mu + \omega_0 + \frac{\mathbf{P}^2}{4M} + \frac{\mathbf{q}'^2}{M} + i\delta \quad \left(E = \frac{\mathbf{q}'^2}{M} + i\delta\right)$$

and the quadratic term vanishes identically as a result of the elimination of the two terms in the curly brackets, so that

$$\Gamma_{\alpha\beta}\left(\mathbf{q}, \mathbf{q}'; \mathbf{P}, \omega_0 - 2\mu + \frac{\mathbf{P}^2}{4M} + \frac{\mathbf{q}'^2}{M} + i\delta\right) = \frac{1}{M} f_{\alpha\beta}(\mathbf{q}, \mathbf{q}'). \tag{3.20}$$

We note that the imaginary part of the four-pole Γ reverses sign on going through the real axis, whereas the real part is continuous (this follows from (3.3) with due allowance for the condition for the unitarity of the amplitude $f_{\alpha\beta}$).

On going off the mass shell, expression (3.20) acquires a certain increment. Let us estimate this increment, assuming for concreteness that we go off the mass shell into the upper half-plane. When the deviation amounts to $\Delta\epsilon$, the elimination of the terms in the curly brackets in (3.3) ceases. It is easy to see that for small differences between the vectors $\mathbf{q}$ and $\mathbf{q}'$ the quadratic term can be written in the form $M^{-1}f(\mathbf{q}, \mathbf{q})\Psi$, where Ψ is a certain function of $\Delta\epsilon\rho_0^3/g^2$. For small values of this argument the quadratic term is of the order of

$$\frac{1}{M} f(\mathbf{q}, \mathbf{q}) \frac{\rho_0^3}{g^2} \Delta\varepsilon. \tag{3.21}$$

This estimate allows us, as shown henceforth, to neglect under certain conditions the quadratic term and to use for Γ the expression $\Gamma_{\alpha\beta} = M^{-1}f_{\alpha\beta}$.

Let us proceed to determine the quantities $\Gamma^{(1)}$ and $\Gamma^{(2)}$. From relation (3.6) it follows that $S^{(-)}_{\alpha\beta} = (S^{(+)}_{\alpha\beta})$. For small-angle scattering it therefore follows from (3.10) and (3.15) that

$$f^{(-)}_{\alpha\beta} = -(f^{(+)}_{\alpha\beta})^*. \tag{3.22}$$

Taking the foregoing into account, we obtain the following final expressions for the four-poles $\Gamma^{(1)}$ and $\Gamma^{(2)}$, when the region near the mass shell is important:

$$\Gamma^{(1)}_{\alpha\beta}(\mathbf{q}+\mathbf{k}, \mathbf{q}; P) = \frac{i}{M}\operatorname{Im} f_{\alpha\beta}(\mathbf{q}+\mathbf{k}, \mathbf{q}) = i\pi g^2\left(\mu_1\delta_{\alpha\beta} + \mu_2\frac{q_\alpha q_\beta}{\mathbf{q}^2}\right), \tag{3.23}$$

$$\Gamma^{(2)}_{\alpha\beta}(\mathbf{q}+\mathbf{k}, \mathbf{q}; P) = \frac{1}{M}\operatorname{Re} f_{\alpha\beta}(\mathbf{q}+\mathbf{k}, \mathbf{q}) = -\pi g^2\left(\mu_3\delta_{\alpha\beta} + \mu_4\frac{q_\alpha q_\beta}{\mathbf{q}^2} - 4\frac{k_\alpha k_\beta}{\mathbf{k}^2}\right). \tag{3.24}$$

4. DIELECTRIC CONSTANT

Let us determine the Green's function G of the atoms. In this section we shall consider a region of temperatures satisfying the inequality (1.6). The number of excited molecules is then small. Therefore the influence of collisions on the distribution of the unexcited atoms can be neglected. Consequently, the Green's function of the unexcited atoms retains the form (2.6). For the excited atoms, the self-energy part is determined in the gas-approximation by the two diagrams of Fig. 4, or, analytically,

$$\Sigma_{\alpha\beta}(p) = -T\sum_{\omega_{p'}}\int d\mathbf{p}' G(p')[\Gamma^{(1)}_{\alpha\beta}(\mathbf{q}, \mathbf{q}; P) + \Gamma^{(2)}_{\alpha\beta}(-\mathbf{q}, \mathbf{q}; P)], \tag{4.1}$$

where $\mathbf{q} = (\mathbf{p} - \mathbf{p}')/2$ and $\mathbf{P} = \mathbf{p} + \mathbf{p}'$.

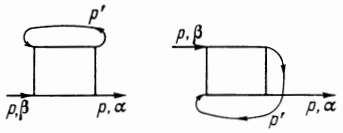

FIG. 4.

The second term contains the backward-scattering amplitude and can be discarded. The summation over ω_p, in (4.1) reduces to replacement of $G(p')$ by the Maxwellian distribution function $n\varphi(\mathbf{p}')$ and replacement of the variable E in the expression for Γ (3.3) by the quantity $i\omega_p + \mu + \omega_0 - \mathbf{p}^2/2M + \mathbf{q}^2/2M$. Therefore the characteristic features of the dependence of Γ on the fourth component of the momentum will be valid also for Σ. In the analytic continuation in the variable $i\omega_p$ from the upper half-plane to the real axis on the

mass shell $i\omega_p = -\mu + \omega_0 + p^2/2M$, so that we can use the expression (3.24) for $\Gamma^{(1)}$. Integrating with respect to $\mathbf{p}'$, we get

$$\Sigma_{\alpha\beta}(\mathbf{p}) = \left(\delta_{\alpha\beta} - \frac{p_\alpha p_\beta}{\mathbf{p}^2}\right)\Sigma_1 + \frac{p_\alpha p_\beta}{\mathbf{p}^2}\Sigma_2; \qquad (4.2)$$

$$\Sigma_1 = i\pi g^2 n\,[\mu_1 + \mu_2(1-\chi(x))/2],$$

$$\Sigma_2 = i\pi g^2 n\,[\mu_1 + \mu_2\chi(x)]. \qquad (4.3)$$

$\chi(x)$ stands for

$$\chi(x) = 1 - \frac{1}{x^2} + \frac{1}{x^3}e^{-x^2}\int_0^x e^{y^2}dy, \quad x = \left(\frac{\mathbf{p}^2}{2MT}\right)^{1/2}. \qquad (4.4)$$

On approaching the mass shell from the lower half-plane, $\Gamma^{(1)}_{\alpha\beta}$, and consequently also $\Sigma_{\alpha\beta}$, reverses sign. When $\Delta\epsilon$ is off the mass shell by an amount of the order of the line width ng^2, the relative value of the correction to Σ is, in accordance with the estimate (3.21), of the order $\rho_0^3 g^{-2}\Delta\epsilon \sim n\rho_0^3 \ll 1$. The expression obtained for Σ when the deviation from the mass shell is $\Delta\epsilon > g^2/\rho_0^3$ will be given in Sec. 5.

Substituting (4.2) in the Dyson equation

$$G_{\alpha\beta} = G^{(0)}_{\alpha\beta} + G^{(0)}_{\alpha\gamma}\Sigma_{\gamma\delta}G_{\delta\beta},$$

we obtain the Green's function $G_{\alpha\beta}$ of the excited atoms:

$$G_{\alpha\beta}(p) = \frac{\delta_{\alpha\beta} - p_\alpha p_\beta/\mathbf{p}^2}{i\omega_p - \mathbf{p}^2/2M + \mu - \omega_0 - \Sigma_1} + \frac{p_\alpha p_\beta}{\mathbf{p}^2}\frac{1}{i\omega_p - \omega_0 + \mu - \mathbf{p}^2/2M - \Sigma_2}. \qquad (4.5)$$

Let us proceed to calculate the two-particle Green's function $K_{\alpha\beta}$ which enters in the definition (2.9) of the polarization operator $\Pi(k)$. The function $K_{\alpha\beta}$ is determined by the graphic equation of Fig. 5.

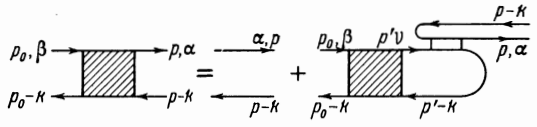

FIG. 5.

The analytic form of the equation is

$$K_{\alpha\beta}(p,k;p_0) = K^{(0)}_{\alpha\beta}(p,k)(2\pi)^3\delta(\mathbf{p}-\mathbf{p}_0)\delta_{\omega_p\omega_{p_0}} - K^{(0)}_{\alpha\gamma}(p,k)T\sum_{\omega_{p'}}\int d\mathbf{p}'\,\Gamma^{(2)}_{\gamma\nu}\left(\mathbf{q}-\frac{\mathbf{k}}{2}, \mathbf{q}+\frac{\mathbf{k}}{2}, P-k\right) \times K_{\nu\beta}(p',k;p_0), \qquad (4.6)$$

where

$$K^{(0)}_{\alpha\beta}(p,k) = G_{\alpha\beta}(p)G^{(0)}(p-k) \qquad (4.7)$$

(see (4.5) and (2.6)) and $\mathbf{q} = (\mathbf{p}' - \mathbf{p})/2$. We have left out from Fig. 5 diagrams corresponding to non-exchange scattering. It is easy to see that these diagrams will contain $\Gamma^{(1)}(-\mathbf{q} + \mathbf{k}/2, \mathbf{q} + \mathbf{k}/2; P - k)$. It follows from (1.7) and (3.18) that

$$k \sim 1/\lambda \ll 1/\rho_0 \sim p\vartheta_{\text{eff}}, \qquad (4.8)$$

On the other hand, the relative momentum of the colliding particles $\mathbf{q} = (\mathbf{p}' - \mathbf{p})/2$ is of the order of the thermal momentum of the atom and is consequently much larger than k. Therefore the non-exchange diagrams will contain the amplitude of scattering through angles close to π. By virtue of the same inequality (4.8) we can use for the exchange four-pole $\Gamma^{(2)}$ the expression (3.24). We omit here the term quadratic in the amplitudes. Indeed, the main contribution to the exchange four-pole is made by the real part of Γ, which remains continuous as it goes through the real axis. Since the function $K^{(0)}$ has a maximum of width ng^2 with respect to the variable ω_p, the relative magnitude of the quadratic term in this region is of the order of $n\rho_0^3 \ll 1$.

The four-pole (3.24) contains the Born amplitude $4\pi g^2 k_\alpha k_\beta/\mathbf{k}^2$. Diagrams corresponding to this amplitude (see Fig. 1b) are reducible and should be discarded. We shall henceforth use for $\Gamma^{(2)}$ the expression

$$\Gamma^{(2)}_{\alpha\beta}\left(\mathbf{q}-\frac{\mathbf{k}}{2}, \mathbf{q}+\frac{\mathbf{k}}{2}; P-k\right) = -\pi g^2\left(\mu_3\delta_{\alpha\beta} + \mu_4\frac{q_\alpha q_\beta}{\mathbf{q}^2}\right). \qquad (4.9)$$

We note that in calculating the transverse part of the D-function the Born terms drop out from the equation automatically.

Since $\Gamma^{(2)}$ (Eq. (4.9)) does not depend on $\omega_{p'}$, Eq. (4.1) can be summed with respect to this variable. Summing also with respect to the variables pertaining to the initial momentum p_0, we obtain an equation for the function $K_{\alpha\beta}(\mathbf{p}, k)$, defined by the equality

$$K_{\alpha\beta}(\mathbf{p},k) = T\sum_{\omega_p,\,\omega_{p_0}}\int d\mathbf{p}_0\,K_{\alpha\beta}(p,k;p_0); \qquad (4.10)$$

$$K_{\alpha\beta}(\mathbf{p},k) = -n\varphi(\mathbf{p})\left(\frac{\delta_{\alpha\gamma} - p_\alpha p_\gamma/\mathbf{p}^2}{i\omega_k - \omega_0 - \Sigma_1} + \frac{p_\alpha p_\gamma/\mathbf{p}^2}{i\omega_k - \omega_0 - \Sigma_2}\right) \times\left\{\delta_{\gamma\beta} + \pi g^2\int dp'\left[\mu_3\delta_{\gamma\nu} + \mu_4\frac{(p-p')_\gamma(p-p')_\nu}{(\mathbf{p}-\mathbf{p}')^2}\right] \times K_{\nu\beta}(\mathbf{p}',k)\right\}, \qquad (4.11)$$

where $\Sigma = \Sigma(\mathbf{p}, i\omega_k - \mu + \mathbf{p}^2/2M)$ and $\varphi(\mathbf{p})$ is the Maxwellian distribution function.

In accordance with the initial assumption, we have left out from the denominators of (4.11) the

terms $\mathbf{p}\cdot\mathbf{k}/M$, which lead to Doppler broadening of the lines. Consequently, in this approximation the function $K_{\alpha\beta}$ does not depend on the vector $\mathbf{k}$, i.e., $K_{\alpha\beta}(\mathbf{p}, k) = K_{\alpha\beta}(\mathbf{p}, i\omega_k)$. Expression (4.11) is defined on a discrete set of points $i\omega_k = i2\pi Tn$. According to the Blaschke theorem,[8] we can perform an analytic continuation from the set of points $\omega_k > 0$ to the entire upper half-plane, the coefficients of the equation being analytic functions in this half-plane with a discontinuity on the real axis. Consequently, the analytic continuation of the solution of (4.11) will also be analytic in the upper half-plane, and will satisfy the equation with analytically continued coefficients. Therefore the analytic continuation of $K_{\alpha\beta}$ to the real axis will be sought as a solution of Eq. (4.11) in which $i\omega_k$ is replaced by ω.

Let us consider the values of ω lying in a region of width of the order ng^2 near ω_0. This region corresponds to the mass shell, where we can use for Σ and $\Gamma^{(2)}$ the expressions (4.3) and (4.9). It is convenient to seek the function $K_{\alpha\beta}$ in the form

$$K_{\alpha\beta}(\mathbf{p},\omega)=\left(\delta_{\alpha\beta}-\frac{p_\alpha p_\beta}{\mathbf{p}^2}\right)K_1(|\mathbf{p}|,\omega)+\frac{p_\alpha p_\beta}{\mathbf{p}^2}K_2(|\mathbf{p}|,\omega). \tag{4.12}$$

Substituting (4.12) in (4.11) (with the changes as indicated above) and integrating along the direction of the vector $\mathbf{p}'$, we obtain the system of equations

$$F(x,\omega)\left[\frac{\omega-\omega_0}{\pi ng^2}-i\left(\mu_1+\mu_2\frac{1-\chi}{2}\right)\right]$$
$$=1-\frac{4}{\sqrt{\pi}}\int dy\, e^{-y^2}y^2\left\{a_1\left(\frac{x}{y}\right)F(y,\omega)+a_2\left(\frac{x}{y}\right)\Phi(y,\omega)\right\},$$
$$\Phi(x,\omega)\left[\frac{\omega-\omega_0}{\pi ng^2}-i(\mu_1+\mu_2\chi)\right]=1-\frac{4}{\sqrt{\pi}}$$
$$\times\int dy\, e^{-y^2}y^2\left\{b_1\left(\frac{x}{y}\right)F(y,\omega)+b_2\left(\frac{x}{y}\right)\Phi(y,\omega)\right\}, \tag{4.13}$$

where

$$K_1(|\mathbf{p}|,\omega)=-\left(\frac{2\pi}{MT}\right)^{3/2}\frac{1}{\pi g^2}e^{-x^2}F(x,\omega),\quad x^2=\frac{\mathbf{p}^2}{2MT}, \tag{4.14}$$

$$K_2(|\mathbf{p}|,\omega)=-\left(\frac{2\pi}{MT}\right)^{3/2}\frac{1}{\pi g^2}e^{-x^2}\Phi(x,\omega);$$

$$a_1(z)=\frac{2\mu_3}{3}-\frac{\mu_4}{24}+\frac{u(z)}{16},\quad a_2(z)=\frac{\mu_3}{3}+\frac{\mu_4}{6}-\frac{\mu_4 v(z)}{16};$$

$$b_1(z)=\frac{2\mu_3}{3}+\frac{\mu_4}{3}+\frac{\mu_4 v(z)}{8},\quad b_2(z)=\frac{\mu_3}{3}+\frac{5\mu_4}{12}-\frac{\mu_4 u(z)}{8}. \tag{4.15}$$

The functions u(z) and v(z) are given by

$$u(z)=\frac{1+z^4}{z^2}-\frac{1}{2}\frac{(1-z^2)(1-z^4)}{z^3}\ln\left|\frac{1+z}{1-z}\right|,$$
$$v(z)=\frac{z^4-1}{z^2}+\frac{1}{2}\frac{(1-z^2)^3}{z^3}\ln\left|\frac{1+z}{1-z}\right|. \tag{4.16}$$

The function $\chi(x)$ was defined earlier (see (4.4)).

Using (2.1), (2.9), (4.12), and (4.14) we obtain an expression for the dielectric constant directly in terms of the solution of the system (4.13):

$$\varepsilon(\omega)=1-\frac{16}{3\sqrt{\pi}}\int_0^\infty x^2 e^{-x^2}[2F(x,\omega)+\Phi(x,\omega)]\,dx. \tag{4.17}$$

Equations (4.13) contain a single parameter s, equal to

$$s=(\omega-\omega_0)/\pi ng^2. \tag{4.18}$$

Consequently, the dielectric constant is a universal function of this parameter $\epsilon(s)$. To determine this function it is necessary to solve the system (4.13). We were unable to obtain an exact analytic solution of this system. An approximate solution can be obtained by using the slow variation of the functions u, v, and χ. Replacing these functions with their values at zero argument, we obtain for $\epsilon(s)$ the expression

$$\varepsilon(s)=1-4/(s+1.11+2.33i). \tag{4.19}$$

When the functions u, v, and χ are replaced by their values at unity argument, the numbers in the denominator of (4.19) are 1.16 + 2.36i. Expression (4.19) can also be obtained when the tensor $q_\alpha q_\beta/\mathbf{q}^2$ is replaced by its angle-averaged value $\delta_{\alpha\beta}/3$ in the expressions (3.23) and (3.24) for $\Gamma^{(1)}$ and $\Gamma^{(2)}$. After this substitution, Eq. (4.11) can be solved exactly (see also [9]). The system (4.13) was solved numerically with an electronic computer. The numerical solution agreed with (4.19) within 1–2%.

We note that, as seen from (4.11), the function $K_{\alpha\beta}$, and consequently also the polarization operator Π, are odd and not even functions of ω_k. This is connected with the following circumstance. It is necessary to add to the obtained function $K_{\alpha\beta}$ a quantity which is determined by a system of diagrams which differ from those employed above in that all the line directions are reversed and $\mathbf{k}$ is replaced by $-\mathbf{k}$. It is easy to see that the function obtained in this manner is $K_{\alpha\beta}(\mathbf{p}, -i\omega_k)$, so that the polarization operator will be even. The resultant expression for ϵ satisfies all the necessary requirements. However, the additional term arising in the expression for ϵ does not have a pole character at $\omega \approx \omega_0$. Its order of magnitude in this

region is ng^2/ω_0 and it can be discarded.

The imaginary part of the dielectric constant Im ϵ, as is well known (see, for example, [10]) determines the absorption-line shape.

5. GENERALIZATION OF RESULTS

The results of the preceding section admit of a number of generalizations. We consider first the wings of the spectral line. At large values of the difference $\omega - \omega_0$ ($|\omega - \omega_0| \gg ng^2$) the integral term of (4.6) can be discarded. This means that the polarization operator is determined by a simple loop. The quantity Im ϵ, which is of greatest interest, can then be written in the form

$$\operatorname{Im}\varepsilon = \frac{4\pi^2 g^2}{3}\frac{n^2}{(\omega-\omega_0)^2}\int d\mathbf{p}\,d\mathbf{p}'\,\varphi(\mathbf{p})\varphi(\mathbf{p}')\cdot \times \frac{1}{2M^2}\int d\varkappa\,\delta\left(\omega-\omega_0+\frac{\mathbf{q}^2}{M}-\frac{\varkappa^2}{M}\right)[|f^{(+)}_{\alpha\gamma}(\mathbf{q},\varkappa)|^2 + |f^{(-)}_{\alpha\gamma}(\mathbf{q},\varkappa)|^2], \quad (5.1)$$

where $\varphi(\mathbf{p})$ is the Maxwellian distribution function. This expression can be obtained from (4.1) by taking into account the second term in the curly brackets of (3.3) and the imaginary part of the amplitude (we have carried out an analytic continuation to the real axis with respect to $i\omega_k$).

As seen from (5.1), the imaginary part of ϵ is determined by the total collision cross section off the mass shell. The departure from the mass shell is equal to $\omega - \omega_0$.

As follows from Sec. 3, when $|\omega - \omega_0| \ll g^2/\rho_0^3$ this cross section coincides with the cross section on the mass shell. In this case, Im ϵ is obtained from (4.19) with $s \gg 1$. When $|\omega - \omega_0| \gg g^2/\rho_0^3$, impact parameters which are much smaller than ρ_0 are significant in the collision. This makes it possible to use the quasiclassical solution of (3.8) (see [5]). The principal term of the cross section is then independent of $\omega - \omega_0$, so that the imaginary part of ϵ retains the previous dependence on $\omega - \omega_0$, but with a different coefficient:

$$\operatorname{Im}\varepsilon = 4\pi g^2\frac{8\pi^2 g^2 n^2}{9(\omega-\omega_0)^2} = 4\frac{2.79}{s^2}. \quad (5.2)$$

We note that the departure off the mass shell in the collision cross section, as shown by Yakimets,[10] corresponds to a transition from the impact to the statistical mechanism of line broadening. As is well known, for a potential $V \sim R^{-3}$ the statistical theory leads to a quadratic decrease in the line intensity (see, for example, [11]). It should be noted that formula (5.2) is valid also in the case when the central part of the line is determined by the Doppler broadening. In this case, the region of applicability of (5.2) is given by the inequality $|\omega - \omega_0| \gg k\sqrt{T/M}$.

Let us consider now a case corresponding to a transition between two excited states, the lower level being connected with the main resonant transition $1 \to 0$ ($0 \to 1$). We assume as before that the temperature is small compared with the distance between the ground and excited states. Under these conditions, only the lower excited level is broadened by collisions (with the unperturbed atoms). The upper excited state cannot be transferred by resonance to another atom. This means that the mass operator Σ differs from zero only for the lower excited state, and the integral term of (4.6) can be discarded. The expression for ϵ can then be obtained for an arbitrary ratio between the Doppler line width $k\sqrt{T/M}$ and the collision width ng^2:

$$\varepsilon = 1 - 4\pi g^2 n\int d\mathbf{p}\frac{\varphi(\mathbf{p})}{\omega-\omega_0+2.33\,i\pi n_0 g_0^2+\mathbf{kp}/M}; \quad (5.3)$$

Here n_0 and n are the densities of the atoms in ground state and at the lower excited level, respectively; g_0 and g are the reduced matrix elements of the dipole transitions between the ground and excited states and accordingly between the excited states. If the transition between the excited states is not dipole, this leads only to a corresponding change in the numerator of (5.3).

Expression (5.3) does not contain a line shift; this is a direct consequence of neglecting the integral term of (4.6). Formula (5.3) is based on the assumption that the Doppler width is small compared with the width of the energy surface, $k\sqrt{T/M} \ll g^2/\rho_0^3$. In the opposite limiting case, and also at large values of $|\omega - \omega_0|$, the coefficient 2.33 preceding the imaginary part of the numerator in (5.3) should be replaced by 2.79 (see (5.2)).

The dispersion width corresponding to such a transition was measured for helium by Kuhn and Vaughan[12] and for argon by Stacey and Vaughan,[13] and less accurately by Hindmursh and Thomas.[14] Our results (5.3) correspond to a coefficient $K = 1.48$ in the formula given for the line width in [12, 13]. The results agree with the experimental data for helium when the oscillator strength $f = 4\pi g_0^2 c/\lambda$ corresponding to the resonant transition $1^1S - 2^1P$ with $\lambda = 584$ Å is equal to 0.275. This value is very close to that calculated by Shiff and Pekeris[15] ($f = 0.276$). The agreement with the data on argon, $\lambda = 1048$ Å, is obtained when $f = 0.24$, whereas it follows from the calculations (Knox[16]) that $f = 0.20$. The poorer agreement in

this case is possibly due to the influence exerted on the transitions by the neighboring weak resonance line $\lambda = 1067$ Å. In addition, the oscillator strengths themselves can be calculated for argon with lower accuracy than for helium.

Finally, let us consider the case when the transitions under investigation occur between levels whose populations are comparable in magnitude. We shall take into account collisions between atoms at these levels, with resonant energy transfer. Under these conditions, the mass operators of both states differ from zero. Assuming, for concreteness, that the angular momentum of the upper and lower states are unity and zero, respectively, we rewrite (4.4) in the form

$$K_{\alpha\beta}(\mathbf{p}, k) = -\left(n_0 - \frac{n_1}{3}\right)\varphi(\mathbf{p}) \times\left(\frac{\delta_{\alpha\gamma} - p_\alpha p_\gamma/\mathbf{p}^2}{\omega - \omega_0 - \Sigma^{(0)} - \Sigma_1^{(1)}} + \frac{p_\alpha p_\gamma/\mathbf{p}^2}{\omega - \omega_0 - \Sigma^{(0)} - \Sigma_2^{(1)}}\right) \times\left\{\delta_{\gamma\beta} + \pi g^2 \int d\mathbf{p}'\left[\mu_3\delta_{\gamma\nu} + \mu_4\frac{(p-p')_\gamma (p-p')_\nu}{(\mathbf{p}-\mathbf{p}')^2}\right] \times K_{\nu\beta}(\mathbf{p}', k)\right\}, \quad (5.4)$$

where n_0 and n_1 are the densities of the atoms in the lower and upper states, and

$$\Sigma^{(0)} = i\pi g^2 n_1(\mu_1 + {}^1/_3\mu_2). \quad (5.5)$$

The expressions for $\Sigma^{(1)}_{1,2}$ are obtained from (4.3) by replacing n with n_0.

Using the procedure for the approximate solution of the equation, we obtain the following expression for ϵ:

$$\varepsilon = 1 - 4\pi g^2(n_0 - {}^1/_3 n_1)[\omega - \omega_0 + (\mu_3 + {}^1/_3\mu_4) \times \pi g^2(n_0 - {}^1/_3 n_1) - i(\mu_1 + {}^1/_3\mu_2)\pi g^2(n_0 + n_1)]^{-1}. \quad (5.6)$$

For large $\omega - \omega_0$ ($|\omega - \omega_0| \gg g^2/\rho_0^3$), the expression for Im ϵ is

$$\operatorname{Im}\varepsilon = \frac{4\pi^2 g^4}{(\omega - \omega_0)^2}[1.55\, n_0 n_1 + 2.79(n_0^2 - {}^1/_3 n_1^2)]. \quad (5.7)$$

Formula (5.6) is valid also for the inverted level population $n_- = n_1/3 - n_0 > 0$. Considering for this case the dispersion equation for transverse waves

$$\omega^2\varepsilon(\omega) = \omega_k^2, \quad \omega_k = |\mathbf{k}|c, \quad (5.8)$$

we can easily show that the imaginary part of ω becomes positive in a definite interval of $|k|$, i.e., generation of electromagnetic waves takes place. The generated waves lie near a value of ω_k, equal to

$$\bar{\omega}_k = \omega_0 + \pi g^2 n_-(\mu_3 + {}^1/_3\mu_4), \quad (5.9)$$

in an interval

$$|\omega_k - \bar{\omega}_k| \leqslant 2\sqrt{2\omega_0\pi g^2 n_-}. \quad (5.10)$$

This result coincides, apart from a numerical coefficient, with the results obtained by Alekseev and the authors.[17]

We present also the frequency interval of the generated waves and the growth increment b, and also the gain κ at the center of the generated line:

$$|\omega - \bar{\omega}| \leqslant b = \sqrt{2\pi\omega_0 g^2 n_-}, \quad (5.11)$$

$$\bar{\omega} = \omega_0 + \pi g^2 n_-(\mu_3 + {}^1/_3\mu_4 - 2);$$

$$\varkappa = \frac{2}{(\mu_1 + {}^1/_3\mu_2)}\frac{n_-}{n}\frac{\omega}{c} \text{ for } n = (n_0 + n_1) \gg n_-. \quad (5.12)$$

6. CONCLUSION

The influence of resonant collisions on the shape of the spectral line was considered recently by Ali and Griem[18] and by Byron and Foley.[19] These results pertain actually to transitions between two excited states. In addition, resonant collisions were considered in these papers by approximate methods, which give rise to appreciable errors. D'yakonov and Perel'[20] investigated the influence of resonant collisions on the equalization of the populations of Zeeman sublevels. The cross sections for excitation transfer by collision were calculated by Watanabe,[21] Omont,[22], and in [5]. Before submitting this article, we also became acquainted with a preprint by Kazantsev,[9] which was graciously furnished to us by the author. He obtained a generalized Boltzmann equation for the density matrix, and a value close to ours (see (4.19)) for the transition line width. However, the line shift obtained by him is apparently in error, since no account was taken of the non-analyticity of the real part of the amplitude (3.17), and an error was made in the sign of the coefficient corresponding to our μ_4.

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