RADIATIVE POLARIZATION OF ELECTRONS IN A MAGNETIC FIELD

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Submitted to JETP editor December 26, 1966

J. Theoret. Phys. (U.S.S.R.) 52, 1422—1426 (May, 1967)

Electron polarization due to radiation in a nonuniform magnetic field is investigated by applying the operator formulation of the quasiclassical approximation. A general expression is obtained for the probability of radiative transition with spin flip in an arbitrary magnetic field.

ELECTRONS and positrons moving in a magnetic field can become polarized as a result of radiation. The polarization arises because the probability of radiative transition with spin flip depends on the orientation of the initial spin. The existence of radiative polarization in a homogeneous magnetic field was first pointed out by Sokolov, Ternov, and co-workers[1,2]. Radiative polarization was considered also by the authors in[3], where we formulated an approach that takes essentially into account the quasiclassical character of the motion of high-energy electrons in a magnetic field and permits, in principle, to consider radiative polarization in an inhomogeneous magnetic field.

In this paper we develop an operator method for the investigation of spin phenomena in a quasiclassical approximation[1)]. This method has turned out to be adequate for our problem and yields the probability of radiative transition with spin flip in an arbitrary magnetic field, whereas hitherto this problem could not be solved even within the framework in perturbation theory in terms of the field inhomogeneity.

It must be noted that the characteristic time of the radiative polarization is of the same order as the operating time of colliding-beam accelerators, and therefore the problem of radiative polarization in an inhomogeneous magnetic field is of great practical interest.

The motion of a high-energy electron in a magnetic field can be considered quasiclassically if the energy of the emitted photons is much lower than the electron energy

$$\hbar\omega \ll E, \qquad \omega \sim \omega_0\gamma^3, \tag{1}$$

where

$$\gamma = \frac{E}{mc^2}, \quad \omega_0 = \frac{v_t}{R}, \quad R = \frac{cp_t}{eH}, \tag{2}$$

R is the instantaneous radius of curvature, and H is the magnetic field. In this case we can describe the motion of the electron with the aid of classical characteristics. Inasmuch as in all the existing installations the inequality (1) is satisfied with a large margin, we confine ourselves to this case.

The matrix element of the transition to the initial state of the particle in an external electromagnetic field $|i\rangle$ to the corresponding final state $|f\rangle$ with emission of a photon will be written, in lowest order of perturbation theory (henceforth we put c = 1)

$$U_{fi} = -\left\langle f\left|\int \{j_\mu, A^\mu\}dt\right| i\right\rangle = \left\langle f\left|\frac{e}{(2\pi)^{3/2}\sqrt{2\hbar\omega}}\int e^{i\omega t}M(t)\,dt\right| i\right\rangle, \tag{3}$$

where

$$eM(t) = \{\mathbf{j}(t)\mathbf{e}, e^{-i\mathbf{kr}(t)}\}. \tag{4}$$

Here $\mathbf{j}(t)$ and $\mathbf{r}(t)$ are respectively the operators of the current and of the particle coordinate, $\mathbf{e}$ is the photon polarization vector (we choose a gauge with $e_0 = 0$), and the braces { } denote the symmetrized operator product.

Summing the transition probability over all final states of the particle, we obtain the following expression for the radiative-transition probability

$$dw = \frac{\alpha}{(2\pi)^2}\frac{d^3k}{\omega}\left\langle i\left|\int dt_1\int dt_2 e^{i\omega(t_1-t_2)}M(t_1)M^*(t_2)\right| i\right\rangle. \tag{5}$$

The foregoing expression can be used to study a number of phenomena occurring when a particle emits a photon in an external electromagnetic field. In the case of interest to us, of electrons (positrons) moving in an external electromagnetic field, we represent (4) in the form

[1)]A similar procedure was used by Schwinger [4] to find the quantum correction to the intensity of electron radiation in a magnetic field.

$$M(t) = u^+(\zeta_f)(\alpha e)e^{-i\mathbf{kr}(t)}u(\zeta_i), \quad (6)$$

where $u(\zeta_f)$ and $u(\zeta_i)$ are the solutions of the Dirac equation in an arbitrary electromagnetic field in operator form; ζ_i and ζ_f characterize the initial and final spin states.

The quantum effects occurring when an ultra-relativistic electron moves in an external magnetic field are of two types. The first is connected with the quantum character of the electron motion itself in the magnetic field. Recognizing that in the first order in ħ we have

$$[\mathbf{vv}] = i\frac{\hbar e}{E^2}[(1-\mathbf{v}^2)\mathbf{H} + (\mathbf{vH})\mathbf{v}], \quad (7)$$

it is clear that the uncertainty in the determination of the electron velocity components is

$$\Delta v_1 \Delta v_2 \sim \hbar e H/E^2 = \hbar\omega_0/E. \quad (8)$$

From this it follows obviously that as the energy increases the motion of the electron in the magnetic field becomes more and more "classical," since the velocity components become defined.[4]

The second type is produced by the recoil of the electron when the photon is emitted, and is therefore of the order of $\hbar\omega/E$. Since $\omega \sim \omega_0\gamma^3$ and we are interested only in the principal term of the expansion in ħ and in $1/\gamma$, the non-commutation of the velocity components can be neglected, and we take into account only the commutation of the operators of the dynamic variables of the electron with the field of the emitted photon.

Taking the foregoing into account, the operator solution of the Dirac equation is

$$u(\zeta) = \sqrt{\frac{H+m}{2H}}\begin{pmatrix} \varphi(\zeta(t)) \\ \dfrac{\sigma\mathbf{P}(t)}{H+m}\varphi(\zeta(t)) \end{pmatrix}, \quad (9)$$

where $H = \sqrt{\mathbf{P}^2 + m^2}$ and $\varphi(\zeta(t))$ is a two-component spinor describing the spin states of the electron at the instant of time t.

We are interested in this paper in the probability of radiative transition with spin flip, so that it is convenient to put

$$\varphi(\zeta_f) = e^{i\sigma a\pi/2}\varphi(\zeta_i) = i(\sigma\mathbf{a})\varphi(\zeta_i), \quad (10)$$

where **a** is a unit vector perpendicular to the spin quantization axis, $\zeta_i \equiv \zeta$.

Taking into account the fact that

$$\varphi_i\varphi_f^+ = -\tfrac{1}{2}i(\mathbf{a} + i[\zeta\mathbf{a}])\sigma \equiv -\tfrac{1}{2}i(\mathbf{b}\sigma), \quad (11)^*$$

and performing the required commutations, we readily obtain for the matrix element

$$M(t) = \frac{\hbar}{2H}e^{-i\mathbf{kr}(t)}(\mathbf{b}(t)[\mathbf{q}(t)\mathbf{e}]), \quad (12)$$

where

$$\mathbf{q}(t) = \mathbf{P}(t)\omega/(H+m) - \mathbf{k}. \quad (13)$$

We see that expression (12) for the matrix element is proportional to ħ, so that the non-commutation of the operators contained in it can be neglected, for allowance for the non-commutation results of corrections of higher order in ħ, which do not interest us. Therefore all the operators in (5) in the initial-state brackets can be replaced by their classical values.

We change over in the integral of (5) to new variables:

$$t = (t_1 + t_2)/2, \quad \tau = t_2 - t_1. \quad (14)$$

Since we are interested in the transition probability per unit time dw/dt, it remains to integrate in (5) (after integrating over the final states of the photon) only with respect to the relative time τ. It will be seen in what follows that the main contribution to the transition probability is made by the region $|\dot{\mathbf{v}}|\tau \sim 1/\gamma$,[2)] and we shall therefore expand all the quantities in powers of $|\dot{\mathbf{v}}|\tau$, corresponding to expansion in $1/\gamma$, and retain only the higher-order terms of the expansion. In addition, we shall neglect the quantities

$$|\dot{\mathbf{H}}|\tau/|\mathbf{H}| \ll 1, \quad (15)$$

where $|\dot{\mathrm{H}}|$ characterizes the change of the magnetic field on the trajectory. If the field is described in terms of the inhomogeneity index n, then condition (15) takes the form $n/\gamma \ll 1$.

Performing this expansion for the quantities in $\mathbf{b}(t_1)$ and in $\mathbf{b}(t_2)$, and estimating the higher terms of the expansion with the aid of the equations of motion of the spin $\zeta(t)$ in an external magnetic field[5], it is easy to show that these terms make a contribution $\sim 1/\gamma$, so that we can put

$$\mathbf{b}(t_1) = \mathbf{b}(t_2) = \mathbf{b}(t). \quad (16)$$

Naturally, the result does not depend on the direction of the vector **a** in a plane perpendicular to the vector **ζ**, so that it is convenient to use the summation formula

$$\tfrac{1}{2}\sum_{\lambda=1} b_i^{(\lambda)}b_j^{(\lambda)*} = \delta_{ij} - \zeta_i\zeta_j - i\varepsilon_{ijk}\zeta_k. \quad (17)$$

Summing also over the photon polarizations, we get

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*$[\zeta\mathbf{a}] \equiv \zeta \times \mathbf{a}$.

2)It is more convenient to use the characteristic $\dot{\mathrm{v}}$ in lieu of the revolution frequency ω_0.

$$\sum M_1M_2^* = \frac{\hbar^2}{4E^2} e^{i\mathbf{k}(\mathbf{r}_2-\mathbf{r}_1)}\left\{ (\mathbf{q}_1\mathbf{q}_2)\left(1-\frac{(\boldsymbol{\zeta}\mathbf{k})^2}{\omega^2}\right)+\frac{1}{\omega^2}(\boldsymbol{\zeta}\mathbf{k}) \times[(\mathbf{q}_1\boldsymbol{\zeta})(\mathbf{q}_2\mathbf{k})+(\mathbf{q}_2\boldsymbol{\zeta})(\mathbf{q}_1\mathbf{k})]-i\left(\left(\boldsymbol{\zeta}-\frac{(\boldsymbol{\zeta}\mathbf{k})\mathbf{k}}{\omega^2}\right)[\mathbf{q}_1\mathbf{q}_2]\right)\right\}, \tag{18}$$

where the indices 1 and 2 denote respectively the dependence on t_1 and t_2. The quantities in (18) can be expanded as follows:

$$\mathbf{q}_1 = (\omega\mathbf{v}-\mathbf{k})-\frac{\omega\tau}{2}\dot{\mathbf{v}}-\frac{\omega}{\gamma}\mathbf{v}+\ldots,$$

$$\mathbf{q}_2 = (\omega\mathbf{v}-\mathbf{k})+\frac{\omega\tau}{2}\dot{\mathbf{v}}-\frac{\omega}{\gamma}\mathbf{v}+\ldots,$$

$$\mathbf{r}_2-\mathbf{r}_1 = \tau\mathbf{v}+\frac{\tau^3}{24}\ddot{\mathbf{v}}+\ldots \tag{19}$$

To obtain the total probability of radiative transition with spin flip it is necessary to integrate over the photon momentum. This is conveniently done prior to integration with respect to τ, using the formula

$$\int e^{-i(ky)} f(k_\mu)\frac{d^3k}{\omega} = -f(i\partial_\mu)\frac{4\pi}{(y_0-i\varepsilon)^2-\mathbf{y}^2}, \tag{20}$$

where

$$y_0 = \tau = t_2 - t_1, \quad \mathbf{y} = \mathbf{r}_2-\mathbf{r}_1,$$
$$y^2 = y_0^2-\mathbf{y}^2 = \tau^2\left(\frac{1}{\gamma^2}+\frac{\tau^2}{12}\dot{\mathbf{v}}^2\right)+\ldots \tag{21}$$

Taking (19)—(21) into account, we can obtain after simple derivations the following expression for the total transition probability:

$$W^\zeta(t) \equiv \frac{dw}{dt} = \frac{\alpha}{\pi}\frac{\hbar^2}{m^2}\gamma^5|\dot{\mathbf{v}}|^3 \times \oint \frac{dz}{(1+z^2/12)^3}\left[\frac{3}{z^4}-\frac{5}{12z^2}+\left(\frac{1}{z^4}+\frac{5}{12z^2}\right)(\boldsymbol{\zeta}\mathbf{v})^2 - \frac{2i}{z^3|\dot{\mathbf{v}}|}(\boldsymbol{\zeta}[\mathbf{v}\dot{\mathbf{v}}])\right], \tag{22}$$

where the substitution $z = \tau\gamma|\dot{\mathbf{v}}|$ was made and the integration contour was drawn below the real axis. We see therefore that the main contribution to the integral was made by the region $|\mathbf{v}|\tau \sim 1/\gamma$. The contour integrals in (22) can be readily obtained from the following universal formula:

$$I_{nm} \equiv \oint \frac{dz}{z^n(1+z^2/12)^m} = \frac{i^n\pi(\sqrt{12})^{1-n}}{(m-1)!}\left(\frac{n+1}{2}\right)\left(\frac{n+1}{2}+1\right)\ldots\left(\frac{n+1}{2}+m-2\right), \quad m \geqslant 1. \tag{23}$$

Ultimately we obtain the following formula for the total probability of radiative transition with spin flip per unit time:

$$W^\zeta = \frac{5\sqrt{3}}{16}\alpha\frac{\hbar^2}{m^2}\gamma^5|\dot{\mathbf{v}}|^3\left\{1-\frac{2}{9}(\boldsymbol{\zeta}\mathbf{v})^2-\frac{8\sqrt{3}}{15|\dot{\mathbf{v}}|}(\boldsymbol{\zeta}[\mathbf{v}\dot{\mathbf{v}}])\right\}. \tag{24}$$

In a homogeneous magnetic field the expression (24) goes over to the well known probability of radiative transition with spin flip for the case of transverse polarization $(\boldsymbol{\zeta}\cdot\mathbf{v}) = 0$ and longitudinal polarization $(\boldsymbol{\zeta}\cdot\mathbf{v}) = 1$ for electrons ($e < 0$) and positrons ($e > 0$).[2] The conclusion that the probability of radiative transition with spin flip does not depend on the spin orientation for longitudinal polarization and does depend, in general, on the spin in the case of transverse polarization remains in force in an inhomogeneous magnetic field.

We see from the foregoing calculation that the inhomogeneity n of the magnetic field enters into the problem in the form of the combination n/γ, and so long as we are considering the case $n/\gamma \ll 1$ the radiation process has the same character as in a homogeneous field. This is connected with the fact that the photon emission occurs over a length much smaller than the characteristic length of the magnetic-field inhomogeneity.

Expression (24) contains time-dependent quantities. We, naturally, are interested in averages over the time. For a general analysis of the radiative polarization in concrete conditions, it is necessary to solve the classical equations of motion of the particle and of the spin vector[5], substitute them in (24), and average over the time. In the case of an axially-symmetrical weak-focusing inhomogeneous magnetic field, the time-averaged expression for the probability of radiative transition with spin flip, apart from correction terms $a^2/\overline{R}^2$ (a is the amplitude of the transverse oscillations, and $\overline{R}$ is the average radius of curvature of the orbit), is of the same as in a homogeneous field ($\overline{R}$ plays the role of the radius). The quantities $a^2/\overline{R}^2$ are quite small (10^{-3}—10^{-4}) for all modern installations.

Thus, the effect of radiative polarization takes place, in general, also in an inhomogeneous field and consequently can be observed in modern storage rings if the influence of depolarizing factors is eliminated[6].

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Translated by J. G. Adashko
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