VORTEX STRUCTURE IN SUPERCONDUCTING FILMS IN A PARALLEL FIELD

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It is shown that a one-dimensional vortex structure of current and magnetic field distribution arises in superconducting films (magnetic field parallel to the surface) in a certain range of thicknesses (exceeding the coherence radius $\xi(T) \sim \hbar v_0/\Delta(T)$, but less than the penetration depth $\delta(T)$). The possibility of observing such a structure by optical diffraction is discussed.

It is well known that superconductivity arises in bulk metallic samples upon decrease of the magnetic field through the formation of a two-dimensional periodic structure of Abrikosov vortex filaments.[1] Such a structure is stable if the corresponding critical field, the so-called upper critical field H_{c2}, exceeds the thermodynamic critical field H_{cm} of the superconductor. Near the sample surface, superconductivity remains at fields exceeding H_{c2}, up to the value $H_{c3} = 1.69\, H_{c2}$ (for $T \to T_c$).[2,3] For films, it is H_{c3} (which depends on the film thickness d) which begins to serve as the upper critical field.[1)] Further, as Abrikosov[3] has remarked, the solution for the order parameter $\psi(\mathbf{r})$, corresponding to H_{c3}, is generally asymmetric relative to the center of the film.[2)] As we shall show, this causes the periodic vortex structure of the "superconducting electron" density distribution, the superconducting current, and the magnetic field longitudinal to the film. In contrast to bulk samples, such a structure in films is one- rather than two-dimensional. It can also occur for values of the Ginzburg-Landau parameter κ characterizing a superconductor of the first kind (for example, for $1/1.7\sqrt{2} < \kappa < 1/\sqrt{2}$).[3)]

Taking the direction of the magnetic field along the y axis and normal to the surface of the film (which is along the z axis), we write the linearized Ginzburg-Landau equation[4,5] as

$$\frac{1}{2m}\left(\frac{\hbar}{i}\frac{\partial}{\partial x}-\frac{eH}{c}z\right)^2\psi-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial z^2}=\alpha\psi, \qquad \left.\frac{\partial\psi}{\partial z}=0\right|_{z=\pm d}, \tag{1}$$

where 2d is the film thickness, and e and m are the charge and mass of the Cooper pair.

If this equation has a solution of the form $\varphi(z)e^{ikx}$ with $\varphi(z) \neq \varphi(-z)$, then $\varphi(-z)e^{-ikx}$ also satisfies the equation. Thus we must take a linear combination of these functions, $C_1\varphi(z)e^{ikx} + C_2\varphi(-z)e^{-ikx}$. Further, it is clear that the minimum free energy corresponds to the solution for which $|C_1| = |C_2|$.[4)] Therefore, without loss of generality, one can take this solution to be (correct to a normalization factor)

$$\psi(x,z)=\varphi(z)e^{ikx}+\varphi(-z)e^{-ikx}, \tag{2}$$

where $\varphi(z)$ satisfies

$$-\frac{d^2\varphi}{dz^2}+\left(k-\frac{eH_0}{\hbar c}z\right)^2\varphi=\frac{2m\alpha}{\hbar^2}\varphi, \qquad \left.\frac{d\varphi}{dz}=0\right|_{z=\pm d} \tag{3}$$

(here we have put $H_{c3}(d) = H_0$).

The vector current density corresponding to (2) has the components

$$j_x=\frac{e\hbar k}{m}[\varphi^2(z)-\varphi^2(-z)]-\frac{e^2}{mc}H_0z[2\varphi(z)\varphi(-z)\cos 2kx + \varphi^2(z)+\varphi^2(-z)], \tag{4}$$

$$j_z=\frac{e\hbar}{m}[\varphi(z)\varphi'(-z)+\varphi'(z)\varphi(-z)]\sin 2kx. \tag{5}$$

Consequently, j_z is not identically zero except at k = 0. Therefore, the line of flow is not parallel to

1)For a second order phase transition (this certainly occurs for a sufficiently large value of the Ginzburg-Landau [4,5] or for small thickness d.).

2)We assume, for simplicity, that the temperature is sufficiently close to T_c so that the Ginzburg-Landau equations apply. The qualitative picture of the effect considered later is not connected with this assumption.

3)As was shown in a previous paper, [6] an analogous vortex structure arises also for "weak" superconductivity (for a Josephson junction [7] placed in an external magnetic field parallel to its surface).

4)The calculation of the coefficients C_1 and C_2 is carried out in the Appendix (see (A.1)).

the surface of the sample in this case. It has the pattern shown schematically in the figure.

The existence of closed lines of flow is easy to deduce by expanding $\varphi(z)$ in (4) and (5), for small z, in a power series about the point $z = 0$. Examining, for simplicity, the case of a "thick" film ($d \gg \xi = \hbar/\sqrt{2m\alpha}$), one has $|\varphi'(0)| \gg \varphi(0)/\xi$, and the indicated expansion takes the form

$$j_z = j_s \sin 2kx, \quad j_x = 2j_s kz,$$
$$j_s = \frac{2e\hbar}{m}\varphi(0)\varphi'(0). \tag{6}$$

Hence, we obtain as the equation for the flow lines

$$k^2z^2 = \sin^2 kx + C, \tag{7}$$

where C is a constant ($C > -1$). For $C > 0$, open trajectories result, the directions of which are parallel to the surface of the film; closed trajectories occur for $C < 0$ (see the figure). The equation of the trajectory which separates the open from the closed flow lines is $kz = \pm \sin kx$.[5]

The pattern described above corresponds to a field equal to the critical field: $H = H_0$. However, it is clear that it will also result for $H \neq H_0$, but $H_0 - H \ll H_0$. Further, as in the theory of Abrikosov,[1] the magnetic field in the film becomes inhomogeneous (oscillating with the x coordinate). If one writes it as $H + \delta H$, where H is the external field, then δH equals

$$\delta H(x, z) = h_1(z) + h_2(z)\cos 2kx, \tag{8}$$

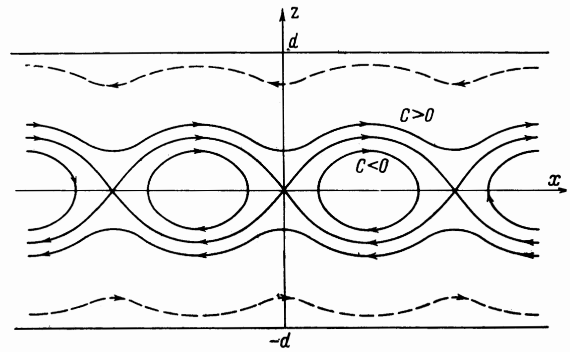
where

$$h_1(z) = -\frac{4\pi e\hbar}{mc}[f(z) + f(-z)],$$
$$h_2(z) = -\frac{2\pi e\hbar}{mck}[\varphi(z)\varphi'(-z) + \varphi'(z)\varphi(-z)], \tag{9}$$

and the function f(z) is

$$f(z) = \int_{-d}^{z}\left(k - \frac{eH_0}{\hbar c}z\right)\varphi^2(z)\,dz. \tag{10}$$

At the surface of the sample, $\delta H(x, y)$ becomes zero. For $h_2(z)$ this is obvious since $\varphi'(\pm d) = 0$.



The function $h_1(z)$ also goes to zero for $z = \pm d$ because of the identity

$$\int_{-d}^{d}\left(k - \frac{eH_0}{\hbar c}z\right)\varphi^2(z)\,dz = 0, \tag{11}$$

which follows from (3) for the maximum possible H_0 ($H_0 = \max H_0(k)$).[6]

As is seen from (8), δH is a periodic function of x with period $a = \pi/k$. The value of k must be foun found from the solution of Eq. (3) for the greatest possible $H_0 = H_{c3}(d)$. For $d \gg \xi$, we have, according to [2], $k = k_0 = eH_{c2}\xi/\hbar c$. The case $d \gg \xi$ is of little interest, of course, since ψ is appreciably different from zero only near the surface, and the oscillating current, field, etc., which result because of the interference of the solutions from opposite surfaces of the film, are exponentially small. For smaller d, the amplitude of oscillation at first grows (although it becomes sizeable far from $H_{c3}(d)$), but then necessarily decreases and goes to zero at a certain critical thickness d_c (for $d \le d_c$ the symmetric solution $\varphi(z) = \varphi(-z)$ corresponds to the maximum field). According to Abrikosov[3], as $T \to T_c$, the value of d_c equals

$$d_c(T) = \frac{1}{2}\sqrt{\frac{5}{2}}\frac{\delta(T)}{\varkappa} \sim \xi(T), \tag{12}$$

where $\delta(T)$ is the penetration depth.

We remark that for $d \ll d_c$ the function $\varphi(z)$ has the simple form

$$\varphi(z) = \text{const}\left[1 - \frac{z^2}{2\xi^2}\left(1 - \frac{z^2}{2d^2}\right)\right], \tag{13}$$

substitution of (13) into (3) gives the well-known result of Ginzburg and Landau,[3, 4]

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[5] We call attention to the analogy between the relations (6) and the corresponding formulas from the theory of the Josephson effect.[7] The flow of superconducting current j_z across a region of (normal) metal situated between regions near the surface of thickness $\sim \xi$, in which ψ is appreciably different from zero, can also be interpreted as "weak" superconductivity. In complete analogy with [6,7], the potential difference at the boundaries of the film induces a motion of the periodic structure, shown in the figure, in a direction parallel to the surface.

[6] The proof of this assertion is given in the Appendix (A.2). We note that the relation (11) guarantees that the total current $J = \int_{-d}^{d} j_x dz$ flowing along the film is equal to zero. For surface superconductivity with $d = \infty$, this feature was pointed out by Saint-James and de Gennes [2] and by Abrikosov.[3] This property holds even for finite thickness.

$$H_{c3}=\sqrt{6}\frac{\delta}{d}H_{cm}. \qquad (14)$$

Experimental observation of the periodic structure of the magnetic field in a film (for $d > d_c(T)$) can be accomplished, in principle, by the same methods as used for the observation of the Abrikosov structure in bulk samples, for example, by nuclear magnetic resonance[8, 9] or neutron diffraction.[10,11] However, a new possibility arises for a film. If the film is sufficiently thin, such that it is significantly transparent to visible light (2d ~ 300–500 Å) then,[7)] inasmuch as the period a of the vortex structure is commensurate with the wavelength of the light, such a film will act as a diffraction grating if the coefficient of absorption of the light depends on the magnetic field. The latter can, for example, be due to the rotation of the plane of polarization (for this, the experiment must be conducted in polarized light with an inclined beam), or due to the Zeeman effect (for example, by dissolving in a metal rare-earth impurities having relatively narrow optical bands exhibiting Zeeman splitting). I thank V. V. Eremenko for a discussion of this question.

APPENDIX

1. To determine the coefficients C_1 and C_2, we write the solution of the Ginzburg-Landau equation for $H \neq H_0$ but $(H_0 - H)/H_0 = \epsilon \ll 1$, as $\psi = \psi_0 + \psi_1$, where $\psi_1 \ll \psi_0$, in which

$$\psi_0(x,z)=C_1\varphi(z)e^{ikx}+C_2\varphi(-z)e^{-ikx}, \qquad (A.1)$$

and $\psi_1(x, z)$ satisfies the equation

$$\frac{1}{2m}\left(\frac{\hbar}{i}\frac{\partial}{\partial x}-\frac{eH_0}{c}z\right)^2\psi_1-\frac{\hbar^2}{2m}\frac{\partial^2\psi_1}{\partial z^2}-\alpha\psi_1=-\beta|\psi_0|^2\psi_0$$
$$-\frac{eH_0}{mc}\varepsilon z\left(\frac{\hbar}{i}\frac{\partial}{\partial x}-\frac{eH_0}{c}z\right)\psi_0$$
$$+\frac{e}{mc}\delta A(x,z)\left(\frac{\hbar}{i}\frac{\partial}{\partial x}-\frac{eH_0}{c}z\right)\psi_0$$
$$+\frac{e}{2mc}\psi_0(x,z)\frac{\hbar}{i}\frac{\partial}{\partial x}\delta A(x,z). \qquad (A.2)$$

Here $\delta A(x, z)$ is the addition to the vector potential, given by

$$\frac{\partial}{\partial z}\delta A(x,z)=\delta H(x,z),$$

where the equation for $\delta H(x, z)$ in this instance is (compare with (8))

$$\delta H(x,z)=-\frac{4\pi e\hbar}{mc}[|C_1|^2f(z)+|C_2|^2f(-z)]$$
$$-\frac{2\pi e\hbar}{mck}[\varphi'(z)\varphi(-z)+\varphi'(-z)\varphi(z)]\operatorname{Re}C_1C_2^*e^{2ikx}. \qquad (A.3)$$

Using the orthogonality of the right side of (A.2) to the solutions of the homogeneous equation for $\varphi(z)e^{ikx}$ and $\varphi(-z)e^{-ikx}$, we obtain the equations which determine C_1 and C_2:

$$C_1(A|C_1|^2+B|C_2|^2)=\varepsilon\eta^2C_1,$$
$$C_2(A|C_2|^2+B|C_1|^2)=\varepsilon\eta^2C_2, \qquad (A.4)$$

where A, B, and η are defined by

$$A=\beta\int_{-d}^{d}\varphi^4(z)\,dz-4\pi\mu^2\int_{-d}^{d}f^2(z)\,dz, \quad \mu=\frac{e\hbar}{mc}; \qquad (A.5)$$

$$B=2\beta\int_{-d}^{d}\varphi^2(z)\varphi^2(-z)\,dz-4\pi\mu^2\int_{-d}^{d}f(z)f(-z)\,dz$$
$$+\left(\frac{\pi\mu}{2k}\right)^2\int_{-d}^{d}[\varphi'(z)\varphi(-z)+\varphi(z)\varphi'(-z)]^2\,dz; \qquad (A.6)$$

$$\eta^2=\frac{\hbar^2}{m}\int_{-d}^{d}\left(k-\frac{eH_0}{\hbar c}z\right)^2\varphi^2(z)\,dz. \qquad (A.7)$$

The equations (A.4) allow three solutions (besides the trivial one $C_1 = C_2 = 0$):

$$\begin{aligned}&1)\ C_1=0, \quad |C_2|=\eta(\varepsilon/A)^{1/2};\\&2)\ C_2=0, \quad |C_1|=\eta(\varepsilon/A)^{1/2};\\&3)\ |C_1|=|C_2|=\eta(\varepsilon/(A+B))^{1/2}.\end{aligned} \qquad (A.8)$$

Without loss of generality, we can assume that the ψ-functions corresponding to these solutions have the forms, for $\epsilon \to 0$,

$$\psi_0^{(1)}=\eta(\varepsilon/A)^{1/2}\varphi(z)e^{ikx}, \quad \psi_0^{(2)}=\eta(\varepsilon/A)^{1/2}\varphi(-z)e^{-ikx},$$
$$\psi_0^{(3)}=\eta(\varepsilon/(A+B))^{1/2}[\varphi(z)e^{ikx}+\varphi(-z)e^{-ikx}]. \qquad (A.9)$$

Further, it is necessary to select from these solutions, the one which corresponds (for a given external field H) to the minimum free energy, $F_S(H)$, of the superconductor. Even without performing a calculation, it is obvious that the minimum free energy will correspond to the solution $\psi_0^{(3)}$, for which $|C_1| = |C_2|$. For example, for large thicknesses ($d \gg \xi$), $\varphi(z)$ is the solution corresponding to surface superconductivity. Then $\psi_0^{(1)}$ and $\psi_0^{(2)}$ describe the formation of superconductivity near only one of the surfaces of the film, and $\psi_0^{(3)}$ near both surfaces. Since the formation of superconducting correlations leads to a decrease in

[7)]In the ultraviolet region, for frequencies satisfying the condition $\omega > \omega_0$ (ω_0 is the plasma frequency), the thickness of a film which is transparent to light increases substantially.

the free energy, it is clear that in the second case this decrease will be approximately twice as great (for $d \gg \xi$).

We note that the condition $A + B \geq 0$ determines the range of values of κ and d for which a phase transition of the second kind occurs. The value of $A + B$ is necessarily positive for sufficiently large κ or small d. Actually, in Eqs. (A.5) and (A.6), $\beta \sim \kappa^2$, and the negative terms do not contain β, i.e., are independent of κ.

2. For the proof of relation (11) (which was also used in the deduction of Eq. (A.4)), we designate by k_0 that value of k for which $H_0(k)$ takes its maximum value $H_0(k_0)$, and by $\varphi_0(z)$ that function $\varphi(z)$ which is the solution of Eq. (3) with $k = k_0$ and $H_0 = H_0(k_0)$. If k is increased by an infinitesimal amount δk the corresponding increment in $\varphi(z)$ satisfies

$$-\frac{d^2}{dz^2}\delta\varphi + \left(k_0 - \frac{eH_0(k_0)}{\hbar c}z\right)^2\delta\varphi - \frac{2m\alpha}{\hbar^2}\delta\varphi = -2\left(k_0 - \frac{eH_0(k_0)}{\hbar c}z\right)\delta k\varphi_0(z). \quad (A.10)$$

Here we have used the fact that k_0 is the point at which $H_0(k)$ is a maximum, so that a change in $H_0(k)$ is of second order in δk and can be disregarded. The solvability condition for Eq. (A.10), which consists in the orthogonality of the right side of (A.10) to the solution of the homogeneous equation for $\varphi_0(z)$, gives

$$\int_{-d}^{d}\left(k_0 - \frac{eH_0(k_0)}{\hbar c}z\right)\varphi_0^2(z)\,dz = 0, \quad (A.11)$$

which agrees with Eq. (11).

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