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TURBULENT HEATING OF IONS IN A CURRENT NONISOTHERMAL PLASMA

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Equations are derived and analyzed which describe the behavior of the electron and ion distribution functions and the energy spectral density of plasma noise. The derivation is based on the solution of the nonlinear problem of instability of a nonisothermal plasma situated in an external electric field. It is shown that in a quasistationary regime limitation of the growth of ion-acoustic oscillations is accompanied by a sharp increase of the plasma resistance and by heating of both the electronic and ionic plasma components. The most effective is heating of resonant ions for which the effective temperature grows appreciably faster than the electron temperature and in a number of cases may considerably exceed the latter. Under certain conditions (sufficiently strong electric fields) the process is accompanied by the appearance of high energy runaway electrons and ions.

In our preceding paper[1] we considered, within the framework of a theory that takes both pair collisions and particle scattering by ion-sound noise into account, the nonlinear problem of current instability in a nonisothermal plasma in an external electric field, and we showed, in particular, that limitation of the plasma-noise growth is accompanied by a rise in both the electron and ion temperatures. However, when the anisotropy of the ion distribution function is neglected, the results obtained in that paper are valid, strictly speaking, only when the ion-ion collision frequency is large enough so that the principal role is assumed by the collision wave absorption, and not by Cerenkov absorption. In addition, we circumvented in that paper the entire question of heating of the resonant ions, since this question cannot be analyzed in detail without taking the ion-absorption anisotropy into account, and requires a simultaneous solution of the equations for both the electron and ion distribution functions. Such an analysis is, however, of interest from the point of view of the different mechanisms for plasma heating, which in turn is of great importance to the controlled-fusion problem.

The purpose of the present paper is to generalize the results obtained in[1] to the case both collisions and Cerenkov absorption of the waves by the ions are significant, and also to derive and analyze equations that describe the heating of the ionic and electronic components of the plasma.

1. FORMULATION OF PROBLEM AND FUNDAMENTAL EQUATIONS

Let us consider a spatially-homogeneous fully-ionized non-isothermal plasma with $T_e \gg T_i$, situated in an external electric field E whose direction is chosen in the negative z direction.

We denote by $W(\mathbf{k}, t)$ and $\mathbf{P} = \mathbf{k}W/\omega$ the spectral densities of the energy and momentum of the ion-sound noise in the plasma, and by $f_e(\mathbf{v}, t)$ and $f_i(\mathbf{v}, t)$ the electronic (e) and ionic (i) distribution functions normalized to unity. The system of equations describing the behavior of the functions f_e, f_i, and W in the lowest order in the nonlinearity (the nonlinear approximation) is conveniently written in a coordinate frame in which the oscillation frequency $\omega(\mathbf{k})$ is an isotropic function of the wave vector $\mathbf{k}$, i.e., in a coordinate frame in which the average velocity $\mathbf{v}_{Ti}$ of the thermal ions responsible for the dispersion law is equal to zero. Accordingly, the quantities f_e, f_i, and W will henceforth be taken to mean the corresponding distribution functions in just this coordinate frame, which does not coincide with the laboratory frame, and which moves relative to the latter along the Oz axis with a certain velocity $V_0(t)$ which, in general, is a function of the time. Consequently we must include in the expressions for the forces $F_{e,i}$ acting on the electrons and the ions, besides the force connected directly with the electric field, also the inertia forces, i.e., we must put

$$F_{e,i} = \pm eE - m_{e,i} dV_0/dt \quad (e > 0), \tag{1}$$

where m_e and m_i are respectively the electron and ion masses.

All ions can be divided into two groups, depending on the value of the velocity v. Ions with velocities $v < s_0$, where s_0 is the phase velocity of the ion-sound waves excited in the plasma, will be called thermal, and those with $v > s_0$ resonant. The thermal ions determine the dispersion law and the magnitude of the collision damping, while the resonant ones are wholly responsible for the Cerenkov absorption of sound by the ions. Accordingly, the quantities pertaining to thermal ions will be marked by a subscript T, and the resonant ones by r; for example, the densities of the thermal and resonant ions are

$$n_T = n \int_{v<s_0} f_i d\mathbf{v}, \quad n_r = n \int_{v>s_0} f_i d\mathbf{v}. \tag{2}$$

We can similarly introduce the concepts of average random-motion kinetic energies w_{Te}, w_{Ti}, and w_r (or the temperatures T_e, T_i, and T_r) of the electrons and the thermal and resonant ions, and accordingly their mean-square velocities v_{Te}, v_{Ti}, and v_r:

$$w_{Te} = \frac{3}{2} m_e v_{Te}^2 = \frac{3}{2} T_e = \int \frac{m_e v^2}{2} f_e \, d\mathbf{v},$$

$$w_{Ti} = \frac{3}{2} m_i v_{Ti}^2 = \frac{3}{2} T_i = \frac{n}{n_T} \int_{v<s_0} \frac{m_i v^2}{2} f_i \, d\mathbf{v},$$

$$w_r = \frac{3}{2} m_i v_r^2 = \frac{3}{2} T_r = \frac{n}{n_r} \int_{v>s_0} \frac{m_i v^2}{2} f_i d\mathbf{v}. \tag{3}$$

The system equations for the distribution functions of the electrons and resonant ions, and for the spectral density of the noise energy, will take the form (see, for example,[2])

$$\frac{\partial f_{e,i}}{\partial t} + \frac{F_{e,i}}{m_{e,i}} \frac{\partial f_{e,i}}{\partial v_z} = \frac{\partial}{\partial v_i} D_{ij}^{(e,i)} \frac{\partial f_{e,i}}{\partial v_j} + \mathrm{St}_{e,i}(f_{e,i}), \quad v \geqslant s_0, \tag{4}$$

$$dW/dt = \Gamma(\mathbf{k}) W, \quad \Gamma(\mathbf{k}) = \gamma_e + \gamma_i - \gamma_{\mathrm{col}} . \tag{5}$$

The first term in the right side of (4), which is proportional to

$$D_{ij}^{(e,i)}(\mathbf{v}) = \frac{\pi}{n_T} \frac{m_i}{m_{e,i}^2} \int d\mathbf{k} s^2(k) k_i k_j W(\mathbf{k}) \delta(ks - \mathbf{kv}), \tag{6}$$

takes into account the scattering of the particles by the ion-sound noise, and the second term takes into account the pair collisions. The first two terms of (5), which are proportional to

$$\gamma_{e,i} = \pi \frac{n}{n_T} \frac{m_i}{m_{e,i}} k s^3 \frac{\partial}{\partial s} \int \delta\left(s - \frac{\mathbf{kv}}{k} \right) f_{e,i} \, d\mathbf{v}, \tag{7}$$

describe the Cerenkov excitation (absorption) of the ion sound by the electrons and ions of the plasma, and the last term

$$\gamma_{\mathrm{col}} = \frac{1}{2\sqrt{\pi}} \frac{n}{n_T} \left(\frac{m_e}{m_i} \frac{T_e}{T_i} \right)^{1/2} \frac{s_m^2}{s^2} \nu_T, \tag{8}$$

describes the attenuation connected with the ion-ion collisions.[3] In formulas (4)—(8) **v** is the particle velocity, **k** the wave vector, $s = s(k) = \omega(k)/k$ the phase velocity, $s_m = (n_T T_e / n m_i)^{1/2}$ the maximum phase velocity of the ion-sound waves, and $\nu_T = \nu_e(v_{Te})$, where $\nu_e(v) = 4\pi e^4 n_T L / m_e^2 v^3$ is the frequency of the electron-ion collisions (L is the Coulomb logarithm).

In order to make Eqs. (4)—(8) a closed system they must be supplemented with an equation for the thermal-ion distribution function. Recognizing, however, that the system (4)—(8) contains only the temperature of the thermal ions, this equation can be replaced by the simpler equation for the average energy w_{Ti} of the thermal ions. The latter, on the other hand, can be readily obtained by integrating the corresponding kinetic equation, and obviously is of the form

$$\frac{d}{dt} n_T w_{Ti} = \gamma_{\mathrm{col}} \mathscr{E} + \int_{v<s_0} \frac{m_i v^2}{2} \mathrm{St}_i \, d\mathbf{v}, \tag{9}$$

where the first term, $\gamma_{\mathrm{coll}} \mathscr{E}$ ($\mathscr{E} = \int W d\mathbf{k}$ is the total energy density of the plasma noise) takes into account the increase in the thermal-ion energy by wave absorption resulting from ion-ion collision, and the second term represents the energy exchange that occurs in collisions of thermal ions with electrons and resonant ions.

It should be noted here that inasmuch as Eqs. (4) and (5) take into account only the so-called resonant interaction of the particles with the waves, and do not take the adiabatic interaction into account,[4] the quantities w_{Te} and w_r defined in accord with (9) do not include the kinetic energy of the vibrational motion of the particles. Accordingly, we shall take to quantity w_{Ti} (or T_i) to mean likewise only the "true" temperature of the thermal ions, and not including the kinetic energy of the oscillations, which can be readily shown to equal $\mathscr{E}/2$.

It remains for us to connect the forces $F_{e,i}$ with the value of the external electric field **E**. This can be readily done by recognizing that the total change in the momentum of the thermal ions should vanish identically in our chosen coordinate system:

$$\frac{d}{dt} n_T m_i |\mathbf{v}_{Ti}| = n_T [F_i + F_{\mathrm{fr}}] + \gamma_{\mathrm{col}} \mathscr{P} + \frac{d\mathscr{P}}{dt} \equiv 0. \tag{10}$$

Here

$$\mathscr{P} = \left| \int \mathbf{P} d\mathbf{k} \right| = \left| \int \frac{\mathbf{k}}{\omega} W d\mathbf{k} \right| \tag{11}$$

denotes the total momentum density of the ion-sound waves in the plasma. The direction of the momentum, as is clear even from symmetry considerations, is opposite to the direction of the external electric field vector **E**, i.e., it coincides with the direction of the Oz axis.

The physical meaning of the different terms of (10) is obvious: The first describes the change in momentum under the influence of the external force F_i, the second under the influence of the friction force resulting from the pair collisions of the thermal ions with the electrons and the resonant ions, the third the change in momentum due to wave absorption resulting from the presence of ion-ion collisions, and the last the change of momentum due to the presence of adiabatic interaction of the thermal ions with the ion-sound waves. The first two terms arise even in the linear approximation (in the noise amplitude), whereas the last two are connected just with the allowance for the nonlinear interaction of the particles with the waves, and do not arise in the linear theory.

Eliminating now the velocity from (1) with the aid of (10) and taking (5) into account, we get

$$F_i = -F_{\text{fr}} - \frac{1}{n_T}\int [\gamma_e + \gamma_i] P_z d\mathbf{k}, \qquad (12)$$

$$F_e = eE + \frac{m_e}{m_i}[eE + F_i],$$

where P_z denotes the projection of the vector $\mathbf{P}$ on the z axis.[1)]

2. SOLUTION OF KINETIC EQUATIONS AND EQUATIONS FOR THE TEMPERATURES

Thus, the problem consists of simultaneously solving the system of nonlinear equations (4) and (5), supplemented with Eq. (9) for the temperature of the thermal ions. We do not attempt here, however, to find the complete solution of this system, and confine ourselves only to a more or less detailed analysis of the system and to the derivation of equations for the average kinetic energies (or temperatures) of the electrons and of the thermal and resonant ions, and of the equations for the electron and ion currents.

The system (4) and (5) is analogous in many respects to the initial system of our preceding paper[1]. The only difference lies in the presence of anisotropy of the ion absorption, an anisotropy connected with the distortion of the distribution function of the resonant ion as a result of their scattering by the plasma noise. The latter, in turn, requires the simultaneous solution of Eq. (5) for the spectral energy density of the ion-sound noise and of the kinetic equation, not only for the electron distribution function, but also for the ion distribution function. This solution, however, can be obtained by a method which is perfectly similar to that used by us earlier[1] to find the quasistationary solutions for the electron distribution function and the spectrum of the plasma noise.

Indeed, one of the major factors that enables us in[1] to simplify greatly the initial equations and, in final analysis, to find the solution for the electron distribution function, was the fact that the main contribution to the Cerenkov radiation (or absorption) was made by particles with velocities $v \gg s$, by virtue of the fact that the thermal velocity v_{Te} of the electrons greatly exceeds the speed of sound s. On the other hand, if we use the results of[1] and estimate the rate of increase of the resonant-ion energy w_r, then it is easy to verify that they are heated much more effectively than the electrons and the thermal ions.[2)] It is therefore natural to assume, just as in[1], that after the lapse of a certain time there is established a certain quasistationary state, in which the anisotropic part of both the electron and the ion distribution functions will vary quite slowly, and the average thermal velocity of the resonant ions in this state will greatly exceed the phase velocity s of the ion-sound waves excited in the plasma. In other words, the main contribution to the Cerenkov absorption will be made, just as for the electrons, by ions with velocities $v \gg s$, and the solution of the equation for the distribution function of the resonant ions can be obtained in exactly the same manner as used in[1] for the electron distribution function.

However, before we proceed to solve Eqs. (4) and (5), we note that the waves that will grow fastest in the unstable mode will obviously be those with phase velocities s (or wave numbers k) corresponding to the maximum value of the total increment Γ. On the other hand, just as was done earlier in[1], we can show that the increment Γ(k) cannot assume a maximum value for a finite range of values of s (or k), and only for one value $s = s_0$ (or $k = k_0$). We can therefore assume that in the quasistationary mode considered by us, when $\Gamma t \gg 1$, the plasma will contain only waves with phase velocities $s = s_0$. Since we are interested only in non-trivial solutions, i.e., solutions corresponding to nonzero total noise energy $\mathscr{E}$, we must put

$$W(\mathbf{k}) = W_0\left(\frac{\mathbf{k}}{k}\right)\frac{\delta(k-k_0)}{k^2}, \qquad (13)$$

where k_0 and s_0 are the values of the wave number and of the phase velocity for which the total increment $\Gamma = \gamma_e + \gamma_i - \gamma_{\text{col}}$ is a maximum.

1)By virtue of the axial symmetry of the problem, the terms proportional to other components of the vector $\mathbf{P}$ vanish after integration in $\mathbf{k}$-space.

2)Simple estimates show that the temperature T_r of the resonant ions reaches a value of the same order as the electron T_e already after the time $\tau_e^{(a)} \approx m_e s_0/eE$ needed to establish the electron distribution function.

Thus, going over in velocity and wave-vector space to spherical coordinates,

$$\mathbf{v} = \{v, \theta, \varphi\}, \quad \mathbf{k} = \{k, \theta', \varphi'\},$$

putting $f_{e,i} = f_{e,i}^{(0)}(v) + f_{e,i}^{(1)}(v, \theta)$, where $f_{e,i}^{(1)} \ll f_{e,i}^{(0)}$, neglecting small quantities of the order of $s_0/v_{Te} \ll 1$ and $s_0/v_r \ll 1$, and omitting the intermediate steps, which are perfectly analogous to those given in[1], we obtain for the functions $f_{e,i}^{(0)}$ and $f_{e,i}^{(1)}$ the following equations:

$$\frac{\partial f_{e,i}^{(0)}}{\partial t} = \mathrm{St}_{e,i}(f_{e,i}^{(0)}) + \frac{1}{2}\frac{1}{v^2}\frac{\partial}{\partial v}\left\{v^2 \nu_{e,i}(v)\frac{\partial f_{e,i}^{(0)}}{\partial v}\int_{-1}^{1}\frac{1-\xi^2}{2}d\xi\left[C(\xi) + \frac{u_{e,i}^2(v) - B^2(\xi)}{1 + A(\xi)}\right]\right\}, \tag{14}$$

$$\frac{\partial f_{e,i}^{(1)}}{\partial \xi} = -\frac{u_{e,i}(v) + B(\xi)}{1 + A(\xi)}\frac{\partial f_{e,i}^{(0)}}{\partial v}, \tag{15}$$

where, in accordance with the definition (6) and relation (13), we have for $v > s_0$

$$A(\xi) = \frac{4\pi m_i k_0 s_0^2}{n_T m_e^2 v_{Te}^3 \nu_T}\frac{1}{1-\xi^2}\int_{-\sqrt{1-\xi^2}}^{\sqrt{1-\xi^2}}\frac{W_0(x)x^2 dx}{(1-\xi^2-x^2)^{1/2}}$$

$$B(\xi) = \frac{4\pi m_i k_0 s_0^3}{n_T m_e^2 v_{Te}^3 \nu_T}\frac{1}{1-\xi^2}\int_{-\sqrt{1-\xi^2}}^{\sqrt{1-\xi^2}}\frac{W_0(x)x dx}{(1-\xi^2-x^2)^{1/2}},$$

$$C(\xi) = \frac{4\pi m_i k_0 s_0^4}{n_T m_e^2 v_{Te}^3 \nu_T}\frac{1}{1-\xi^2}\int_{-\sqrt{1-\xi^2}}^{\sqrt{1-\xi^2}}\frac{W_0(x) dx}{(1-\xi^2-x^2)^{1/2}},$$

$$u_{e,i}(v) = \frac{F_{e,i}}{m_{e,i}\nu_{e,i}(v)}, \quad \nu_{e,i}(v) = \frac{4\pi e^4 n_T L}{m_{e,i}^2 v^3},$$

$$\nu_T = \nu_e(v_{Te}), \tag{16}$$

and for $v < s_0$ we have $A = B = C \equiv 0$. Here $\xi = \cos\theta$, $x = \cos\theta'$, and $\theta = \theta'$ are respectively the angles between the Oz axis and the vectors **v** and **k**.

In the derivation of (14) and (15) we used a collision integral in the Landau form[5,6] and took account of the fact that in the velocity region $v_{Ti} \ll v \ll v_{Te}$ of interest to us this integral can be represented with sufficient accuracy in the form

$$\mathrm{St}_{e,i}(f_{e,i}) = \mathrm{St}_{e,i}(f_{e,i}^{(0)}) + \frac{\partial}{\partial \xi}\frac{1-\xi^2}{2}\nu_{e,i}(v)\frac{\partial f_{e,i}^{(1)}}{\partial \xi}, \tag{17}$$

where $\nu_{e,i}(v)$ denotes the frequency of collision of the electrons (ν_e) or of the resonant ions (ν_i) with the thermal ions[3)].

To determine s_0 we need expressions for the increments γ_e and γ_i in the vicinity of the point $s = s_0$. It must be borne in mind here that since the spectral energy density W(k, x) of the ion-sound noise has in the approximation considered here a δ-like character, the diffusion coefficients $D_{ij}^{(e,i)}$ are discontinuous functions of the velocity v. Consequently, while the distribution functions themselves are continuous, their derivatives with respect to the velocity, and consequently the derivatives of the increments γ_e and γ_i with respect to s are discontinuous at the point $s = s_0$. We present accordingly the values of the increments at $s \to s_0 - 0$ and $s \to s_0 + 0$. Going over to spherical coordinates in (7) neglecting as before the small quantities of the order of s_0/v_{Te} and s_0/v_r, and taking into account the expressions (14) for $\partial f_{e,i}^{(1)}/\partial\xi$, we get for γ_e and γ_i the following expressions:

$$\gamma_{e,i}(s) = \gamma_{e,i}^{(0)}\frac{ks^4}{k_0 s_0^4}\begin{cases}\dfrac{Q_{e,i}(x)}{s} - 1 & \text{for } s \to s_0 + 0\\[2ex] \dfrac{Q_{e,i}(x)}{s} - \dfrac{f_{e,i}^{(0)}(s)}{f_{e,i}^{(0)}(s_0)} & \text{for } s \to s_0 - 0\end{cases}, \tag{18}$$

where

$$\gamma_{e,i}^{(0)} = 2\pi^2\frac{n}{n_T}\frac{m_i}{m_{e,i}}k_0 s_0^4 f_{e,i}^{(0)}(s_0),$$

$$Q_{e,i}(x) = \frac{2x}{\pi}\int_0^1\frac{d\mu}{\sqrt{1-\mu^2}}q_{e,i}(\mu\sqrt{1-x^2}),$$

$$q_{e,i}(\xi) = \frac{u_{e,i}^{(0)} + B(\xi)}{1 + A(\xi)}, \quad u_e^{(0)} = \frac{3}{4\pi}\frac{F_e}{m_e}\frac{1}{v_{Te}^3\nu_T}\frac{1}{f_e^{(0)}(s_0)}$$

$$u_i^{(0)} = \frac{3}{4\pi}\frac{F_i}{m_e}\frac{1}{v_{Te}^3\nu_T}\frac{n_T}{n}\frac{m_i}{m_e}\frac{1}{f_i^{(0)}(s_0)}. \tag{19}$$

Consequently, the total increment Γ(s, x) can be represented in the form

$$\Gamma(s, x) = \begin{cases}[\gamma_e^{(0)} + \gamma_i^{(0)}]\left[\dfrac{Q^*(x)}{s} - 1\right]\dfrac{ks^4}{k_0 s_0^4} - \gamma_{\mathrm{col}}(s) \text{ for } s \to s_0 + 0\\[2ex] [\gamma_e^{(0)} + \gamma_i^{(0)}]\dfrac{Q^*(x)}{s}\dfrac{ks^4}{k_0 s_0^4}\\[2ex] -\left[\gamma_e^{(0)}\dfrac{f_e^{(0)}(s)}{f_e^{(0)}(s_0)} + \gamma_i^{(0)}\dfrac{f_i^{(0)}(s)}{f_i^{(0)}(s_0)}\right]\dfrac{ks^4}{k_0 s_0^4} - \gamma_{\mathrm{col}}(s) \text{ for } s \to s_0 - 0,\end{cases} \tag{20}$$

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3)We have neglected here the insignificant difference between the Coulomb logarithms L which enter in the expressions for the collision frequencies ν_e and ν_i, and assumed for simplicity that the ions are singly-charged.

where

$$Q^*(x) = \frac{2x}{\pi}\int_0^1 \frac{d\mu}{\sqrt{1-\mu^2}} q^*(\mu\sqrt{1-x^2}),$$

$$q_*(\xi) = \frac{u^* + B(\xi)}{1 + A(\xi)},$$

$$u^* = \frac{\gamma_e^{(0)} u_e^{(0)} + \gamma_i^{(0)} u_i^{(0)}}{\gamma_e^{(0)} + \gamma_i^{(0)}} = \frac{u_e^{(0)} f_e^{(0)}(s_0) + (m_e/m_i) u_i^{(0)} f_i^{(0)}(s_0)}{f_e^{(0)}(s_0) + (m_e/m_i) f_i^{(0)}(s_0)}. \quad (21)$$

Using these expressions it is easy to obtain an equation for the time variation of the total momentum density $\mathcal{P}$ of the plasma waves.

Indeed, let us assume for concreteness that the total increment $\Gamma(s_0, x)$ is positive when $x > x_0$ and negative when $x < x_0$. We assume accordingly that the noise intensity $W_0(x)$ differs from zero only inside a cone with a vertex angle $\theta' = \theta_0$, i.e., only when $x > x_0 = \cos\theta_0$, and is equal to zero when $x \le x_0$, i.e., outside this cone. We assume also that the angle θ_0 is close to $\pi/2$, so that $x_0^2 \ll 1$. Such a situation, as follows from[1], takes place if the electric field E is at least several times larger than the quantity

$$E_{\text{lim}} = \frac{1}{3}\sqrt{\frac{2}{\pi}}\frac{m_e v_{Te} \nu_T}{e}\sqrt{\frac{m_e}{m_i}}\frac{s_0}{s_m},$$

below which there is no instability at all[4]. We now multiply (5) by **k** and integrate over **k**-space, taking (20) into account. Then, bearing in mind the identity

$$1 = \frac{2}{\pi}\int_0^1 \frac{d\mu}{\sqrt{1-\mu^2}} = \frac{2}{\pi}\int_{|x|}^1 \frac{y\,dy}{\sqrt{1-y^2}\sqrt{y^2-x^2}}$$

reversing the order of integration in the double integrals, and neglecting small quantities of the order of $x_0^4 \ll 1$, we obtain for $\mathcal{P}$ the following equation:

$$\frac{d\mathcal{P}}{dt} = [nF_e + n_r F_i]\left(1 - \frac{3}{2}\tilde{x}_0^2\right) - \gamma_{\text{col}}\mathcal{P}, \quad (22)$$

where

$$\tilde{x}_0^2 = \frac{1}{u^*}\int_{x_0} \frac{y^3\,dy}{\sqrt{1-y^2}} q^*(\sqrt{1-y^2}). \quad (23)$$

Since we are considering here the case when the noise intensity is much higher than thermodynamic equilibrium value, we have $A \gg 1$ and consequently $\tilde{x}_0^2 \ll 1$ (in the limit as $W \to \infty$ we have $\tilde{x}_0^2 \to \sim s_0/u^* \ll 1$). We can therefore neglect the noise intensity in the first approximation, and we get for $\mathcal{P}$ a linear differential equation, the solution of which can be obtained in a trivial manner. From this equation it follows, in particular, that if we neglect pair collisions (i.e., when $\gamma_{\text{col}} = 0$), the total momentum of the ion-sound waves increases linearly with time. From the physical point of view, incidentally, this result is perfectly natural and is a direct consequence of the law of conservation of the total momentum of the system.

Taking (22) into account, we can now find the final expression for the forces $F_{e,i}$. Using the collision integral in the Landau form (see also the paper of Sivukhin[6]) and taking into account expression (15) for $\partial f_{e,i}^{(1)}/\partial\xi$, we get after simple calculations

$$F_{\text{fr}} = \frac{n}{n_T}\int_{v<s_0} m_i v \xi\, \text{St}_i\, d\mathbf{v} = \frac{3}{2}\frac{[nF_e + n_p F_i]}{n_T}\tilde{x}_0^2. \quad (24)$$

Substituting further this expression (12) and recognizing that $n_T + n_r = n$ and that according to (22)

$$\int [\gamma_e + \gamma_i]\frac{x}{s_0} W\,d\mathbf{k} = \frac{d\mathcal{P}}{dt} + \gamma_{\text{col}}\mathcal{P}$$

$$= [nF_e + n_p F_i]\left(1 - \frac{3}{2}\tilde{x}_0^2\right)$$

we get

$$F_i = -F_e = -eE. \quad (25)$$

Just as the equation for the total noise momentum $\mathcal{P}$, the equations for the average kinetic energies (or temperatures) of the electrons (w_{Te}), thermal ions (w_{Ti}), and resonant ions (w_r) can be readily obtained even without first solving Eq. (5) for the spectral noise density $W(s, x)$. However, the coefficients of these equations will depend, albeit weakly, on the concrete form of the function $W_0(x)$. Therefore, in order to impart a concrete significance to these coefficients, we shall first solve (5) and find the explicit form of $W_0(x)$.

An analysis of (5) shows that during the first instants after the occurrence of the instability the fastest to increase are the waves propagating along the Oz axis, i.e., corresponding to $x = 1$. However, as the amplitudes of these oscillations increase, their growth increments begins to decrease, whereas the amplitudes of the waves propagating at an angle to the Oz axis continue to grow at almost the same rate as initially. Further increase of the oscillation energy is accompanied by a still greater decrease of the increment, and this decrease is

[4] We note that the assumption that x_0 is small is not of principal significance. In the opposite case, when $x_0 \sim 1$, all the calculations can also be carried through to conclusion, but the resultant formulas are more cumbersome. Furthermore, greatest interest attaches precisely to the case of large supercriticality, when $E \gg E_{\text{lim}}$.

most intense when x is close to unity. In other words, the increments corresponding to different angles θ' (or to different $x = \cos\theta'$) tend to become equalized. It is therefore natural to assume that after the lapse of a certain time (of the order of several reciprocal increments) there sets in a state in which the total increment $\Gamma(s_0, x)$ will not depend on the angle θ' at all values of x of the corresponding instability region, and the amplitudes of the waves propagating in different directions will increase at the same rate. It is precisely this state, which we shall call "quasistationary," which will be the subject of our further research.

Thus, let us assume that at values $x \geq x_0$ the total increment $\Gamma(s_0, x)$ does not depend on x. It is easy to show that the necessary and sufficient condition for this is the equality[5)]

$$q^*(\xi) = v_0/(1-\xi^2) \quad \text{for } \xi^2 < 1 - x_0^2, \tag{26}$$

where v_0 is a certain constant to be determined, and which in general can depend on the time t. Taking into account the definition (21) of the function $q^*(\xi)$, we can regard (26) as an integral equation for the determination of the unknown function $W_0(x)$. The solution of a perfectly analogous equation was obtained in[1], and we can simply use the formulas of that paper. As a result we get

$$W_0(x) = \frac{\mathscr{E}_0}{2\pi}\begin{cases}\varphi(x) & \text{for } x \geqslant x_0 \\ 0 & \text{for } x \leqslant x_0\end{cases}, \tag{27}$$

where

$$\mathscr{E}_0 = \frac{eEs_0 n_T}{\gamma_e^{(0)} + \gamma_i^{(0)}}, \quad x_0^2 = \frac{s_0}{\lambda u^*}, \quad \lambda = \frac{s_0}{v_0},$$

$$\varphi(x) = \frac{\lambda}{x(1-\lambda x)^2}\Big\{(x^2 - x_0^2)^{1/2}[(x^2 - x_0^2) + 3x^2(1-\lambda x)]$$
$$-\frac{3}{4}\lambda x_0^4 \ln\frac{x + \sqrt{x^2 - x_0^2}}{x_0}\Big\}. \tag{28}$$

Using these expressions, we can now find the total energy and momentum densities of the noise:

$$\mathscr{E} = \mathscr{E}_0 J_{\mathscr{E}}(\lambda), \quad \mathscr{P} = \mathscr{P}_0 J_r(\lambda),$$

$$\mathscr{P}_0 = \frac{n_T e E}{\gamma_e^{(0)} + \gamma_i^{(0)}}\left[1 - \frac{3}{2}x_0^2\right],$$

$$J_{\mathscr{E}}(\lambda) = \frac{1}{\lambda^2(1-\lambda)}\Big\{1 + (1-\lambda)\ln\frac{1}{1-\lambda}$$
$$-\frac{9}{2}(1-\lambda) + 5(1-\lambda)^2 - \frac{3}{2}(1-\lambda)^3\Big\}$$
$$-\frac{3}{2}\frac{x_0^2}{1-\lambda}\left[\lambda + (1-\lambda)\ln\frac{1}{1-\lambda}\right] + O(x_0^4),$$

$$J_r(\lambda) = \frac{\lambda}{1-\lambda} + O(x_0^4). \tag{29}$$

Substituting, finally, the expression given above for $\mathscr{P}$ into (22) and recognizing that $\tilde{x}_0^2 = x_0^2\sqrt{1 - x_0^2}$ by virtue of (26), we obtain an expression for the time dependence of $J_r(\lambda)$ (and consequently of λ):

$$\frac{d}{dt}\frac{J_r}{\gamma_e^{(0)} + \gamma_i^{(0)}} = 1 - \frac{\gamma_{\text{col}}(s_0)}{\gamma_e^{(0)} + \gamma_i^{(0)}}J_r. \tag{30}$$

In order for formulas (13) and (27)–(30) to determine completely the spectral energy density of the ion-sound noise, they must be supplemented by an equation for the oscillation frequency or for the oscillation phase velocity s_0. This equation can be obtained from the condition that the total increment be a maximum at $s = s_0$, and is of the form

$$\frac{\partial}{\partial s_0}\frac{\Gamma(s_0 + 0) + \Gamma(s_0 - 0)}{2} = 0, \tag{31}$$

where the functions $\Gamma(s_0 \pm 0)$ are defined by (20). Without stopping to investigate the roots of this equation, a task entailing no fundamental difficulties, we proceed now to derive and analyze equations for the average kinetics equations and expressions for the average translational velocities of the electrons and of the resonant ions.

We first rewrite (14) and (15) in a somewhat more convenient form. Taking (27) into account and making simple transformations, we get

$$\frac{\partial f_{e,i}^{(0)}}{\partial t} = \text{St}_{e,i}(J_{e,i}) + \frac{1}{v^2}\frac{\partial}{\partial v}\Big\{v^2\frac{\partial f_{e,i}^{(0)}}{\partial v}\nu_{e,i}(v)\frac{u^* s_0}{3}$$
$$\times\Big[\beta + \frac{3}{2}\Big(\alpha + \frac{x_0^2}{4\lambda}\Big)\Big(\frac{v}{v_{e,i}^*}\Big)^6\Big]\Big\}, \tag{32}$$

$$\frac{\partial f_{e,i}^{(1)}}{\partial \xi} = -\frac{\pm(v/v_{e,i}^*)^3 + {}^4/_3\pi^{-1}I_1(y)/y^2}{\lambda x_0^2 + {}^4/_3\pi^{-1}I_2(y)/y^2}\Big|_{y=\sqrt{1-\xi^2}} s_0\frac{\partial f_{e,i}^{(0)}}{\partial v}, \tag{33}$$

where

$$\frac{1}{(v_{e,i}^*)^3} = \frac{4\pi}{3}f_{e,i}^{(0)}(s_0)\frac{\gamma_e^{(0)} + \gamma_i^{(0)}}{\gamma_{e,i}^{(0)}}\frac{n}{n_T}, \quad (v_i^*)^3 = \frac{m_e}{m_i}(v_e^*)^3, \tag{34}$$

and the constants $\alpha(x_0)$ and $\beta(x_0)$ and the functions $I_n(y)$ (n = 0, 1, 2) are determined as follows:[6)]

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5)Recognizing that when $x < x_0$ the noise intensity is practically zero, and consequently $q^* = u^*$ when $\xi^2 > 1 - x_0^2$, it is easy to verify that the total increment $\Gamma(s_0, x)$ is negative for values $x < x_0$.

6)The coefficients α and β are obviously not independent. For example, in the case when $\varphi(x)$ is given by (28) we have

$$\lambda\beta + (1-\lambda)J\ (\lambda) = {}^3/_2[\sqrt{1 - x_0^2} - \lambda\alpha].$$

$$\alpha=\frac{3\pi}{4}\int_{x_0}^{1}\frac{y^5\,dy}{\sqrt{1-y^2}}\frac{1}{{}^3/_4\pi\lambda x_0^2y^2+I_2(y)}$$

$$\beta=\frac{2}{\pi}\int_{x_0}^{1}\frac{y\,dy}{\sqrt{1-y^2}}\left[I_0(y)-\frac{I_1^2(y)}{{}^3/_4\pi\lambda x_0^2y^2+I_2(y)}\right]\quad I_n(y)$$

$$=\int_{x_0}^{y}\frac{\varphi(x)x^n\,dx}{\sqrt{y^2-x^2}} \tag{35}$$

The exact values of these coefficients depend on the parameters λ and x_0^2 and can be obtained by numerical integration. However, in the case when $x_0^2 \ll 1$, $\varphi(x)$ is defined by (28), and $\lambda \approx 1$ they can be easily estimated, and we get

$$\beta \approx 0.4\lambda,\qquad \alpha\lambda \approx 1-0.9\lambda^2.$$

Using (32), we can easily obtain now equations for w_{Te} and w_r. Multiplying the right and left sides of (32) by $m_{e,i}v^2/2$ and integrating with respect to the velocities, we get

$$\frac{d}{dt}nw_{Te}=eEs_0n_T\frac{\gamma_e^{(0)}}{\gamma_e^{(0)}+\gamma_i^{(0)}}\left[\beta+\left(\alpha+\frac{x_0^2}{4\lambda}\right)\frac{9}{4\pi}\frac{\overline{v_e^3}}{v_e^{*6}f_e^{(0)}(s_0)}\right]+\frac{2}{3}\sqrt{\frac{2}{\pi}}\frac{m_e}{m_i}v_T\left[n_T(w_{Ti}-w_{Te})+n_r(w_r-w_{Te})\right]\frac{n}{n_T}, \tag{36}$$

$$\frac{d}{dt}n_rw_r=eEs_0n_T\frac{\gamma_i^{(0)}}{\gamma_e^{(0)}+\gamma_e^{(0)}}\left[\beta+\left(\alpha+\frac{x_0^2}{4\lambda}\right)\frac{9}{4\pi}\frac{n_r}{n}\frac{\overline{v_r^3}}{v_i^{*6}f^{(0)}(s_0)}\right]+\frac{2}{3}\sqrt{\frac{2}{\pi}}\frac{m_e}{m_i}v_T\left[n_r(w_{Te}-w_r)+\frac{m_e}{m_i}\frac{v_{Te}^3}{v_r^3}\frac{n_r}{n}n_T(w_{Ti}-w_r)\right]\frac{n}{n_T}, \tag{37}$$

where

$$\overline{v_e^3}=\int v^3f_e^{(0)}d\mathbf{v},\qquad \overline{v_r^3}=\frac{n}{n_r}\int_{v>s_0}v^3f_i^{(0)}d\mathbf{v}$$

are suitably normalized third moments, equal to $8\sqrt{2/\pi}v_{Te}^3$ and $8\sqrt{2/\pi}v_r^3$ respectively in the case of a Maxwellian distribution.

Equations (36) and (37) must be supplemented by an equation for the average energy w_{Ti} of the thermal ions, which assumes the following form after calculating the term that takes into account the energy exchange in pair collisions of the thermal ions with electrons and with resonant ions:

$$\frac{d}{dt}n_Tw_{Ti}=eEs_0n_T\frac{\gamma_{col}(s_0)}{\gamma_e^{(0)}+\gamma_i^{(0)}}J_{\mathscr{E}}(\lambda)+\frac{2}{3}\sqrt{\frac{2}{\pi}}\frac{m_e}{m_i}v_T\times\left[n_T(w_{Te}-w_{Ti})+\frac{m_e}{m_i}\frac{v_{Te}^3}{v_r^3}\frac{n_r}{n}n_T(w_r-w_{Ti})\right]\frac{n}{n_T}. \tag{38}$$

It follows from (30) that the ratio $\gamma_{col}(s_0)J_{\mathscr{E}}(\lambda)/(\gamma_e^{(0)}+\gamma_i^{(0)})$ is always smaller than unity.

Taking (33) into account, it is easy to obtain expressions for the mean translational velocities of the electrons

$$\bar{v}_{ze}=\int v\xi f_e\,d\mathbf{v},$$

and of the resonant ions

$$\bar{v}_{zr}=\frac{n}{n_r}\int_{v>s_0}v\xi f_i\,d\mathbf{v},$$

and also for the total density $j=-en\bar{v}_{ze}+en_r\bar{v}_{zr}$ of the current flowing through the plasma. It takes the form[7)]

$$\bar{v}_{ze,r}=s_0\left[\frac{3}{2}\left(\frac{\sqrt{1-x_0^2}}{\lambda}-\alpha\right)\pm3\left(\alpha+\frac{x_0^2}{4\lambda}\right)\frac{\overline{v_{e,r}^3}}{v_{e,i}^{*3}}\right],$$

$$j=-en_Ts_0\left\{\frac{3}{2}\left(\frac{\sqrt{1-x_0^2}}{\lambda}-\alpha\right)+3\left(\alpha+\frac{x_0^2}{4\lambda}\right)\left[\frac{n}{n_T}\frac{\overline{v_e^3}}{v_e^{*3}}+\frac{n_r}{n_T}\frac{\overline{v_r^3}}{v_i^{*3}}\right]\right\} \tag{39}$$

It follows immediately from these expressions, in particular, that for sufficiently strong fields E, when $x_0 \ll 1$, the current density does not depend explicitly on the value of E. In other words, the conductivity of the unstable plasma is inversely proportional to the applied field. Comparison of (39) with the corresponding expression obtained for the current density neglecting the interaction of the waves with the particles shows that allowance for the particle scattering by the plasma noise produced in the unstable plasma leads to a decrease of the effective conductivity by approximately $x_0^{-2} \approx eE/m_ev_Ts_0$ times.

3. DISCUSSION OF RESULTS AND BRIEF CONCLUSIONS

For lack of space, we shall not present here a detailed analysis or solution of Eq. (22) (or (30)) for the function $J_r(\lambda)$ (or $\lambda(t)$), which determines the dependence of the noise energy on the time. We shall merely indicate that the total energy density and momentum density of the noise increases

[7)]We note that equations that are almost analogous to (32) – (39) can be obtained even without specifying the concrete explicit form of the function $\varphi(x)$ (see (28)). Moreover, this can be done even when s_0 depends on x, as is the case, for example, when the first order of smallness in s_0/v_{Te} and s_0/v_r is taken into account, or when the nonlinear interaction between the waves is taken into account (see below). Of course, the numerical coefficients α and β still remain unknown here.

monotonically, as follows from these equations, and that

$$\mathscr{E} \approx s_0 \mathscr{P}, \quad \mathscr{P} = eEn_T \begin{cases} t & \text{if} \quad \gamma_{col}(s_0) t \ll 1 \\ \sim \gamma_{col}^{-1}(s_0), & \text{if} \quad \gamma_{col}(s_0) t \gg 1 \end{cases}. \qquad (40)$$

It should also be noted that in solving the problem we have neglected completely the nonlinear interaction of the waves with one another, and in particular the s-s scattering[2] [8]). At first glance this seems perfectly valid, for inasmuch as the probability of the s-s scattering is proportional to the difference between the frequencies of the interacting waves, it follows that the corresponding nonlinear increment $\gamma_n(s_0)$ vanishes when (13) is substituted in it. Such a situation occurs, however, only if k_0 (or s_0) does not depend on the angle θ' (or x). Further, since the derivative of the nonlinear increment with respect to s differs from zero at the point $s = s_0$, even if s_0 does not depend on x, it follows that allowance for the nonlinear interaction in (5) leads to a dependence of the quantity s_0 on x, and consequently, to a change in the angular dependence of the noise spectrum.[9)] Estimates show that this change becomes significant only when $\gamma_n(s_0) \gg \gamma_{col}$ and becomes manifest primarily at small angles, i.e., when $x \approx 1$. As a result, when $\gamma_{col} \ll \gamma_n$, the expression obtained here for the noise spectrum is valid, strictly speaking, only for sufficiently short times, when $\gamma_n(s_0)t \lesssim 1$, where, as shown by estimates, $\gamma_n(s_0) \lesssim 0.1 k_0 s_0 \mathscr{E}/nm_i s_0^2$. However, Eqs. (32)—(39) have a much wider range of applicability since, as already mentioned, they depend quite weakly on the concrete form of the noise spectrum.

We turn now to a discussion of Eqs. (32)—(39) for particles. We consider first Eqs. (32) and (36) for the electronic plasma component. We note first that since we have assumed in the derivation of these equations that a) $f_e^{(1)} \ll f_e^{(0)}$ and b) the characteristic frequency of the variation of the function $f_e^{(1)}$ is much smaller than the effective collision frequency, i.e., that

$$\frac{\partial \ln f_e^{(1)}}{\partial t} \approx \frac{\partial \ln f_e^{(0)}}{\partial t} \ll \nu_{eff}^{(e)} = A\nu_e(v) \approx \frac{u^*}{s_0} \nu_e,$$

it follows that Eqs. (32) and (33) for the electrons are valid only at velocities satisfying the condition

$$v^2 \ll \frac{v_e^{*3}}{s_0} \approx 3 v_{Te}^2 \sqrt{\frac{m_i}{m_e} \frac{s_m}{s_0}} \gg v_{Te}^2. \qquad (41)$$

In addition, it is easy to verify that if the electric field is not very strong, so that[10)]

$$\frac{E}{E_{cr}} \ll \sqrt{\frac{m_i}{m_e} \frac{s_m}{s_0}}, \quad E_{cr} = \frac{m_e v_{Te} \nu_T}{e}, \qquad (42)$$

then at velocities

$$v^2 \ll 3 v_{Te}^2 \frac{s_m}{s_0} \sqrt{\frac{m_i}{m_e} \frac{E_{cr}}{E}}$$

the electronic distribution function is close to Maxwellian with a temperature that can be determined from (36) and increases monotonically with time. (Since the ratio $\overline{v_e^3}/v_e^{*3}$ does not depend on the time, we have here $w_{Te} \sim T_e \sim t^\epsilon$, where $1 < \epsilon < 2$.) Outside this region, the distribution function can differ quite strongly from Maxwellian, and may in particular decrease with increasing velocity much more slowly than $\exp(-v^2/2v_{Te}^2)$.

The foregoing circumstance is formally connected with the presence in (32) of a term proportional to $(v/v_e^*)^6$ and leading to the occurrence of a "tail" of high-energy electrons in the distribution function. The number of these electrons increases with time. This must be borne in mind in estimating the flux of runaway electrons, otherwise the value obtained will be too low. However, when condition (42) is satisfied, the number of particles in this "tail" is exponentially small, so that in first approximation it can be disregarded. On the other hand, if the field is sufficiently large and condition (42) is violated, then the distribution function will be far from Maxwellian in the entire velocity region and the rise of the electron temperature will be accompanied by the appearance of a large number of runaway electrons, produced within a time[11)] $t_0^{(e)} \approx m_e v_{Te}^2/eEs_0$.

Going over to Eq. (38) for the thermal-ion temperature, it must be pointed out that since the decrement γ_{col} of the collision damping was taken by us from the linear theory, in which it is assumed that the ion distribution function is nearly Maxwellian, it follows that Eq. (38) is valid, strictly speaking, only when $E/E_{cr} \ll (s_m/s_0)\sqrt{T_e/T_i}$. However, it is clear even from simple physical con-

[8)]We emphasize that the theory developed here is applicable, of course, only in the case of weak turbulence, when $\mathcal{E}/nT_e \ll 1$.

[9)]This was pointed out to us by L. Rudakov.

[10)]E_{cr} is the so-called critical field, starting with which all the electrons would, according to the theory of stable plasmas, go over into the continuous acceleration mode [7,8].

[11)]This was first indicated by Korablev and Rudakov [9], who considered a similar problem but neglected both pair collisions (corresponding formally to the condition $E/E_{cr} \to \infty$) and Landau damping by the ions.

siderations that even when this condition is violated the energy absorbed per unit time by the ions as a result of the presence of ion-ion collisions will remain, as before, of the order of $\gamma_{col}\,\mathscr{E}$, and Eq. (38) will thus remain practically unchanged. At the same time, since $J_{\mathscr{E}} \approx J_r$ and consequently, according to (22) and (30)

$$\frac{\gamma_{col}(s_0) J_{\mathscr{E}}}{\gamma_e^{(0)} + \gamma_i^{(0)}} = \begin{cases} \gamma_{col}(s_0)t, & \text{if} \quad \gamma_{col}(s_0)\,t \ll 1 \\ \sim 1, & \text{if} \quad \gamma_{col}(s_0)\,t \gg 1 \end{cases},$$

it follows from (38) that when $E \gg (s_m/s_0)\sqrt{m_e/m_i}E_{cr}$ the temperature of the thermal ions first increases quite slowly, and then (if $\gamma_{col}(s_0)t \gg 1$) it increases at almost the same rate as the electron temperature.

It is easy to verify in this case that

$$\frac{dn\,w_{Te}}{dn_T\,w_{Ti}} = \frac{\gamma_e^{(0)}}{\gamma_e^{(0)} + \gamma_i^{(0)}}\left\{\beta + \frac{16}{\pi}\alpha\left[\frac{\gamma_e^{(0)} + \gamma_i^{(0)}}{\gamma_e^{(0)}}\frac{n}{n_T}\right]^2\right\}. \quad (43)$$

It follows from this equation, in particular, that the temperature ratio $z(t) = T_e/T_i$ tends to a certain finite limit as $t \to \infty$. Indeed, let the particle velocity distribution be nearly Maxwellian, and let the ratio

$$\frac{n\gamma_i^{(0)}}{n_T\gamma_e^{(0)}} = \left(\frac{m_i}{m_e}\right)^{1/2}\left(\frac{T_e}{T_i}\right)^{3/2}\exp(-s_0^2/2v_{Ti}^2)$$

at the initial instant of time not exceed, say, unity. Then, since we have $\beta + 16\alpha/\pi \lesssim 2$ when $\lambda \sim 1$ and $z(0) \gg 2$, it follows that z(t) is a decreasing function of the time (or of T_i). Further, it is easy to check by using (31) that at sufficiently small z and large T_i we have $s_0 \to s_m = \sqrt{T_e/m_i}$ and consequently the right side of (43) becomes a function of z only. Then, putting $n_T \approx n$, we can rewrite (43) in the form

$$\frac{dz}{d\ln T_i} = \psi(z) - z, \quad \psi(z) \approx \sqrt{m_i/m_e}\,z^{3/2}e^{-z/2}, \quad (43')$$

from which it is seen that when $T_i \to \infty$ ($t \to \infty$) z tends to a limiting value z_0, defined by

$$z_0 = \psi(z_0), \quad z_0 \approx \ln\left[\frac{m_i}{m_e}\ln\frac{m_i}{m_e}\right].$$

For a hydrogen plasma with $m_i/m_e \approx 2 \times 10^3$ this yields $z_0 \approx 10$.

Thus, if $E \gg (s_m/s_0)\sqrt{m_e/m_i}E_{cr}$ and $t\gamma_{col}(s_0) \gg 1$, the heating of the thermal ions is due primarily to collision absorption of the waves, and the rate of this heating can exceed the rate of heating by electron-ion collisions by a factor $(E/E_{cr})\sqrt{m_i/m_e}s_0/s_m$. However, the ratio T_i/T_e always remains smaller than $z_0^{-1} \approx 10^{-1}$.

We proceed, finally, to the equations for the resonant ions. It is easy to verify that, just as in the case of electrons (see Eq. (41)), Eqs. (32) and (33) for resonant ions are valid only when

$$v^2 \ll \frac{v_i^{*3}}{s_0} \approx 3\sqrt{\frac{m_i}{m_e}}\,s_m^2\frac{s_m}{s_0} \gg s_m^2. \quad (44)$$

In spite of the formal analogy between the equations for the electrons and the ions, there is an essential difference between them, consisting in the fact that $v_i^* \approx (m_i/m_e)^{1/6}s_m$ does not depend on the temperature of the resonant ions, whereas $v_e^* \approx v_{Te}$. As a result, by virtue of the condition (44), the equations for the ions are valid only so long as

$$T_r \ll 3T_e\sqrt{\frac{m_i}{m_e}\frac{s_m}{s_0}}. \quad (44')$$

An analysis of (32) and (33) shows that the form of the ionic distribution function $f_i^{(0)}$ and the character of the temporal variation of the temperature T_r (or of w_r) depend essentially on the magnitude of the electric field. For example, if the field E is sufficiently weak, so that

$$\frac{E}{E_{cr}} \ll \left(\frac{m_e}{m_i}\right)^{1/6}\frac{s_m}{s_0} \approx 1, \quad (45)$$

then the term proportional to $(v/v_i^*)^6$ in (32) (and accordingly proportional to v_r^3 in (37)) can be neglected, and the distribution function will fall off exponentially with increasing velocity[12)] v. For sufficiently small times

$$t \ll t_E = \nu_T^{-1}\left(\frac{E}{E_{cr}}\frac{s_0}{s_m}\right)^{3/2}\left(\frac{m_i}{m_e}\right)^{5/4},$$

i.e., so long as

$$T_r \ll T_e\frac{E}{E_{cr}}\sqrt{\frac{m_i}{m_e}\frac{s_0}{s_m}},$$

the distribution function in the velocity region $v_r \lesssim v \lesssim v_i^*$ is

$$f_i^{(0)} \sim \frac{1}{\tau^{3/5}}\exp\left(-\frac{v^5}{25\tau}\right), \; \tau = \frac{\nu_i(v)v^3}{3}\int^t u^* s_0 dt \approx \left(\frac{m_e}{m_i}\right)^2 v_{Te}^3 s_0\frac{eEt}{m_e}, \quad (46)$$

and the effective ion temperature is $T_r \approx m_i\tau^{2/5} \sim t^{2/5}$, i.e., it increases quite rapidly. After a time t_E, it reaches a "stationary" value

[12)]It must be borne in mind that, in analogy with the electrons, the presence of a term $\sim v^6$ in (32) leads to the appearance of a "tail" of high-energy ions in the velocity region $v > v_i^*$. However, if condition (45) is satisfied the number of particles in this tail is exponentially small.

$$T_r \approx \frac{E}{E_{cr}}\left(\frac{m_i}{m_e}\right)^{1/2}\frac{s_0}{s_m}T_e \ll 3T_e\left(\frac{m_i}{m_e}\right)^{1/2}\frac{s_m}{s_0}, \tag{47}$$

increasing subsequently only to the extent that the electron temperature T_e increases. The rise in the temperature T_r is accompanied by an increase in the number of resonant ions, the density of which can be obtained for $E \ll E_{cr}s_m^4/s_0^4$ from the relation[13)]

$$\frac{n_r(t)}{n - n_r(t)} = \left[\frac{T_r(t)}{T_i(t)}\right]^{3/2}\exp\left(-\frac{s_0^2(t)}{2v_{Ti}^2(t)}\right). \tag{48}$$

The distribution function in this state (i.e., when $t > t_E$) is nearly Maxwellian with a temperature defined by (47).

On the other hand, if the external field is strong, so that condition (45) is violated, then the temperature of the resonant ions increases at all times more rapidly than T_e, and when $t \ll t_0^{(i)} = t_0^{(0)}(m_i/m_e)^{1/3}$ the distribution function in the region $v < v_i^*$ is determined as before by the relation (46). After a time of the order of $t_0^{(t)}$ the ion temperature reaches a value $T_r \approx (m_i/m_e)^{1/3}T_e$ and an important role is assumed by the term proportional to $(v/v_i^*)^6$. This, in turn, causes the temperature T_r to start increasing even rapidly when $t > t_0^{(i)}$, and the distribution function acquires in the region $v > v_r$, where $f_i^{(0)} \approx 1/v^4$, a "tail" of fast ions, the number of which increases rapidly.

Although the subsequent evolution (starting with $T_r \gtrsim 3T_e(m_i/m_e)^{1/3}s_m/s_0$) of the ion distribution function and of the temperature T_r can not be traced, strictly speaking, qualitatively on the basis of Eqs. (32) and (33), it is clear that further increase in the temperature of the resonant ions will be accompanied by the appearance of a large number of runaway ions, the number of which, when condition (42) is satisfied, can greatly exceed the number of runaway electrons.

We see thus that the instabilities occurring in a nonisothermal plasma are accompanied, on the one hand, by a sharp increase in the resistance of the plasma to electric current, and on the other by an increase in both the ion and the electron temperatures, and the rate of the former may greatly exceed the corresponding value obtained in the theory of a stable plasma. The most rapid increase takes place in the temperature of the resonant ions, which can reach, within a relatively short time, values exceeding the electron temperature by many times.

In conclusion we note that the foregoing results can be generalized without difficulty to include also the case of a partly ionized plasma, provided the number of neutral atoms is not very large, so that their contribution to the collision frequencies $\nu_{e,i}$ is relatively small. To this end it is simply necessary to add in Eqs. (36)—(38) terms that take into account the change in energy upon collision with the neutral atoms (including inelastic collisions), and to supplement them with an equation for the neutral-gas temperature. We point out also that the results remain apparently valid also in the presence of an external electric field H parallel to E, if $H^2 \ll 4\pi n m_i c^2$.

The author is grateful to V. P. Silin for calling his attention to the group of questions discussed here.

13)When $E > E_{cr}s_m^4/s_0^4$ the number n_r of resonant ions increases much more slowly.

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