THE STRUCTURE OF TWO-DIMENSIONAL FLOW AROUND BODIES IN DISPERSIVE MEDIA

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Submitted to JETP editor January 6, 1967

J. Exptl. Theoret. Phys. (U.S.S.R.) 52, 1657—1672 (June, 1967)

Two-dimensional flow around a thin body in a weakly dispersive medium is considered. The main equations describing the flow can be reduced to the Korteveg—de Vries equation. A similarity law is established. The solution is investigated in the linear approximation. Solutions of the Korteveg—de Vries equation are found which describe the flow for zero angles of attack and for values of the similarity parameter σ smaller than some critical number σ_0 ($\sigma_0 \sim 3$). For $\sigma \ll 1$ the solutions go over to those of the linearized equations. It is shown that for $\sigma < \sigma_0$ the structure of the flow remains qualitatively similar to that which is derived in the linear approximation. Solitons (solitary stationary waves) arise in the flow when $\sigma > \sigma_0$. Their number increases with increase of σ.

1. INTRODUCTION

THE flow around bodies placed in media in which the phase velocity of the waves does not coincide with the group velocity has a great number of distinguishing features. In spite of the fact that problems of this type are outwardly different for different dispersive media (motion of a body in a plasma, on the surface of water, etc.), certain common laws exist in all these cases and are determined by a combination of nonlinear and dispersion effects.

It is natural to begin an investigation of the qualitative features of such phenomena with problems in which the nonlinear and dispersive terms in the fundamental equations are sufficiently small. The present paper is devoted to two-dimensional stationary flow around bodies under such conditions. It is related with respect to its results and method to some earlier papers[1,2].

As in[1], we shall assume that the dispersion equation can be represented in the form of a series in odd powers of k:

$$\omega = c_0 k (1 \pm \delta^2 k^2 + \ldots), \qquad (1.1)$$

(c_0 is the phase velocity as k → 0 and δ is the so-called "dispersion length") in which it is sufficient to retain only the terms that have been written out. To this end it is necessary first that the characteristic dimensions of the body λ be large compared with the dispersion length. Certain additional limitations on the region of applicability of the obtained results are given in Sec. 5 of this paper. Another

small parameter we shall use is the quantity ν, which characterizes the relative deviation from the equilibrium state of the medium (for example, $(n - n_0)/n_0$, where n is the plasma density, or $(h - h_0)/h_0$, where h is the height of the liquid in the case of gravitational-capillary surface waves, etc.). Thus, we assume henceforth that

$$\varepsilon = (\delta/\lambda)^2 \ll 1, \quad \nu \ll 1, \qquad (1.2)$$

and we shall henceforth retain in the equations the nonlinear and dispersion terms only in the lowest order in ϵ and ν (terms of order $\epsilon \nu^2$ will be neglected here).

We now consider several concrete examples.

A. Two-dimensional plasma motions transverse to a magnetic field. Let the magnetic field be parallel to the z axis, and let the velocities be parallel to the xy plane and let all the quantities be dependent only on xy. It is assumed also that $H^2/8\pi \gg p$ and that the characteristic lengths exceed the Debye radius. The latter assumption leads to the quasi-neutrality condition $n_e \approx n_i = n$. Eliminating the electric field E from the equations*

$$m_{i,e} d\mathbf{v}_{i,e}/dt = \pm e\mathbf{E} \pm (e/c)[\mathbf{v}_{ie}\mathbf{H}], \quad \partial \mathbf{H}/\partial t = -c \operatorname{rot} \mathbf{E}$$

and using the quasineutrality condition, we obtain the fundamental equations for two-dimensional motion of the plasma across a magnetic field H in the form

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*$[\mathbf{v}_{ie}\mathbf{H}] \equiv \mathbf{v}_{ie} \times \mathbf{H}$.

$$dV / dt \equiv \partial V / \partial t + (V \nabla) V = - (8 \pi m n)^{-1} \nabla H^2, \quad (1.3)$$

$$\frac{\partial H}{\partial t} = - \operatorname{div}(HV) + \frac{m_e c}{e} \operatorname{rot} \frac{d v_e}{dt} - \frac{m_i c}{e} \operatorname{rot} \frac{d V}{dt}, \quad (1.4)$$

$$\frac{\partial H}{\partial y} = \frac{4 \pi n e}{c} (V_x - v_{ex}), \quad \frac{\partial H}{\partial x} = - \frac{4 \pi n e}{c} (V_y - v_{ey}), \quad (1.5)$$

$$\partial n / \partial t + \operatorname{div} n V = 0, \quad (1.6)$$

where we have introduced the plasma "mass" velocity

$$V = (m_i v_i + m_e v_e)/m, \quad m = m_i + m_e$$

and neglected terms of order m_e/m_i compared with unity.

The last two terms in (1.4) are "dispersive" (they lead to a dependence of the phase velocity of the linear waves on k). If these are neglected, then Eqs. (1.3)—(1.6) are transformed into the magneto-hydrodynamics equations, and from (1.4) and (1.6) we get the condition for the "freezing-in" of the magnetic field: $n/n_0 = H/H_0$, where n_0 and H_0 are the unperturbed values of the density and of the field (at $x = - \infty$). Inasmuch as Eqs. (1.3) and (1.6) are written for plasma motion transverse to the magnetic field, they coincide formally with the gasdynamics equations with adiabatic exponent $\gamma = 2$ if the dispersion term is discarded. The linear perturbations propagate in this case at a speed $c_A = (4 \pi n_0 m)^{-1/2} H_0$.

If we take into account the dispersive terms, then no freezing-in takes place, but if these terms are sufficiently small, then deviations from the freezing-in can also be regarded as small (they can be readily obtained by successive approximations). With this, it can be readily verified that the last term in (1.4) is smaller than the principal term by a factor $\nu \epsilon^4 (m_i/m_e)^{1/2}$, and we assume that it can be neglected. Neglecting also terms of order $\nu^2 \epsilon$ and going over to stationary flow in a reference frame in which the body placed in the stream is at rest, we obtain the fundamental equations in the form

$$(V \nabla) V (x, y) = - (c_A{}^2 / 2 H_0 H) \nabla H^2, \quad (1.7)$$

$$\nabla (HV) = 2 \delta^2 V_0 \Delta \partial H / \partial x, \quad (1.8)$$

$$V \to V_0, \quad H \to H_0 \quad (x \to - \infty), \quad (1.9)$$

$$\delta^2 = c^2 / 2 \omega_{0e}{}^2, \quad c_A{}^2 = H_0{}^2 / 4 \pi n m \quad (1.10)$$

(the quantity δ is defined such as to coincide with the dispersion length in the linear dispersion relation (1.1); with this, $c_0 = c_A$).

It follows from (1.7) and the conditions (1.9) that

the flow under consideration is potential, i.e.,

$$V = \nabla \varphi + V_0, \quad \varphi(- \infty) = 0. \quad (1.11)$$

We then obtain for φ, at the assumed degree of accuracy, the equation

$$\varphi_{yy} - (M^2 - 1) \varphi_{xx} - (M/c_A)(3 \varphi_x \varphi_{xx} + 2 \varphi_y \varphi_{xy} + \varphi_x \varphi_{yy})$$

$$+ 2 \delta^2 M^2 \Delta \varphi_{xx} = 0, \quad (1.12)$$

where $M = V_0/c_A$ is the "Mach number." We note that if we put $\delta = 0$ in (1.12), we obtain the well-known expression for the velocity potential of stationary flow of a compressible gas with adiabatic exponent $\gamma = 2$, accurate to second-order terms inclusive (cf. Eq. (106.3) of[3], where they put $\varphi \to \varphi + V_0 x$).

B. Nonlinear ion-sound waves in a plasma without a magnetic field ($T_e \gg T_i$). The dispersion equation again takes the form (1.1), where

$$c_0 = (T/m_i)^{1/2}, \quad \delta^2 = D^2/2 = T/8 \pi e^2 n_0, \quad T = T_e, \quad (1.13)$$

i.e., the dispersion length is determined by the Debye radius D. Assuming for simplicity that $T_i = 0$, we can write the equations of motion for the ions in the form

$$\partial v / \partial t + (v \nabla) v = - (e/m_i) \nabla \Phi, \quad (1.14)$$

$$\partial n / \partial t + \nabla (n v) = 0, \quad (1.15)$$

$$\Delta \Phi = 4 \pi e [n_0 \exp (e \Phi / T) - n], \quad (1.16)$$

where v is the velocity, n the density of the ions, $n_0 = n(- \infty)$. For sufficiently long waves, the left side of (1.16) will be smaller by a factor $\epsilon = (D/\lambda)^2$ than any of the terms on the right side. We can therefore solve (1.16) with respect to Φ by successive approximations. Confining ourselves to terms of order ϵ, we get

$$\exp (e \Phi / T) \approx n / n_0 + (T / 4 \pi e^2 n_0) \Delta \ln (n / n_0) \quad (1.17)$$

Neglecting terms of order $\nu^2 \epsilon$, we get

$$e \nabla \Phi = T \nabla \left[\exp \left(\frac{e \Phi}{T} \right) \right] \exp \left(- \frac{e \Phi}{T} \right) \approx \frac{T}{n} \nabla n + \frac{T}{n_0} D^2 \nabla \Delta n.$$
$$(1.18)$$

Substituting this in (1.14), we have

$$\partial v / \partial t + (v \nabla) v = - (c_0{}^2 / n) \nabla n - 2 c_0{}^2 \delta^2 \nabla \Delta n / n_0, \quad (1.19)$$

where c_0 and δ are defined in (1.13). Equations (1.19) and (1.15) constitute a complete system of equations in the approximation under consideration. The last term in (1.19) is dispersive. If we omit it, we obtain the hydrodynamics equations with adia-

batic exponent $\gamma = 1$, which describe in the linear approximation ion sound without dispersion.

From (1.19) with the condition $n(-\infty) = \text{const} = n_0$ it follows that the flow is potential. Going over to a reference frame in which the body is at rest, and putting $\mathbf{v} = \mathbf{V}_0 + \nabla\varphi$, we obtain for planar flow $(M = V_0/c_0)$:

$$\varphi_{yy} - (M^2 - 1)\varphi_{xx} - (M/c_0)[(3 - M^2)\varphi_x\varphi_{xx} \\ + \varphi_x\varphi_{yy} + 2\varphi_y\varphi_{xy}] + 2M^2\delta^2\Delta\varphi_{xx} = 0 \qquad (1.20)$$

(where we have again confined ourselves only to the principal nonlinear and dispersive terms, of order ν^2 and $\epsilon\nu$ respectively).

C. Let us consider, finally, the last example—gravitational waves on the surface of a liquid. The depth of the unperturbed liquid is here the length parameter defining the dispersion. When $\epsilon = (h_0/\lambda)^2 \ll 1$ the dispersion equation again takes the form (1.1), where

$$\delta^2 = h_0^2/6, \quad c_0 = \sqrt{gh_0}. \qquad (1.21)$$

Expanding the exact equations for the surface waves (see, for example,[4]) in powers of $\nu = (h - h_0)/h_0$ and $\epsilon = (h/\lambda)^2$, and neglecting terms of order ν^3, ϵ^2, and $\nu^2\epsilon$, we get

$$\partial\mathbf{v}/\partial t + (\mathbf{v}\nabla)\mathbf{v} + \nabla(p_0/\rho + gh) = 0, \qquad (1.22)$$

$$\partial h/\partial t + \text{div } h\mathbf{v} + (h_0^3/3)\Delta \text{ div } \mathbf{v} = 0, \qquad (1.23)$$

where p_0 is the pressure on the surface of the liquid, ρ is the density (the liquid is incompressible), and $\mathbf{v}(x, y)$ is the horizontal component of the velocity (parallel to the xy plane). We shall henceforth assume that $p_0 = \text{const}$, so that $\nabla p_0/\rho = 0$. The last term in (1.23) is the dispersive one and leads to the second term of (1.1). If we neglect this term, we obtain the equations for shallow water, which, as is well known, are equivalent to two-dimensional hydrodynamics with $\infty = 2$[3,4]. If $h(-\infty) = h_0$, the flow is potential. Going over to a reference frame in which the body is at rest, and the liquid has at infinity a velocity $\mathbf{V}_0$, and putting $\mathbf{V} = \mathbf{V}_0 + \nabla\varphi$, we obtain, at the same accuracy as in the preceding cases,

$$\varphi_{yy} - (M^2 - 1)\varphi_{xx} - (M/c_0)(3\varphi_x\varphi_{xx} + 2\varphi_y\varphi_{xy} \\ + \varphi_x\varphi_{yy}) + 2\delta^2\Delta^2\varphi = 0. \qquad (1.24)$$

2. FUNDAMENTAL EQUATIONS

Thus, by considering the problem of stationary two-dimensional flow around a body in three different dispersive media, we obtain, in the lowest approximation in the nonlinearity and in the dispersion, the very similar equations (1.12), (1.20), and

(1.24) for the velocity potential. It is easy to verify that they can be represented at the assumed degree of accuracy, in the unified form.

$$\varphi_{yy} - (M^2 - 1)\varphi_{xx} - (M/c_0)\{[2 + M^2(\gamma - 1)]\varphi_x\varphi_{xx} \\ + 2\varphi_y\varphi_{xy}\} + 2SM^4\delta^2\varphi_{xxxx} = 0, \qquad (2.1)$$

where $M = V_0/c_0$ is the Mach number, c_0 the phase velocity of the waves in the linear approximation as $k \to 0$, δ the dispersion length, $S = \pm 1$ depending on the sign used in the dispersion equation (1.1), and γ is the "adiabatic exponent" of the corresponding hydrodynamics. In deriving (2.1) from the aforementioned Eqs. (1.12), (1.20), and (1.24) we made use of the fact that at the assumed accuracy it is possible to replace φ_{yy} in the nonlinear and dispersive terms by the expression that follows from the linearized equation with $\delta = 0$, namely $\varphi_{yy} \approx (M^2 - 1)\varphi_{xx}$. Equation (2.1) with $\delta = 0$ coincides, accurate to nonlinear terms of second order inclusive, with the equation for the velocity potential in two-dimensional flow around a body (see[3], Eq. (106.3)).

We shall consider in this paper only "supersonic" flow, with $M > 1$. To be able to neglect the nonlinear terms of second and higher orders of smallness, it is necessary to stipulate that the body in the stream be sufficiently thin, namely

$$l/b \gg M, \qquad (2.2)$$

where b is the effective thickness and l is the length of the body. In addition, the dispersion length δ must also be small compared with b. The angle of attack α must likewise be small.

We shall henceforth consider Eq. (2.1) subject to a boundary condition wherein the normal velocity component vanishes on the surface of the body, i.e.,

$$[(\varphi_x + V_0)(b/l)f_\pm'(x/l) - \varphi_y]_{y=bf_\pm(x/l)} = 0, \qquad (2.3)$$

where $y = bf_\pm(x/l)$ are the equations of the upper and lower lines of the profile of the body, respectively (Fig. 1).

If there were no dispersion, i.e., $\delta = 0$, then, for a sufficiently thin body the flows in the regions $y > 0$ and $y < 0$ would have the form of simple waves (accurate to terms of third order in φ[3]), i.e., we would have $\varphi_y = F_\pm(u)$, where
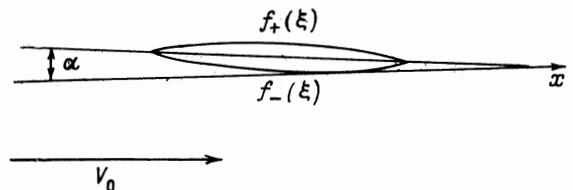

FIG. 1

$$u = \varphi_x, \qquad (2.4)$$

and $F_{\pm}(u)$ are certain functions which have different forms in the upper and lower half-planes. These functions satisfy the same system of equations

$$[(M^2-1)+au]\,\partial u/\partial x - [F'(u)-2(M/c_0)F(u)]\,\partial u/\partial y = 0, \qquad (2.5a)$$

$$F'(u)\,\partial u/\partial x - \partial u/\partial y = 0,$$
$$a = (M/c_0)[2 + M^2(\gamma-1)], \qquad (2.5b)$$

the first of which follows from the relation $\varphi_{yy} = F'(u)\,\partial u/\partial y$ and Eq. (2.1) in which we put $\delta = 0$, while the second is obtained from $\varphi_{xy} = \varphi_{yx}$ and (2.4). In order for this system to have nontrivial solutions for $\partial u/\partial x$ and $\partial u/\partial y$, its determinant must vanish, from which we get the following equation for $F(u)$:

$$F'^2 - 2(M/c_0)F'F - (au + M^2 - 1) = 0.$$

A solution of this equation, accurate to terms of second order in u inclusive, is

$$F_{\pm}(u) = \mp\sqrt{M^2-1}\,u \mp u^2 M^3(\gamma+1)\,/\,4\sqrt{M^2-1}\,c_0, \quad (2.6)$$

where the upper and lower signs are taken for the half-planes $y > 0$ and $y < 0$ respectively. Substitution of (2.6) in (2.5b) yields the equation

$$u_y \pm \sqrt{M^2-1}\,u_x \pm uu_x M^3(\gamma+1)/2\sqrt{M^2-1}\,c_0 = 0, \quad (2.7)$$

the solution of which describes the aforementioned simple waves.

Let us consider now Eq. (2.1) with $\delta \neq 0$. Owing to the dispersive term, the quantity φ_y will no longer be a function of $u = \varphi_x$ only. Instead of this, we must put

$$\varphi_y = F(u) + \varepsilon\psi(x,y), \qquad (2.8)$$

where $\varepsilon = (\delta/b)^2 \ll 1$ and $\psi(x, y)$ is a certain function of order of unity; with this, $F(u)$, as a zeroth-order term relative to ε, is determined as before by formula (2.6). To determine ψ we substitute (2.8) in (2.1) and also in the relation $\varphi_{yx} = \varphi_{xy}$. We then obtain a system of equations for u_x and u_y,

$$(M^2 - 1 + au)u_x - [F_{\pm}'(u) - 2(M/c_0)F_{\pm}(u)]u_y$$
$$= 2SM^4\delta^2 u_{xxx} \mp \varepsilon\sqrt{M^2-1}\,\psi_x, \qquad (2.9)$$

$$F_{\pm}'(u)u_x - u_y = -\varepsilon\psi_x, \qquad (2.10)$$

which differs from (2.5) only in that the right sides do not vanish. Since its determinant is equal to zero, it is not compatible for arbitrary right sides. Substitution of (2.6) in (2.9) and elementary manipulations yield a compatibility condition in the form

$$\varepsilon\psi_x = \pm(2SM^4\delta^2/\sqrt{M^2-1})u_{xxx} \qquad (2.11)$$

(in the derivation of (2.11) it is necessary to con-

fine oneself to terms of lowest order in u and ϵ). Substituting (2.11) in (2.10) we obtain for u the equation

$$u_y \pm \sqrt{M^2-1}\,u_x \pm [(\gamma+1)M^3/2\sqrt{M^2-1}\,c_0]\,uu_x$$
$$\mp (S\delta^2 M^4/\sqrt{M^2-1})u_{xxx} = 0, \qquad (2.12)$$

where the sign preceding the radical coincides with the sign of y.

The boundary condition for this equation should be taken to be (2.3) in which φ_y is replaced by (2.8), and F(u) and ψ are determined respectively by expressions (2.6) and (2.11), i.e.,

$$[f_{\pm}'(x/l)\,(u + V_0)\,b/l \pm \sqrt{M^2-1}\,u \pm u^2 M^3(\gamma+1)/4c_0\sqrt{M^2-1}$$
$$\mp 2SM^4\delta^2/\sqrt{M^2-1}\,u_{xx}]_{y=bf_{\pm}(x/l)} = 0. \qquad (2.13)$$

Since we shall henceforth be interested only in effects due to nonlinear terms in the equations, we replace (2.13) by the approximate boundary condition

$$u = \mp V_0 b/\sqrt{M^2-1}\,lf_{\pm}'(x/l) \quad (y = \pm 0), \qquad (2.14)$$

which is obtained if one omits from (2.13) the terms of second and higher order of smallness relative to the quantity $f_{\pm}'(x)$, which is small for a sufficiently thin body.[1]

We now change over to new (dimensionless) variables

$$\xi = (\pm\sqrt{M^2-1}\,y - x)/l, \quad \tau = y(\gamma+1)M^4 b/2(M^2-1)l^2,$$
$$\eta = \mp u\sqrt{M^2-1}\,l/bMc_0. \qquad (2.15)$$

Then the fundamental equation (2.12) and the boundary conditions take the form

$$\eta_\tau + \eta\eta_\xi + \mu\eta_{\xi\xi\xi} = 0, \qquad (2.16)$$

$$\eta(\tau = 0, \xi) = \chi_{\pm}(\xi), \quad (\chi_{\pm}(\xi) = f_{\pm}'(-\xi)), \qquad (2.17)$$

$$\eta(\tau, \xi = \infty) = 0, \qquad (2.18)$$

$$\mu = \pm 2S\delta^2\sqrt{M^2-1}/bl(\gamma+1). \qquad (2.19)$$

Equation (2.16) coincides with the Korteveg—de Vries equation, which has already been investigated in a number of papers[1,2,5-7].

By the same token we have obtained the following similarity law: All flows around similar contours (i.e., contours defined by identical dimensionless

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[1] The corrections to (2.14) can be obtained by solving (2.13) by successive approximations and regarding (2.14) as the first approximation. With this, it is sufficient to use the first-approximation calculation of the solution increment with the second approximation in the boundary condition.

functions $f_\pm(\xi)$) and having identical numbers μ are similar.

We note also that the similarity parameter μ is connected with the quantity σ introduced in[1] by the relation

$$\sigma = |\mu|^{-1/2}. \qquad (2.20)$$

The quantity σ defines the "degree of nonlinearity" of the problem: the larger σ, the larger the nonlinear effects are compared with the dispersive effects, and vice-versa.

3. LINEAR APPROXIMATION

We consider first the solution of (2.16) under conditions (2.17) and (2.18) in the linear approximation. Discarding the nonlinear term in (2.16), we obtain (see, for example,[2]):

$$\eta(\tau, \xi) = \frac{1}{\sqrt{\pi}\,(3\mu\tau)^{1/3}} \int_{-\infty}^{\infty} \Phi\left(\frac{\xi - \xi'}{(3\mu\tau)^{1/3}}\right)\chi(\xi')\,d\xi', \qquad (3.1)$$

where

$$\Phi(z) = \pi^{-1} \int_{0}^{\infty} \cos(u^3/3 + uz)\,du$$

is the Airy function, which attenuates exponentially as $z \to \infty$.

When $\mu\tau \to 0$ we get $\pi^{-1/2}(3\mu\tau)^{-1/3}\Phi[(3\mu\tau)^{-1/3}\xi] \to \delta(\xi)$ and (3.1) goes over into the well-known relation for the linear approximation in ordinary gas dynamics (see[3], Sec. 116).

Of greatest interest to us is the limiting case of sufficiently large τ and ξ. To obtain the asymptotic relations it is convenient to proceed as follows: Using the fact that $\chi(\xi')$ vanishes when ξ' is sufficiently large, we expand $\Phi(z - z')$ in (3.1) in powers of z'. Then

$$\eta = \pi^{-1/2} \sum_{m=0}^{\infty} \frac{(-1)^m p_m}{m!}(3\mu\tau)^{-(m+1)/3}\Phi^{(m)}(z), \qquad (3.2)$$

where

$$z = (3\mu\tau)^{-1/3}\xi, \quad p_m = \int_{-\infty}^{\infty} \xi^m \chi(\xi)\,d\xi. \qquad (3.3)$$

Using the asymptotic form of the Airy function, we obtain for its derivative when $|z| \gg 1$

$$(-1)^m \Phi^{(m)}(z)$$

$$\approx \begin{cases} \dfrac{1}{2} z^{(2m-1)/4} \exp\left(-\dfrac{2}{3}z^{3/2}\right) & (z > 0) \\[2mm] (-1)^m |z|^{(2m-1)/4} \cos\left[\dfrac{2}{3}|z|^{3/2} + \dfrac{\pi}{2}\left(m - \dfrac{1}{2}\right)\right] & (z < 0). \end{cases}$$

$$(3.4)$$

Substitution of (3.4) in (3.2) leads to a certain asymptotic expansion for $\eta(\tau, \xi)$, which, say for $z < 0$, takes the form ($|z| \gg 1$):

$$\eta \approx \pi^{-1/2} \sum_{m=0}^{\infty} \frac{p_m}{m!}(3\mu\tau)^{-(m+1)/3} z^{(2m-1)/4}$$

$$\times \cos\left[\frac{2}{3}|z|^{3/2} + \frac{\pi}{2}\left(m - \frac{1}{2}\right)\right]. \qquad (3.5)$$

This series can be summed by introducing the Fourier component $\tilde{\chi}(k)$ of the function $\chi(\xi)$ and expanding it in powers of k:

$$\tilde{\chi}(k) = \int_{-\infty}^{\infty} \chi(\xi)\exp(-ik\xi)\,d\xi = \sum_{m=0}^{\infty} (-ik)^m p_m/m!. \qquad (3.6)$$

Comparing (3.5) and (3.6), we can easily verify that when $z < 0$ and $|z| \gg 1$ we get

$$\eta \approx |3\pi^2\mu\tau\xi|^{-1/4} \mathrm{Re}\left\{\tilde{\chi}\left(-\left|\frac{\xi}{3\mu\tau}\right|^{1/2}\right)\exp i\left(\frac{2}{3}z^{3/2} - \frac{\pi}{4}\right)\right\}. \qquad (3.7)$$

Similarly, for $z > 0$ we get

$$\eta \approx \frac{1}{2}(3\pi^2\mu\tau\xi)^{-1/4}\tilde{\chi}[i(\xi/3\mu\tau)^{1/2}]\exp\left(-\frac{2}{3}z^{3/2}\right). \qquad (3.8)$$

(It is assumed here that the Fourier component $\tilde{\chi}(k)$ can be analytically continued in the complex plane, which is certainly the case if $\chi(\xi)$ differs from zero only in a finite interval of ξ).

We see from (3.7) and (3.8) that the function $\eta(\tau, \xi)$ at fixed value of τ is a wave packet which spreads with increasing τ. The width of the packet is determined by the size of the interval Δk in which the function $\tilde{\chi}(k)$ is essentially different from zero, and by virtue of (3.6) it is inverse to the width of the function $\chi(\xi)$. Thus, $\Delta k \sim 1$, and the effective width of the packet is of the order of $|\mu\tau|$.[2] The wavelengths in the packet increase with increasing τ like $(3\mu\tau)^{1/3}$.

The criterion for the accuracy of the linear approximation is

$$s^2 = 2a/3\mu k^2 \ll 1, \qquad (3.9)$$

where $k(\tau, \xi)$ and $a(\tau, \xi)$ are the "local" wave number and the amplitude of the wave packet (3.7). The quantity s^2, the meaning of which was discussed in detail in[1] (see formula (5.12) of[1] and (3.9)), is equal in order of magnitude to the ratio of the nonlinear term in the Korteveg—de Vries equation to the dispersive term. It follows from (3.7) that

$$a \sim |3\mu\tau\xi|^{-1/4}|\tilde{\chi}(k)|, \quad k = |\xi/3\mu\tau|^{1/2}. \qquad (3.10)$$

Substitution of these expressions in (3.9) yields

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[2])It follows therefore that in order for expression (3.7) to describe a sufficiently large part of the packet, it is necessary that the values $\xi \approx 3\mu\tau$ lie within the region of applicability of (3.7). This yields $(3\mu\tau)^{2/3} \gg 1$.

$$s^2 \sim |\mu^{-1}\tau^3\xi^{-5}|^{1/4}\widetilde{\chi}(|\xi/3\mu\tau|^{1/2}). \qquad (3.11)$$

Let us consider now the "motion" of a point with a constant oscillation phase, for example, the crest of a wave in the packet. The equation of motion has, according to (3.7), the form z = const or ξ = const $(\mu\tau)^{1/3}$. Substituting this in (3.11) we get the dependence of s^2 on the "time" τ for a point with fixed phase:

$$s^2(\tau) \sim \mu^{-1}(\mu\tau)^{1/3}|\chi(k)|, \qquad (3.12)$$

$$k = \text{const}\,(\mu\tau)^{-1/3}. \qquad (3.13)$$

Substituting (3.13) in (3.6), and the latter in (3.12), we obtain the following asymptotic expression for $s(\tau)$ when $|\tau| \gg 1$

$$s^2(\tau) \sim p_r\mu^{-1}(\mu\tau)^{(1-r)/3}, \qquad (3.14)$$

where p_r is the first nonzero moment of the function $\chi(\xi)$ (i.e., $p_n = 0$ (n < r), $p_r \neq 0$). In particular, if

$$p_0 = \int_{-\infty}^{\infty}\eta(0,\xi)d\xi \neq 0, \qquad (3.15)$$

then $s \rightarrow \infty$ as $\tau \rightarrow \infty$, i.e., the nonlinear effects become significant at sufficiently large τ, no matter how small the initial value of η.

Starting from the results of numerical integration of the Korteweg—de Vries equation[1], we can propose that these nonlinear effects should become manifest in the formation of solitons—solitary stationary waves—at sufficiently large values of τ, even if the similarity parameter σ (2.20) is smaller than its critical value σ_c introduced in[1] (in the latter case there should be produced, besides the solitons, also a significant part which does not break up into solitons). It must be borne in mind, however, that at very small values of σ the nonlinear effects set in at such large values of τ, that the wave packet constituting the solution has already spread out sufficiently, so that the most essential part of the process can be regarded in the linear approximation.

As applied to flow around a thin body, relation (3.15) holds true in the case when the "angle of attack" differs from zero, as can be readily verified by substituting (2.17) in (3.15). In this case, apparently, among the waves produced behind the body, there will always be present waves with nondecreasing amplitude in the form of solitary waves (solitons). However, at very small attack angles, the solitons should be produced quite far behind the body, and should have a very small amplitude. (The general picture of the flow in the presence and in the absence of solitons is considered in Sec. 5.)

If $p_0 = 0$ and $p_1 \neq 0$ (this case corresponds to zero angle of attack), then it follows from (3.14) that when $\tau \gg 1$ we get $s^2 \sim \mu^{-1}p_1 = \text{const}$. Thus, in this case the quantity s^2 and with it the nonlinear effects will be small for all τ, if they were small when $\tau \lesssim 1$. No solitons appear in this case, in agreement with the results of[1]. This is certainly the case when $p_0 = p_1 = 0$.

4. QUASILINEAR SOLUTIONS OF THE KORTEVEG—DE VRIES EQUATION AND FLOW WITH ZERO ANGLE OF ATTACK

Let us consider the flow around a long body at zero attack angle. According to (2.17) and (3.3) we have in this case

$$p_0 = 0, \quad p_1 = \int_{-\infty}^{\infty}\xi\chi_{\pm}(\xi)d\xi = \int_{-\infty}^{\infty}f_{\pm}(\xi)d\xi, \qquad (4.1)$$

where $f_{\pm}(\xi)$ is the equation of the upper or lower part of the contour of the body. From the results of[1] it follows that in this case the qualitative form of the solution of the Korteweg—de Vries equation at large values of τ should be close to the similar solution, the form of which is

$$\eta_0(\xi,\tau) = \mu(3\mu\tau)^{-2/3}\psi_0\{(3\mu\tau)^{-1/3}\xi\}. \qquad (4.2)$$

The function $\psi_0(z)$ was investigated in detail in[2], and satisfies the equation

$$\psi_0'''(z) - z\psi_0' + \psi_0\psi_0' - 2\psi_0 = 0. \qquad (4.3)$$

Starting from this, we shall seek the solution of the Korteweg—de Vries equation in the form

$$\eta(\xi,\tau) \equiv F(z,1/\xi) = \xi^n\sum_{m=0}^{\infty}f_m(z)\xi^{-m}$$

$$= \mu(3\mu\tau)^{n/3}\sum_{m=0}^{\infty}\psi_m(z)(3\mu\tau)^{-m/3}, \qquad (4.5)$$

where $z = (3\mu\tau)^{-4/3}\xi$, and n and $\psi_m(z) \equiv z^{n-m}f_m(z)$ are unknown quantities.

Substituting (4.5) in (2.16) and stipulating that all terms of this series satisfy the boundary condition (2.18) (i.e., that they vanish as $z \rightarrow \infty$), we get n = −2 and that the $\psi_k(z)$ must satisfy the equations

$$\psi_k''' - z\psi_k' - (k+2)\psi_k + \sum_{p=0}^{k}\psi_p\psi_{k-p}' = 0 \qquad (4.6)$$

and the following conditions and infinity

$$\psi_k(\infty) = 0, \quad k = 0,1,2,\ldots \qquad (4.7)$$

This system of recurrence equations determines any of the functions $\psi_k(z)$ in terms of the first k − 1 functions.

When k = 0 we get from (4.6) just Eq. (4.3), i.e., the first term of the expansion (4.5) is the self-similar solution (inasmuch as n = -2). The remaining terms of (4.5) should supplement the self-similar solution in such a way that the entire sum becomes the complete solution of the Korteveg—de Vries equation. When $\tau \to \infty$ and the values of z are bounded, all the terms of the series (4.5) with k > 0 decrease more rapidly than the first term, so that the entire sum approaches asymptotically the self-similar solution.

It is interesting to note that the system (4.6) has the following particular solution

$$\psi_k = \frac{\lambda^k}{k!} \frac{d^k \psi_0(z)}{dz^k}, \quad k = 1, 2, \ldots, \quad (4.8)$$

where λ is an arbitrary constant, as can be readily verified by direct substitution. The solution (4.8) has a very simple meaning. In order to clarify it, we substitute (4.8) in (4.5) (where n = -2). Then

$$\eta(\xi, \tau) = \mu(3\mu\tau)^{-2/3} \sum_{k=0}^{\infty} (3\mu\tau)^{-k/3} \frac{\lambda^k}{k!} \psi_0^{(k)}[(3\mu\tau)^{-1/3} \xi]$$

$$= \mu(3\mu\tau)^{-2/3} \psi_0[(3\mu\tau)^{-1/3}(\xi + \lambda)].$$

We have obtained again the self-similar solution, but shifted by an amount λ along the ξ axis. The existence of such a solution is perfectly natural, since the Korteveg—de Vries equation (2.16) is invariant to the translations $\xi \to \xi + \lambda$. The foregoing example shows that (4.5) has the character of a "multipole expansion." We now proceed to investigate the asymptotic behavior of the solutions of the system (4.6). For large positive z, owing to the conditions (4.7), we can neglect the nonlinear terms and the system (4.6) can be replaced by

$$\psi_k''' - z\psi_k' - (k+2)\psi_k = 0 \quad (k = 0, 1, 2, \ldots). \quad (4.9)$$

The general solutions of (4.9) are obtained in the Appendix. We shall consider here only those solutions which satisfy the conditions (4.7) and vanish exponentially as z → ∞. As shown in the Appendix, they take the form

$$\psi_k^{(L)}(z) = A_k d^{k+1} \Phi(z)/dz^{k+1}, \quad (4.10)$$

where $\Phi(z)$ is the Airy function and A_k are arbitrary constants. Thus, the asymptotic form of the functions $\psi_k(z)$ at large positive z is

$$\psi_k(z) \approx \frac{A_k}{2} (-1)^k z^{k/2+1/4} \exp(-2/3 z^{3/2}). \quad (4.11)$$

In the region of large negative z, the functions (4.10) oscillate rapidly, and their amplitude increases like $z^{k/2 + 1/4}$, so that in general it is necessary also to take the nonlinear terms into account

in (4.6). However, we can reason here in the same manner as in[2] (Sec. 3), where Eq. (4.3), defining the self-similar solution, was investigated for the function $\psi_0(z)$. Namely, if the constants A_k are sufficiently small, then the nonlinear terms in (4.6) come into play when the functions $\psi_k(z)$ already assume their oscillating asymptotic values. Then, using the method of induction and the results of[2] (Sec. 3), we can easily verify that for sufficiently small A_k and as z → -∞ the principal term in the asymptotic form of $\psi_k(z)$ becomes

$$\psi_k^{(0)}(z) = |z|^{k/2+1/4} \left[\alpha_k \cos \left(\frac{2}{3} |z|^{3/2} + \pi/4 \right) \right.$$

$$\left. + \beta_k \sin \left(\frac{2}{3} |z|^{3/2} + \pi/4 \right) \right], \quad (-z \gg 1), \quad (4.12)$$

where α_k and β_k are constants determined by the initial conditions (the constant terms in the phases have been added for convenience).

Let us compare now the considered solutions with the solution of the linearized equation (3.1). It must be noted here first that the nonlinear solution (4.5) goes over in the limit into (3.1) only for those initial conditions which satisfy relation (4.1), as can be verified by comparing (3.2) with (4.5) (where, as shown above, we must put n = -2). This circumstance is closely related with the result obtained in Sec. 3, namely that if $p_0 \neq 0$, then the nonlinear corrections are small when $\sigma \ll 1$.

If we now assume that $p_0 = 0$ and compare (3.2), in which we have substituted the asymptotic expressions (3.4), with the expansions (4.5) and (4.11) and (4.12), then we find that, for sufficiently small values of the nonlinearity parameter σ defined in (2.20), or accordingly for large μ, the coefficients A_k, α_k, and β_k take the form

$$A_k = \frac{(-1)^k}{\sqrt{\pi} \mu (k+1)!} p_{k+1} + O(\mu^{-1}) \quad (k = 0, 1, 2, \ldots),$$

$$\alpha_{2m} = \frac{(-1)^m p_{2m+1}}{\sqrt{\pi} \mu (2m+1)!} + Q(\mu^{-1}), \quad \alpha_{2m+1} = O(\mu^{-1}), \quad (4.13)$$

$$\beta_{2m+1} = \frac{(-1)^m p_{2m+2}}{\sqrt{\pi} \mu (2m+2)!} + O(\mu^{-1}), \quad \beta_{2m} = O(\mu^{-1}),$$

$$m = 0, 1, 2, \ldots \quad (4.14)$$

Expression (4.12) determines the principal term in the asymptotic form of $\psi_k(z)$ at large negative z. The next terms can be obtained by iteration. Confining ourselves to one iteration, we get

$$\psi_k(z) \approx \psi_k^{(0)}(z) + z^{(k-1)/2}[a_k \sin 4/3 |z|^{3/2}$$

$$+ b_k \cos 4/3 |z|^{3/2} + c_k], \quad (4.15)$$

where $\psi_k^{(0)}(z)$ is defined in (4.12) and

$$a_k = -\frac{1}{16}\sum_{p=0}^{k}(\alpha_{k-p}\,\alpha_p - \beta_{k-p}\,\beta_p),$$

$$b_k = \frac{1}{8}\sum_{p=0}^{k}\alpha_{k-p}\,\beta_p,$$

$$c_k = -\frac{1}{12}\sum_{p=0}^{k}(\alpha_{k-p}\,\alpha_p + \beta_{k-p}\,\beta_p). \qquad (4.16)$$

When k = 0 the expressions (4.15) and (4.16) coincide with the formulas obtained in[2] for the asymptotic form of the self-similar solution (formula (3.10) of that paper contains a misprint, namely, a nonoscillating term proportional to $z^{-1/2}$ is missing).

If we now substitute (4.15) and (4.16) into formula (4.5) (with n = −2), then we obtain after simple manipulations the following asymptotic expression for a solution of the form (4.5) in the region of large negative z:

$$\eta = \left(\frac{k}{3\mu\tau}\right)^{1/2}\{\Psi_1(k)\cos\,(^2/_3|z|^{3/2}+\pi/4) + \Psi_2(k)$$

$$\times\,\sin\,(^2/_3|z|^{3/2}+\pi/4)\} - \frac{1}{12\mu^2\tau k}\{^1/_4[\Psi_1{}^2(k)$$

$$-\,\Psi_2{}^2(k)]\sin\,^4/_3|z|^{3/2} - ^1/_2\Psi_1(k)\,\Psi_2(k)\cos\,^4/_3|z|^{3/2}$$

$$+\,^1/_3[\Psi_1{}^2(k)+\Psi_2{}^2(k)]\}, \qquad (4.17)$$

where

$$k = |\xi/3\mu\tau|^{1/2}, \quad z = (3\mu\tau)^{-1/3}\xi, \qquad (4.18)$$

$$\Psi_1(k) = \mu\sum_{n=0}^{\infty}\alpha_n k^n, \quad \Psi_2(k) = \mu\sum_{n=0}^{\infty}\beta_n k^n. \quad (4.19)$$

In the limiting case $\sigma \ll 1$ or $\mu \gg 1$, the coefficients α_k and β_k are determined by formulas (4.14), and the functions Ψ_1 and Ψ_2 become, in accord with (3.6),

$$\Psi_1 \to \frac{1}{\sqrt{\pi k}}\,\mathrm{Im}\,\tilde{\chi}(k), \quad \Psi_2 \to \frac{1}{\sqrt{\pi k}}\,\mathrm{Re}\,\tilde{\chi}(k), \quad (4.20)$$

where $\tilde{\chi}(k)$ is the Fourier transform of the initial perturbation $\chi(\xi)$.

When $\sigma \gtrsim 1$ relations (4.20) no longer hold and it is impossible to obtain an explicit expression for the functions Ψ_1 and Ψ_2 in terms of the initial perturbations. Nonetheless, formula (4.17) is useful in this case, too, since it contains important information with respect to the general qualitative characteristics of the solution. It follows from it that the solution for $\xi < 0$, $|z| \gg 1$ has the form of sinusoidal waves, the phases of which depend on ξ and τ like $|z|^{3/2}$, and the amplitude factors of the fundamental

harmonic Ψ_1 and Ψ_2 depend on ξ and τ only in terms of the "local" wave number k, which is defined by formula (4.18).[3] Although the functions Ψ_1 and Ψ_2 differ when $\sigma \gtrsim 1$ from those obtained in the linear approximation, the general dependence of the phases and of the amplitudes on ξ and τ is the same as in the linearized solution. We shall therefore call solutions of the type (4.5) quasilinear solutions of the Korteveg—de Vries equation.[4] The nonlinear effects of these solutions consist in the appearance of multiple harmonics and of a certain average perturbation, which are small when $|z| \gg 1$ and are expressed in terms of the amplitudes Ψ_1 and Ψ_2.

It turns out that the quasilinear solutions exist only when $\sigma < \sigma_0$, where σ_0 is a certain critical value of the number σ. This follows from the fact that all the functions $\psi_k(z)$ are determined by Eqs. (4.6) in terms of the function $\psi_0(z)$, which enters in the self-similar solution. The latter, as follows from the results of[2], is not regular for all values of σ. Namely, as shown in[2], the function $\psi_0(z)$, which takes the form (4.11) (with k = 0) for large positive z, has no singularities on the real axis only when $A_0 < 3.6$. If we now use (4.13) for a rough estimate, then we find that $p_1/\mu < 7$, where p_1, in accord with (4.1), is equal to the area of the contour of the body in the stream in dimensionless units. If the characteristic length l and the thickness b of the contour are chosen in (2.15) in such a way that $p_1 = 1$, then we get for the critical σ, in accordance with (2.20),

$$\sigma_0 \sim 3. \qquad (4.21)$$

When $\sigma \gg 3$, quasilinear solutions of the Korteveg—de Vries equation no longer exist. This result agrees with the numerical solutions of the Korteveg—de Vries equation, which show that for sufficiently large σ (exceeding the critical value) a solution is produced in the frontal part of the profile of the solution; for very large σ several solitons are produced, and these are not described by the quasilinear solution. No solitons were observed for $\sigma < 3$.

By way of an illustration we present Figs. 2—3, which show the numerical solutions of the

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[3]The phase is expressed in terms of the "local" wave number in the same way as in a plane wave satisfying the linearized Korteveg—de Vries equation:

$$^2/_3 z^{3/2} = k\xi - \omega\tau \qquad (\omega = -\mu k^3).$$

[4]As in the linear approximation, the asymptotic expressions (4.17) − (4.19) describe a sufficiently large part of the wave packet when $\mu\tau \gg 1$ (see footnote [3]).
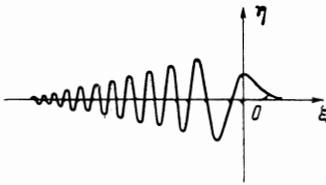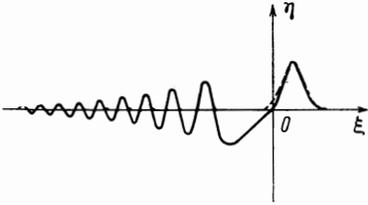
FIG. 2

FIG. 3

Korteveg—de Vries equation (obtained by Yu. A. Berezin) under the initial condition

$$\chi(\xi) = -\frac{\partial}{\partial\xi}\exp(-\xi^2) \qquad (4.22)$$

(this condition is qualitatively close to (2.17) at zero angle of attack) and for fixed sufficiently large τ. Figure 2 shows the form of the solution $\eta(\xi, \tau)$ as a function of ξ for $\sigma = 2$, while Fig. 3 is for $\sigma = 6$. With increasing τ, these curves are altered in such a way that all the crests of the waves, except the frontal wave, move to the left, while the maxima of the frontal waves move to the right (much slower when $\sigma = 2$ than when $\sigma = 6$). The difference between the solutions at $\sigma = 2$ and $\sigma = 6$ lies primarily in the fact that the frontal wave at $\sigma = 6$ has a tendency to become detached from the remaining part of the profile (as $\tau \to \infty$), and during the course of this process its amplitude η_0 ceases to vary. The "experimentally" determined velocity of the motion of the maximum in the frontal wave approaches the velocity V of a soliton having the same amplitude (as is well known, $V = \eta_0/3$), and the profile of the wave coincides quite accurately with the profile of the soliton. For clarity, the dashed line of Fig. 3 shows the profile of the soliton $\eta = \eta_0\mathrm{ch}^{-2}[\eta_0/12\mu)^{1/2}(\xi - \xi_0)]$ with the same amplitude η_0 as the frontal wave, superimposed on the latter. The small discrepancy between them occurs in practice only in the rear part of the profile of the frontal wave; it disappears with increasing τ. All this gives grounds for assuming that when $\sigma = 6$ a soliton is separated from the wave packet representing the solution.

As to the solution for $\sigma = 2$, no tendency was observed here for a stationary form to become established at the frontal wave. The amplitudes of all waves decrease here slowly and the entire packet spreads out.

At sufficiently large σ, the number of solitons exceeds unity. For example, as shown by a numerical solution, not less than two solitons are produced when $\sigma = 40$ (for details see[8]).

5. CONCLUSION

In conclusion, we consider the general form of different characteristics of the flow around a body at zero angle of attack. From the results of Sec. 4 it follows that the structure of the stream will be different for small and large $z = (3\mu\tau)^{-1/3}\xi$. Let us consider these regions in greater detail. We shall assume here that $\mu > 0$ (the transition to negative μ is effected by the substitutions $\eta \to -\eta$, $\xi \to -\xi$, $\tau \to \tau$).

a) $z < 0$, $|z| \gg 1$ (according to formulas (2.15) this is the region within the Mach angle and not too close to the Mach lines). If $\sigma < \sigma_0$ (see (4.21)), then the flow is described by the quasilinear solution of the Korteveg—de Vries equation. In the region under consideration, this solution is given by formulas (4.17)—(4.19). If we neglect multiple harmonics, then the equal-phase lines are determined by the equation z = const or, in x, y coordinates

$$x = \pm\sqrt{M^2 - 1}\,y \pm C\left(\frac{3\delta^2 M^4}{l^3\sqrt{M^2 - 1}}\right)^{1/3}y^{1/3}, \qquad (5.1)$$

where $C = -z$ (the $\pm$ signs are taken for the upper and lower half-planes). In Fig. 4, the curves of this family are tagged II and the Mach lines I. The tangents to the lines (5.1) have a slope that approaches $(M^2 - 1)^{-1/2}$ asymptotically as $y \to \infty$; the value of the parameter C increases in the direction from the Mach line ($z = 0$) to the x axis ($z = -\infty$).

The equation of the lines of constant wave number k = const have in the coordinates x, y the form y = const·x, with k increasing from the Mach line towards the x axis (where $k = \infty$). Thus, the short-wave region is situated far from the Mach lines, whereas the waves that start from the body in the stream are long near the Mach lines.
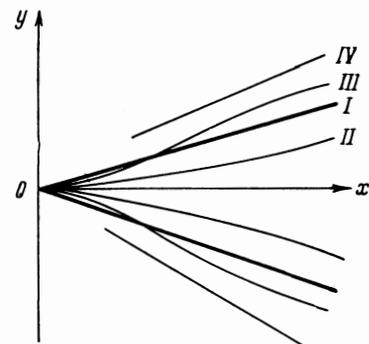

FIG. 4

b) We now proceed to the region of long waves. A qualitative idea of this region can be obtained by considering the points with fixed $z \lesssim 1$ and sufficiently large τ: $(3\mu\tau)^{1/3} \gg 1$. We can then confine ourselves in (4.5) to the first nonvanishing term, which constitutes the self-similar solution (4.2). The behavior of the function $\psi_0(z)$ is qualitatively analogous to $\Phi'(z)$, where $\Phi(z)$ is the Airy function (see Fig. 3 of[2], but we must bear in mind that this figure represents the function $\psi_0(-3^{1/3}z)$).

We denote by z_0 the value of z at which $\psi_0(z)$ has the extreme-right maximum (we shall henceforth call this the first maximum). From the aforementioned Fig. 3 of[2] we see that $z_0 > 0$. With decreasing similarity parameter σ (2.20), the function $\psi_0(z)$ tends to the corresponding solution of the linearized equation, i.e., to $-\Phi'(z)$, while the latter has the first extremum at $z = 0$. Thus, with decreasing σ we get $z_0 \rightarrow +0$. As shown by numerical calculations (see Fig. 3 of[2]), z_0 is nevertheless very small even if $\sigma \rightarrow \sigma_0 \sim 3$, when the quasilinear flows cease to exist. The equation of the line of the first maximum $(3\mu\tau)^{-1/3}\xi = z_0$ in x, y coordinates has the form (5.1), where C is a negative quantity of small absolute value, $C = -z_0$. This curve is designated III in Fig. 4; it goes outside the limits of the Mach angle, and when $y \rightarrow \infty$ its slope decreases and approaches the slope of the Mach line.

c) For sufficiently large σ, exceeding σ_0, solitons appear in the profiles of the solution $\eta(\xi, \tau)$. The peaks of the latter "move" along the lines $\xi = \eta_0\tau/3$ + const, where η_0 is the soliton amplitude. In x,y coordinates the equations of these lines are

$$x = \pm y\sqrt{M^2 - 1}[1 - \eta_0(\gamma + 1)M^4 b/6(M^2 - 1)^{3/2}l] + \text{const},$$

i.e., they constitute straight lines lying outside the Mach angle (Fig. 4, IV). Their slope increases with the amplitude η_0; these amplitudes should not change after the soliton has been formed.

Finally, let us determine more precisely the region of applicability of all our asymptotic expressions.

The corresponding condition is easiest to obtain from Eq. (5.1) by stipulating that the second term in the right side be much smaller than the first. Then, recognizing that $C = -z$, we get[5]

$$|x - \sqrt{M^2 - 1}\,y| \ll \sqrt{M^2 - 1}\,|y|. \qquad (5.2)$$

In addition, the condition $\mu\tau \gg 1$ (see footnote[4]) denotes that

$$y/l \gg (l/\delta)^2 M^4 (M^2 - 1)^{-1/2}. \qquad (5.3)$$

In conclusion, the author is grateful to Yu. A. Berezin for the numerical calculations and to A. V. Gurevich, L. P. Pitaevskiĭ, and R. Z. Sagdeev for useful discussions.

APPENDIX

SOLUTIONS OF EQUATIONS (4.9)

Differentiating (4.9), we get $\psi'_k(z) = \psi_{k+1}(z)$; it follows therefore that $\psi_k(z) = f^{(k+1)}(z)$, where $f(z)$ satisfies the equation

$$f'' - zf = a, \qquad (A.1)$$

and a is an arbitrary constant. The general solution of (A.1) is the sum of the general solution of the homogeneous equation, i.e., the Airy equation, and the particular solution of the inhomogeneous equation. The particular solution of the inhomogeneous equation, as can be readily checked, is

$$f(z) = -a\Phi(z)\int_{z_0}^{z} d\zeta \Phi^{-2}(\zeta)\int_{\zeta}^{\infty}\Phi(\eta)d\eta. \qquad (A.2)$$

When $z \gg 1$ it follows from (A.2) that $f(z) \approx -a/z$; as $z \rightarrow -\infty$, the function $f(z)$ oscillates rapidly, as does $\Phi(z)$. It is easy to verify that if $a \neq 0$, then the solution (4.5) will have at $z \rightarrow \infty$ an asymptotic behavior which differs greatly from the linear one (which is given by (4.1)) even in the case of small values of the similarity parameter σ, when a limiting transition to the linear approximation should take place. It is therefore natural to put $a = 0$. Then the only solution of Eq. (A.1) which attenuates as $z \rightarrow \infty$ is the Airy function $\Phi(z)$, and the corresponding solutions of (4.9) take the form (4.10).

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[5] The condition (5.2) is a reflection of the limited region of applicability of the fundamental equations (2.16).