PAIR PRODUCTION BY A PHOTON AND PHOTON EMISSION BY AN ELECTRON IN THE FIELD OF AN INTENSE ELECTROMAGNETIC WAVE AND IN A CONSTANT FIELD

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The production of an electron-positron pair by a high-energy photon in the field of an intense electromagnetic wave is considered, assuming that a large number of photons from the wave are absorbed in the process. It is observed that there is a significant difference between the differential probabilities for pair production in linearly and circularly polarized waves. This is a result of the much larger angular momentum and relative momentum of the pair in the case of circular polarization. In the limiting situation of a crossed constant field, simple expressions result for the differential probabilities of pair production by a photon and photon emission by an electron. Arguments are presented which allow the use of the same expressions for the case of an arbitrary constant field.

1. INTRODUCTION

THE simplest quantum processes in the field of a plane monochromatic wave were considered in [1-5]. General formulas were obtained there for such processes as the emission of a photon by electrons, pair production by a photon, single-photon annihilation of an electron-positron pair, and the corresponding probabilities were analyzed in limiting situations determined by the parameter[3]1)

$$x = ea/m = Bm/B_0\omega, \tag{1a}$$

characterizing the intensity of the wave (here a and B are the amplitudes of the potential and field strength of the wave, ω is the frequency, and $B_0 = m^2/e$ is the characteristic field strength in quantum electrodynamics). For $x \ll 1$ the expressions for the probabilities become identical with those obtained in perturbation theory, corresponding to the absorption of 1, 2, ... photons from the wave, whereas for $x \gg 1$ the probabilities reduce to those obtained in a crossed constant field.

In the present paper we pay special attention to pair production by a photon in the field of a wave: $\gamma + s\gamma_{\text{wave}} \to e^- + e^+$. This reaction channel opens up when the number s of photons absorbed from the wave becomes larger than the threshold value s_0, which depends on the external parameters:

$$s_0 = -\frac{2m_*^2}{(kl)} = \begin{cases} \dfrac{2x(1+x^2/2)}{\varkappa} \text{— linearly polarized wave} \\ \dfrac{2x(1+x^2)}{\varkappa} \text{— circularly polarized wave.} \end{cases} \tag{1b}$$

Here

$$\varkappa = -\frac{(kl)x}{m^2} = \frac{e\sqrt{(F_{\mu\nu}l_\nu)^2}}{m^3};$$

l_μ, k_μ are the 4-momenta of the incident photon and of the wave, respectively; m_* is the effective mass of the electron. For waves with a frequency much smaller than the electron mass ($\omega \sim 10^{-6}$ m for lasers), the parameter s_0 turns out to be very large for a wide range of incident photon energies, i.e., the process occurs as a result of the absorption of a large number of photons from the wave. This makes it possible to find simple expressions for the differential and total probabilities of pair production for a wide range of the parameter x, including the value $x \sim 1$.2) More precisely, in Secs. 2 and 3 it will be assumed that the parameters x, s_0, or x, $\varkappa$ satisfy the condition

$$s_0 \gg (1+x^2)^{3/2} \quad \text{or} \quad \varkappa \ll x/\sqrt{1+x^2}. \tag{2}$$

1)Units with $\hbar = c = 1$, $e^2/4\pi = 1/137$ are used. Notation: $q_\mu = (\mathbf{q}, iq_0)$, $(kq) = \mathbf{k}\cdot\mathbf{q} - k_0q_0$.

2)For photon emission by electrons one can also derive simple expressions for the probability of emission of sufficiently high-order harmonics, however for $x \sim 1$ their contribution to the total probability will be negligible.

The case $x \gg 1$ and arbitrary κ reduces essentially to the situation of a constant crossed field. In Sec. 4 more detailed data on the differential distributions will be given in the case of pair production by a photon and photon emission by an electron in the case of a crossed field ($F^2_{\mu\nu} = F^*_{\mu\nu} F_{\mu\nu} = 0$).

2. PAIR PRODUCTION BY A PHOTON IN THE FIELD OF A LINEARLY POLARIZED WAVE

The probability for pair production by a photon of momentum l_μ with a polarization parallel or perpendicular to the polarization of the wave is (cf. Eqs. (15), (35) in [3])

$$W_{\parallel,\perp}(x,\varkappa) = \frac{e^2m^2n}{16\pi^2 l_0} \sum_{s>s_0} \int_0^{2\pi} d\varphi \int_1^{s/s_0} \frac{du}{u\sqrt{u(u-1)}} \times \{(1 \mp 1 \pm 2\sigma)A_0^2 + x^2(2u-1\mp 1)(A_1^2 - A_0A_2)\}; \quad (3)$$

where the upper sign corresponds to $W_\parallel$ and the lower one to $W_\perp$. Here n denotes the (numerical) density of the incident photons; $u = (kl)^2/4(kq)(kq')$; k and l are the 4-momenta of the wave and of the incident photon, respectively; q and q$'$ denote the 4-quasimomenta of the electron and positron; φ is the angle between the planes determined by $(\mathbf{k}, \mathbf{q}')$ and $(\mathbf{k}, \mathbf{q})$ in the coordinate frame in which the 3-momenta $\mathbf{k}$ and $\mathbf{l}$ are opposites. For each value of s there is a conservation law $sk + l = q + q'$. The quantity $\sigma = 1 + \tau^2$, where

$$\tau = -\frac{eF_{\mu\nu}{}^* q_\mu' q_\nu}{m^4\varkappa} = \left[\left(1+\frac{x^2}{2}\right)\left(\frac{s}{s_0 u} - 1\right)\right]^{1/2} \sin\varphi. \quad (4)$$

The variables α, and β in the functions $A_n(s, \alpha, \beta)$ are connected with u and φ by means of the relations

$$\alpha = z\cos\varphi, \quad z = \frac{4x^2}{\varkappa}\sqrt{1+\frac{x^2}{2}}\sqrt{u\left(\frac{s}{s_0} - u\right)}, \quad \beta = \frac{x^3 u}{2\varkappa}. \quad (5)$$

The definition and properties of the functions A_n, as well as additional information about the variables used here, can be found in [3].

We note that the probabilities $W_{\parallel,\perp}$ for pair production of scalar particles can be obtained from (3) by removing the term 2u of the second term inside the curly bracket, and by multiplying the right hand side of the resulting expression by $-1/2$. Thus it can be seen that spin effects are quite important in pair production.

In the case $s_0 \gg (1 + x^2)^{3/2}$ the functions $A_n(s, \alpha, \beta)$ can be replaced by their asymptotic expressions, which are obtained from the integral representation

$$A_n(s,\alpha,\beta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\varphi \cos^n\varphi\, e^{f(\varphi)} = \operatorname{Re}\frac{1}{\pi}\int_0^{\pi} d\varphi \cos^n\varphi\, e^{f(\varphi)},$$
$$f(\varphi) = -i(\alpha\sin\varphi - \beta\sin 2\varphi - s\varphi), \quad (6)$$

by the usual method of steepest descent, making use of the second derivative of the function $f(\varphi)$ in the saddle point. In this case

$$A_0 \approx \frac{1}{\pi}\operatorname{Re} e^f \sqrt{\frac{2\pi}{-f''}} = \sqrt{\frac{2}{\pi|f''|}}\, e^{\operatorname{Re} f}\sin o,$$
$$o = \operatorname{Im} f - 1/2 \arg f'', \quad (7)$$

where f and f$''$ are the values of the function $f(\varphi)$ and of its second derivative at the saddle point $\varphi_0 = \psi + i\epsilon$. The position of the latter is determined from the equations

$$\cos\varphi_0 = \frac{\alpha}{8\beta} - i\frac{\sqrt{\sigma}}{x}, \quad \sigma = x^2\left[\frac{s}{4\beta} - \frac{1}{2} - \left(\frac{\alpha}{8\beta}\right)^2\right]. \quad (8)$$

The parameters ψ and ϵ are related to α and β, s via the equations

$$\operatorname{ch}\varepsilon = \frac{1}{2}\left[\left(\frac{1}{2}+\frac{s}{4\beta}+\frac{\alpha}{4\beta}\right)^{1/2} + \left(\frac{1}{2}+\frac{s}{4\beta}-\frac{\alpha}{4\beta}\right)^{1/2}\right],$$
$$\cos\psi = \frac{\alpha/8\beta}{\operatorname{ch}\varepsilon} \quad (9)$$

and define f and f$''$:

$$\operatorname{Re} f = -s\left[\varepsilon - \frac{\operatorname{sh}\varepsilon\operatorname{ch}\varepsilon(1+2\cos^2\psi)}{1+2\cos^2\psi+2\operatorname{sh}^2\varepsilon}\right],$$
$$\operatorname{Im} f = s\left[\psi - \frac{\sin\psi\cos\psi(1+2\operatorname{ch}^2\varepsilon)}{1+2\cos^2\psi+2\operatorname{sh}^2\varepsilon}\right],$$
$$f'' = -\frac{4s\sin\psi\operatorname{sh}\varepsilon}{1+2\cos^2\psi+2\operatorname{sh}^2\varepsilon}(\sin\psi\operatorname{ch}\varepsilon + i\cos\psi\operatorname{sh}\varepsilon). \quad (10)$$

We thus obtain for the functions A_0^2 and $A_1^2 - A_0A_2$ occuring in the expression (3)

$$A_0^2 = \frac{e^{2\operatorname{Re} f}}{\pi|f''|}(1-\cos 2o), \quad (11)$$

$$A_1^2 - A_0A_2 = \frac{2\sigma e^{2\operatorname{Re} f}}{\pi x^2|f''|}. \quad (12)$$

Although the expressions of these functions have been considerably simplified, the complicated dependence of f and f$''$ on u and φ still makes it impossible to carry out the integration over these latter variables in (3). However, it will become clear in the sequel that the most important contribution to the total probability (3) is given by those terms of the sum over s, and those angles φ, for which

$$s - s_0 \lesssim (1+x^2)^{3/2}, \quad (s-s_0)/2s_0 \ll 1,$$
$$\sin^2\varphi \lesssim (1+x^2)^{-1}. \quad (13)$$

At the same time, it is clear from the expressions (5), (9), and (10) that if one disregards the common multiplier s, then the functions f and f$''$ depend on u, s, and s_0 only through the combination δ^2,

$$\delta^2 \equiv \frac{1}{2}\left(\frac{s}{s_0 u} - 1\right) \leqslant \frac{s-s_0}{2s_0}$$

which is smaller than $(s - s_0)/2s_0$ for all values of u in the interval $1 \le u \le s/s_0$, and therefore is effectively a small quantity. Expanding f and f″ in terms of the parameter δ^2, we obtain

$$2\,\mathrm{Re}\,f = -2s\left\{\mathrm{Arth}\,t - \frac{t}{1+t^2} - 2\delta^2 t\left(\frac{t^2}{1+t^2} - \sin^2\varphi\right) + \delta^4\left[\frac{t^3(9+2t^2+t^4)}{2(1+t^2)} - t(3+t^4)\sin^2\varphi + \frac{(1-t^2)^2(1+t^2)}{2t}\sin^4\varphi\right] + \ldots\right\},$$

$$2\,\mathrm{Im}\,f = 2s\left\{\frac{\pi}{2} - 4\delta\frac{\cos\varphi}{\sqrt{1+t^2}} + 2\delta^3\sqrt{1+t^2}\left[\frac{3-t^2}{1+t^2} + \frac{2-t^2}{3}\cos^2\varphi\right]\cos\varphi + \ldots\right\}, \tag{14}$$

$$|f''| = \frac{4st}{1+t^2} + \ldots, \quad \arg f'' = \pi + \delta t\sqrt{1+t^2}\cos\varphi + \ldots,$$

where $t = \tanh\epsilon|_{s=s_0} = (1+x^2)^{-1/2}$, and the dots denote terms of the order δ^2 compared to the smallest terms written out.

Substituting these expansions in the expressions (11) and (12) for the functions A_0^2 and $A_1^2 - A_0A_2$, we note that in the effective domain of s and φ we can retain only the zero order and quadratic terms with respect to δ in the expansion of 2Ref, since the remaining terms are of the order $s\delta^4t^3 \sim \delta^2 \ll 1$. In the expansions of $|f''|$ and arg f″ it is sufficient to retain the first terms. In 2Im f the first two terms suffice only if $s\delta^3 \ll 1$, a condition which is not satisfied if $x^3 \gtrsim \kappa^{-1/2} \gg 1$. Therefore we retain 2Im f unexpanded. The oscillating term cos 2o, which contains 2Im f, leads to rapid oscillations of the distribution with respect to cos φ, with oscillation frequency $\approx 8s\delta(1+t^2)^{-1/2}$, and will be essential only in the case in which the solution with respect to φ is obtained to an accuracy $\Delta\varphi \lesssim (1+t^2)^{1/2}/8s\delta\sin\varphi$. Therefore its contribution to the total probability is negligibly small, with the exception of values of s lying very close to the threshold, when $s\delta \lesssim 1$. As a result we obtain

$$W_{\parallel} = \frac{e^2m^2n}{8\pi^2 l_0}\frac{2+x^2}{s_0\sqrt{1+x^2}}\exp\left\{-2s_0\left(\mathrm{Arsh}\frac{1}{x} - \frac{\sqrt{1+x^2}}{2+x^2}\right)\right\} \times \sum_{s>s_0}\sqrt{\frac{s-s_0}{s_0}}\exp\left\{-2(s-s_0)\left(\mathrm{Arsh}\frac{1}{x} - \frac{1}{\sqrt{1+x^2}}\right)\right\} \times G(x, s-s_0),$$

$$G(x,\Delta) = \exp\left\{-\frac{2\Delta}{\sqrt{1+x^2}(2+x^2)}\right\} \times \int_0^1 d\cos\theta\exp\left\{-\frac{\Delta x^2\sin^2\theta}{\sqrt{1+x^2}(2+x^2)}\right\} I_0\left(\Delta\frac{\sin^2\theta}{\sqrt{1+x^2}}\right), \tag{15}$$

$$W_{\perp} = 2W_{\parallel}.$$

Here $I_0(z)$ is the modified Bessel function, which appears as a result of the integration with respect to φ.

We note that θ and u are related as follows

$$u = \left(1 - \frac{s-s_0}{s}\cos^2\theta\right)^{-1}$$

and θ is the angle between **k** and **q′** in the center-of-mass system. Thus the integrand in (15) is the differential distribution with respect to θ. For $x \ll 1$, when the particles are concentrated near the plane (**k**, **q**), another representation of $G(x, \Delta)$ is useful, namely a representation which is obtained from (15) by replacing θ, φ by the variables θ', φ', where θ' is the angle between **q′** and **k** × **q**, and φ' is the angle between the planes (**k** × **q**, **q′**) and (**k** × **q**, **k**) in the center-of-mass system:

$$G(x,\Delta) = \exp\left\{-\frac{\Delta}{\sqrt{1+x^2}(2+x^2)}\right\}\int_0^1 d\cos\theta' \times \exp\left\{-\frac{\Delta(3+2x^2)\cos^2\theta'}{\sqrt{1+x^2}(2+x^2)}\right\} I_0\left(\frac{\Delta\sin^2\theta'}{\sqrt{1+x^2}(2+x^2)}\right). \tag{16}$$

The function $G(x, \Delta)$ is slowly varying with respect to s, as compared to the exponential function. Therefore the s-distribution is peaked at the threshold and decays with the increase of s according to an exponential law, the width of the distribution being

$$(s-s_0)_{\mathrm{eff}} \sim \frac{1}{2}\left(\mathrm{Arsh}\frac{1}{x} - \frac{1}{\sqrt{1+x^2}}\right)^{-1} = \begin{cases} \frac{1}{2}\ln^{-1}\frac{2}{ex}\left(1 - \frac{3}{4}x^2\ln^{-1}\frac{2}{ex} + \ldots\right), & x \ll 1 \\ \frac{3}{2}x^3\left(1 + \frac{9}{10x^2} + \ldots\right), & x \gg 1 \end{cases} \tag{17}$$

In the case $x \ll 1$ only one or two terms, closest to s_0 (for very small $s - s_0$), are essential in (15), since

$$W_{\parallel s} = \frac{e^2m^2n}{4\pi^2 l_0}\frac{1}{s_0}\sqrt{\frac{s-s_0}{s_0}}\left(\frac{x}{2}\right)^{2s} \times \exp\left\{2s - s_0 - \frac{s_0x^2}{2}\right\} G(0, s-s_0); \tag{15'}$$

it is assumed that $sx^4 \ll 1$, $(s - s_0)x^2 \ll 1$. The factor $G(0, s - s_0) \sim 1$.

For $x \gg 1$

$$G(x,\Delta) \approx \frac{1}{2}\sqrt{\frac{\pi x}{2\Delta}}e^{-\Delta/x^3} I_0\left(\frac{\Delta}{x^3}\right) E_2\left(\sqrt{\frac{2\Delta}{x}}\right), \tag{16'}$$

$$E_2(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-x^2}dx,$$

where $E_2(z)$ is the error function. Then we obtain for the s-distribution and the total probability

$$W_{\parallel s}=W_{\parallel}\frac{4}{3x^3}\exp\left\{-\frac{5(s-s_0)}{3x^3}\right\}I_0\left(\frac{s-s_0}{x^3}\right)E_2\left(\sqrt{\frac{2(s-s_0)}{x}}\right),$$

$$W_{\parallel}=\frac{3e^2m^2n}{32l_0}\left(\frac{\varkappa}{2\pi}\right)^{3/2}\exp\left\{-\frac{8}{3\varkappa}\left(1-\frac{1}{10x^2}\right)\right\},\ W_{\perp s}=2W_{\parallel s}. \tag{15''}$$

It is assumed that $x^4\kappa \gg 1$. For $x^2\kappa \gg 1$ the expression (15″) coincides with the formula obtained by Reiss,[2] cf. also [3].

For the production of a pair of scalar particles the probability $W_{\parallel}$ is given by the expression (15), divided by two, whereas $W_{\perp}$ is given by the same expression (15), but with a function $G(x, \Delta)$ differing from (16) by the additional factor

$$\left(1+\frac{x^2}{2}\right)\frac{s-s_0}{2s_0}\cos^2\theta' \equiv \frac{1}{2}\tau^2$$

in the integrand. Therefore in the production of scalar pairs $W_{\perp}$ is much smaller than $W_{\parallel}$:

$$W_{\perp}\sim W_{\parallel}\begin{cases} s_0^{-1}, & x\lesssim 1\\ \varkappa, & x\gg 1\end{cases}.$$

Thus, the probability $W_{\perp}$ for the production of a positron-electron pair is of the same order as $W_{\parallel}$ due to the spin of the electron or positron, which in this case forms a triplet state with zero projection of the total angular momentum along the direction of the electric field.

3. PAIR PRODUCTION BY A PHOTON IN THE FIELD OF A CIRCULARLY POLARIZED WAVE

In this case we consider the pair production probability averaged over the polarization of the incident photon (cf. Eq. (11) in [6]):

$$W(x,\varkappa)=\frac{e^2m^2n}{8\pi l_0}\sum_{s>s_0}\int_1^{s/s_0}\frac{du}{u\sqrt{u(u-1)}}\left\{J_s^2(z) + x^2(2u-1)\left[\left(\frac{s^2}{z^2}-1\right)J_s^2(z)+J_s'^2(z)\right]\right\}. \tag{18}$$

Here $J_s(z)$ is a Bessel function, $s_0 = -2m_*^2/(kl) = 2x(1+x^2)/\kappa$,

$$z=(4x^2\sqrt{1+x^2}/\varkappa)\sqrt{u(s/s_0-u)},$$

and the remaining notations are the same as in (3).

In the same manner as in Sec. 2, we consider the case $s_0 \gg (1+x^2)^{3/2}$, i.e., pair production due to the the absorption of a large number of photons from the wave. Using in place of the Bessel functions their asymptotic expressions[3])

$$J_s(z)=\frac{e^{-s(\alpha-\operatorname{th}\alpha)}}{(2\pi s\operatorname{th}\alpha)^{1/2}},$$

$$\operatorname{ch}\alpha=\frac{s}{z}=\left[\left(1+\frac{1}{x^2}\right)\Big/\frac{2s_0u}{s}\left(2-\frac{2s_0u}{s}\right)\right]^{1/2}, \tag{19}$$

we obtain in this case

$$W=\frac{e^2m^2n}{16\pi^2l_0}\sum_{s>s_0}\int_1^{s/s_0}\frac{du}{u\sqrt{u(u-1)}}e^{-2s(\alpha-\operatorname{th}\alpha)}\cdot\frac{1+2x^2(2u-1)\operatorname{sh}^2\alpha}{s\operatorname{th}\alpha}. \tag{20}$$

The differential distribution in u is essentially determined by the function $e^{f(u,s)}$, where

$$f(u,s)\equiv sg\left(\frac{2s_0u}{s}\right)=-2s(\alpha-\operatorname{th}\alpha),$$

$$\operatorname{ch}\alpha=\left[\frac{1+x^{-2}}{t(2-t)}\right]^{1/2},\quad t=\frac{2s_0u}{s}. \tag{21}$$

This function has a sharp maximum for $u = u_0 = s/2s_0$, with a width

$$(u-u_0)_{\text{eff}}\sim\left(\frac{2}{|f_u''|}\right)^{1/2}\sim\left(\frac{\sqrt{1+x^2}}{s_0}\right)^{1/2}\ll 1. \tag{22}$$

The maximum is situated in the physical region $1 \leq u \leq s/s_0$ of the variable u if $s/2s_0 > 1$, and is outside this region if $s/2s_0 < 1$. In the latter case the u-distribution attains a maximum for $u = 1$.

It will be shown below that for $x^2 \lesssim \kappa^{-1}$ the most important values of s are situated in the interval

$$-\frac{1}{2}<\frac{s-2s_0}{2s_0}\lesssim\left(\frac{\sqrt{1+x^2}}{s_0}\right)^{1/2}\ll 1, \tag{23}$$

these are the values for which the maximum of the distribution with respect to u is either completely situated to the left of the physical region $1 \leq u \leq s/s_0$, or is completely inside it (within the limits of its width). In this case $f(u, s)$ can be approximated by the expansion around $u = 1$:

$$f(u,s)=f(1,s)+f_u'(1,s)(u-1)+{}^1\!/_2 f_u''(1,s)(u-1)^2.$$

Making use of this expansion in (20), and taking it into account that $(u-1)_{\text{eff}} \ll 1$, we obtain the distribution in s:

$$W_s=\frac{e^2m^2n}{16\pi\sqrt{\pi}\,l_0}\left[\frac{2(s-s_0)}{s^3}\right]^{1/2}\times\frac{(1+2x^2\operatorname{sh}^2\alpha)\exp\{f(s)+y^2/4\}}{\{2s\operatorname{th}^3\alpha[(2s_0/s-1)^2+\operatorname{th}^2\alpha]\}^{1/4}}D_{-1/2}(y), \tag{24}$$

valid for all s situated within the interval (23). Here

$$f(s)\equiv f(1,s)=-2s(\alpha-\operatorname{th}\alpha),$$

$$\operatorname{ch}\alpha=\left[\left(1+\frac{1}{x^2}\right)\Big/\frac{2s_0}{s}\left(2-\frac{2s_0}{s}\right)\right]^{1/2},$$

3)These expressions are valid if $s\tanh^3\alpha \gg 1$, which is equivalent to the condition (2).

$$y=-\frac{s-2s_0}{s}\left[\frac{2s\,\mathrm{th}^3\,\alpha}{(2s_0/s-1)^2+\mathrm{th}^2\,\alpha}\right]^{1/2}, \qquad (25)$$

$D_{-1/2}(y)$ is the parabolic cylinder function with the asymptotic behavior

$$D_{-1/2}(y)=\begin{cases} y^{-1/2}e^{-y^2/4}, & y\to+\infty\\ (-2/y)^{1/2}e^{y^2/4}, & y\to-\infty\end{cases} \qquad (26)$$

and remains finite at the origin.

The characteristic properties of the distribution (24) with respect to s are determined by the exponential function $e^{f(s)}$, since for the most important values of s the function $\exp(y^2/4)D_{-1/2}(y)$ varies slowly compared with $e^{f(s)}$. Therefore the distribution (24) can be simplified in the essential region, by expanding $f(s)$ and $y(s)$ around the point $s = s_c$ where $f(s)$ has a maximum

$$f(s)\approx f(s_c)+{}^1/_2f''(s_c)(s-s_c)^2, \qquad (27)$$

$$y(s)\approx y(s_c)+y'(s_c)(s-s_c),$$

and all other quantities can be taken at the point $s = s_c$ (note that the second term in $y(s)$ is important only for $x^2 \sim \kappa^{-1} \gg 1$).

Thus the s-distribution has a sharp maximum at $s = s_c$, where $f'(s_c) = 0$, or

$$\frac{\mathrm{th}\,\alpha_c}{\alpha_c}=2-\frac{2s_0}{s_c},\quad \mathrm{ch}\,\alpha_c=\left[\left(1+\frac{1}{x^2}\right)\Big/\frac{2s_0}{s_c}\left(2-\frac{2s_0}{s_c}\right)\right]^{1/2},$$

$$s_c=\begin{cases} s_0(1+\ln^{-1}x^{-2}+\ldots), & \ln x^{-2}\gg 1\\ 2s_0(1-1/3x^2+\ldots), & x\gg 1\end{cases} \qquad (28)$$

and falls off to both sides of the maximum as a gaussian of width (for the transitional situation when $x^2 \sim \kappa^{-1}$, cf. infra)

$$(s-s_c)_{\mathrm{eff}}\sim\left[\frac{s_c\,\mathrm{th}\,\alpha_c}{(\alpha_c/\mathrm{th}\,\alpha_c-1)^2+\alpha_c^2}\right]^{1/2}$$

$$=\begin{cases}\sqrt{2s_0}/\ln x^{-2}, & \ln x^{-2}\gg 1\\ \sqrt{2s_0x}, & x\gg 1\end{cases}. \qquad (29)$$

This width is small compared to s_0, so that in units of s_0 the distribution is a sharp peak situated between the threshold and twice the value of the threshold: $s_0 < s_c < 2s_0$.

On the other hand for all x (with the exception of exponentially small ones, when $\ln x^{-2} \gtrsim (2s_0)^{1/2} \gg 1$) the width of the distribution is large compared to one. Therefore the pair production is achieved via the absorption of a large number of photons exceeding the number necessary for overcoming the threshold.

In this case the summation of expressions (24) can be replaced by an integration, which is effected by the formula

$$\int_{-\infty}^{\infty}dy\exp\left\{-\frac{(y-y_c)^2}{2\mu}+\frac{y^2}{4}\right\}D_{-1/2}(y)$$

$$=\left(\frac{2\pi\mu}{\sqrt{1-\mu}}\right)^{1/2}\exp\left\{\frac{y_c^2}{4(1-\mu)}\right\}D_{-1/2}\left(\frac{y_c}{\sqrt{1-\mu}}\right), \qquad (30)$$

with the notation $\mu = y_c'^2|f''|^{-1}$. Taking into account further that $y_c(1-\mu)^{-1/2} \gg 1$, we obtain, finally

$$W=\frac{e^2m^2n}{16\pi l_0}\frac{(1+2x^2\,\mathrm{sh}^2\,\alpha_c)\exp\{-2s_c(\alpha_c-\mathrm{th}\,\alpha_c)\}}{s_c\,\mathrm{th}\,\alpha_c}$$

$$\times\left\{2\left(\frac{\alpha_c}{\mathrm{th}\,\alpha_c}-1\right)\left[\left(\frac{\alpha_c}{\mathrm{th}\,\alpha_c}-1\right)^2+\alpha_c^2\right]\right\}^{-1/2}. \qquad (31)$$

In the case $x^2 \gtrsim \kappa^{-1} \gg 1$ the main contribution is given by values of s situated near the point $2s_0$ in the interval

$$-\left(\frac{\sqrt{1+x^2}}{s_0}\right)^{1/2}\lesssim\frac{s-2s_0}{2s_0}\lesssim\varkappa. \qquad (32)$$

For these s the maximum of the distribution in u is either completely inside the physical region $1 \le u \le s/s_0$, or is outside it at a distance not exceeding its width. In this case, and also for all $s > 2s_0$ it is natural to approximate the function $f(u, s)$ by an expansion about the point $u = u_0 = s/s_0$:

$$f(u,s)\approx f(u_0,s)+{}^1/_2f_u''(u_0,s)(u-u_0)^2.$$

Then we obtain from (20) the distribution

$$W_s=\frac{e^2m^2n}{4\pi\sqrt{\pi}l_0}\frac{s_0}{s}\left(\frac{\sqrt{1+x^2}}{2s}\right)^{5/4}\left(1+2\frac{s-s_0}{s_0}\right)$$

$$\times\exp\left\{-2s\left(\mathrm{Arsh}\frac{1}{x}-\frac{1}{\sqrt{1+x^2}}\right)+\frac{(s-2s_0)^2}{2s\sqrt{1+x^2}}\right\}$$

$$\times D_{-1/2}\left(-\frac{2(s-2s_0)}{(2s\sqrt{1+x^2})^{1/2}}\right), \qquad (33)$$

valid for all s satisfying the left inequality (32), independently of x. For $x^2 \gtrsim \kappa^{-1} \gg 1$, it follows:

$$W=\frac{3e^2m^2n}{8\pi\sqrt{\pi}l_0}\left(\frac{x}{4s_0}\right)^{5/4}\exp\left\{-\frac{4s_0}{3x^3}\left(1-\frac{9}{10x^2}\right)\right\}$$

$$\times\sum_{s>s_0}\exp\left\{-\frac{2(s-2s_0)}{3x^3}-\frac{(s-2s_0)^2}{4s_0x}\right\}$$

$$\times D_{-1/2}\left(-\frac{s-2s_0}{\sqrt{s_0x}}\right). \qquad (34)$$

Taking into account the asymptotic properties (26) of the function $D_{-1/2}(y)$, it is easy to see that the s-distribution has a maximum near the point $s = 2s_0$ and drops off to the left of the maximum as a gaussian, and to the right as an ordinary exponential, with the respective widths

$$\left(\frac{s-2s_0}{2s_0}\right)_{\mathrm{eff}}\sim\begin{cases}-\sqrt{x/2s_0}\approx-\varkappa/2x\sqrt{\varkappa}, & \text{on the left}\\ 3x^3/4s_0\approx 3\varkappa/8, & \text{on the right}\end{cases} \qquad (35)$$

Thus, for $x(\kappa)^{1/2} \gtrsim 1$, the left tail of this distribution becomes narrower than the right tail, and for

$x^2 \gg \kappa^{-1}$ the distribution starts effectively at the point $s = 2s_0$ with a sharp jump, after which it decays in a relatively slow manner, resembling the s-distribution in a linearly polarized wave. In the latter case, however, the distribution starts at $s = s_0$.

We note that in the region $1 \ll x^2 \lesssim \kappa^{-1}$ the distributions (24) and (33) overlap and yield identical results.

Since $(s - 2s_0)_{\text{eff}} \sim x^3 \gg 1$, a replacement of the sum in (34) by an integral, with use of the formula

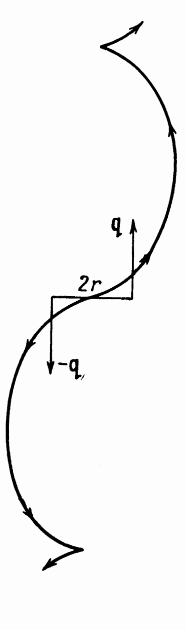
$$\int_{-\infty}^{\infty} dy\, e^{zy - y^2/4} D_{-1/2}(y) = \sqrt{\frac{2\pi}{z}}\, e^{z^2/2}, \quad z > 0, \tag{36}$$

yields

$$W = \frac{3}{64}\sqrt{\frac{3}{2}}\frac{e^2 m^2 n}{\pi l_0}\varkappa \exp\left\{-\frac{8}{3\varkappa}\left(1 - \frac{1}{15x^2}\right)\right\}. \tag{37}$$

This formula is valid for $\kappa \ll 1$, $x^2\kappa \gtrsim 1$, and for $x^2\kappa \gg 1$ it goes over into the pair production probability in a weak constant crossed field, cf. [3].

The preceding discussion shows that the s-distributions are essentially different in linearly and circularly polarized waves. This difference is due to the fact that in absorbing a large number (s) of photons from a circularly polarized wave, the electron-positron pair acquires a large angular momentum $\approx s$ (both total and orbital angular momentum), and in order for its wave function to be significantly different from zero, a relatively large relative momentum becomes necessary, viz. $q_{\text{rel}} = 2q = 2m_*((s - s_0)/s_0)^{1/2} = 2(s(s - s_0))^{1/2}$. Therefore it is clear that the number of photons absorbed above threshold must be large and turns out to be of the order of s_0, whereas the momentum q is of the order of m_*. For $x \gg 1$ the particles are emitted near the plane perpendicular to **k**, with an impact parameter 2r (cf. the figure), where $r = ea/m_*\omega' \approx 1/\omega'$ is the orbit radius (cf. [7], p. 145). Since the frequency ω' in the system in which the electron is on the average at rest (i.e., $\mathbf{q} = 0$) is connected with the frequency ω in the center-of-mass system by means of the relation $\omega' = \omega(s/s_0)^{1/2}$, it follows that $r = \omega^{-1}(s_0/s)^{1/2}$, and the orbital momentum of the pair is $l = 2rq = 2(s_0(s - s_0))^{1/2}$. On the other hand l should equal s, hence $s = 2s_0$—the value of the position of the maximum in the pair production probability which was derived above.



The trajectories of the motions of the electron and of the positron in the center-of-mass system, for $q/ea = 1$, or $x \gg 1$, $s = 2s_0$ (cycloids). For arbitrary q/ea the electron and the positron move along trochoids with average momenta q and −q.

4. DIFFERENTIAL DISTRIBUTIONS IN PAIR PRODUCTION BY A PHOTON AND IN PHOTON EMISSION BY AN ELECTRON IN A CONSTANT FIELD

In [3] the following expression was derived for the probability of pair production by a photon in a constant crossed field:

$$F_{\|,\perp}(\varkappa) = \frac{e^2 m^2 n}{4\pi^2 l_0}\int_1^{\infty}\frac{du}{u\sqrt{u(u-1)}}\left(\frac{2u}{\varkappa}\right)^{1/3}$$
$$\times \int_{-\infty}^{\infty} d\tau \{(1 \pm 1 \pm 2\tau^2)\Phi^2(y) + (1 + \tau^2)$$
$$\times (2u - 1 \mp 1)[\Phi^2 + y^{-1}\Phi'^2]\}, \tag{38}$$

where $\Phi(y)$ is the Airy function, $y = (2u/\kappa)^{2/3}(1 + \tau^2)$; the variables u, τ are the same as in (3), i.e.,

$$u = \varkappa^2/4\chi\chi', \quad \tau = -eF_{\mu\nu}{}^* p_\mu' p_\nu/m^4\varkappa,$$
$$\chi = e\sqrt{(F_{\mu\nu}p_\nu)^2}/m^3, \quad \varkappa = e\sqrt{F_{\mu\nu}l_\nu)^2}/m^3,$$

p and p′ are the momenta (more precisely, the quantum numbers, cf. [3]) of the electron and positron; n is the number density of incident photons, and l_μ their momentum. The conservation laws imply that $\chi + \chi' = \kappa$. The integrand in this formula is the probability distribution in the variables u and τ, and it can be denoted by $dF/du\,d\tau$. The integral of this function with respect to τ obviously yields the distribution in u. This distribution can be brought to a relatively simple form if one makes use of the relation

$$\Phi^2(y) + \frac{1}{y}\Phi'^2(y) = \frac{1}{2y}\frac{d^2}{dy^2}\Phi^2(y) \tag{39}$$

and changes the variable τ into $t = (2u/\kappa)^{2/3}\tau^2$:

$$\frac{dF_{\|,\perp}}{du} = \frac{e^2 m^2 n}{4\pi^2 l_0}\frac{1}{u\sqrt{u(u-1)}}\left\{\int_0^{\infty}\frac{dt}{\sqrt{t}}\left(1 \pm 1 \pm 2\frac{t}{a}\right)\Phi^2(a+t)\right.$$
$$\left. + \frac{2u - 1 \mp 1}{2a}\frac{d^2}{da^2}\int_0^{\infty}\frac{dt}{\sqrt{t}}\Phi^2(a+t)\right\}; \tag{40}$$

here $a = (2u/\kappa)^{2/3}$. Making use of the reduction formula

$$\int_0^\infty dt\, t^\alpha \Phi^2(a+t) = \frac{\alpha}{2(2\alpha+1)}\left(\frac{d^2}{da^2} - 4a\right) \times \int_0^\infty dt\, t^{\alpha-1}\Phi^2(a+t) \quad (41)$$

and of the integral transform[8]

$$\int_0^\infty \frac{dt}{\sqrt{t}}\Phi^2(a+t) = \frac{\sqrt{\pi}}{2}\int_{2^{2/3}a}^\infty dy\,\Phi(y). \quad (42)$$

we obtain from (40) the distribution in u in the simple form

$$\frac{dF_{\parallel,\perp}(\varkappa, u)}{du} = \frac{e^2m^2n}{8\pi\sqrt{\pi}\,l_0}\frac{1}{u\sqrt{u(u-1)}}\left\{\int_z^\infty \Phi(y)\,dy - \frac{4u-2\mp 1}{z}\Phi'(z)\right\}, \quad z = \left(\frac{4u}{\varkappa}\right)^{2/3}. \quad (43)$$

Integrating this distribution with respect to u, making use of two integrations by parts in the first term, we obtain the total probability

$$F_{\parallel,\perp}(\varkappa) = -\frac{e^2m^2n}{64\pi\sqrt{\pi}\,l_0}\varkappa\int_{(4/\varkappa)^{2/3}}^\infty dz\,\frac{8u+1\mp 3}{\sqrt{z}\,u\sqrt{u(u-1)}}\Phi'(z),$$
$$u = \frac{1}{4}\varkappa z^{3/2}. \quad (44)$$

In distinction from (38), the integrand of this expression is no longer a distribution in u, and does not coincide with (43). It is easy to derive from the integral (44) the limiting cases $\kappa \ll 1$ and $\kappa \gg 1$, given in [3].

In order to obtain the pair production probabilities for scalar particles it is necessary to delete the terms 2u and 4u, respectively in the second terms in the curly brackets of (38) and (43) and in (44) it is necessary to subtract 12u from the numerator, after which the resulting expressions should be multiplied by $-1/2$.

It was shown in [3] that the probability of emission of a photon by an electron in a constant crossed field is

$$F(\chi) = \frac{e^2m^2c}{2\pi^2}\int_0^\infty \frac{du}{(1+u)^2}\left(\frac{u}{2\chi}\right)^{1/3}\int_{-\infty}^\infty d\tau\left\{-\Phi^2(y) + (1+\tau^2)\left(1+\frac{u^2}{2(1+u)}\right)[\Phi^2 + y^{-1}\Phi'^2]\right\}. \quad (45)$$

Here

$$y = (u/2\chi)^{2/3}(1+\tau^2), \quad u = \varkappa(\chi-\varkappa)^{-1},$$
$$\tau = -eF_{\mu\nu}{}^*p_\mu' p_\nu/m^4\varkappa = eF_{\mu\nu}{}^*k_\mu' p_\nu/m^4\varkappa,$$
$$\chi = e\sqrt{(F_{\mu\nu}p_\nu)^2}/m^3, \quad \varkappa = e\sqrt{(F_{\mu\nu}k_\nu')^2}/m^3,$$

k′ is the momentum of the emitted photon, p and p′ are the momenta of the electron before and after the emission process, c is the ratio of the numerical density of incident electrons to their energy. The integrand in (45) gives the probability distribution of the variables u, τ.

With the aid of Eqs. (39) and (42) the integral over τ in (45), i.e., the distribution of u, can be represented in the form

$$\frac{dF(\chi,u)}{du} = -\frac{e^2m^2c}{4\pi\sqrt{\pi}}\frac{1}{(1+u)^2}\left\{\int_z^\infty \Phi(y)\,dy + \frac{2}{z}\left(1+\frac{u^2}{2(1+u)}\right)\Phi'(z)\right\}, \quad (46)$$

The integral of this distribution yields the total emission probability

$$F(\chi) = -\frac{e^2m^2c}{8\pi\sqrt{\pi}}\chi\int_0^\infty dz\,\frac{5+7u+5u^2}{\sqrt{z}(1+u)^3}\Phi'(z), \quad u = \chi z^{3/2}. \quad (47)$$

Such a representation of $F(\chi)$ was obtained earlier by Gol'dman.[9] We note that the integrand in (47) is not a probability distribution of u (in distinction from (45)) and does not coincide with (46).

The radiation intensity produced by one particle differs from (45) by the substitution $c \to 1$ and by the presence of the additional factor $u(1+u)^{-1}$ in the integrand (cf. [3], Eq. (30)). Therefore the u-distribution has the form

$$\frac{dI(\chi,u)}{du} = -\frac{e^2m^2}{4\pi\sqrt{\pi}}\frac{u}{(1+u)^3}\left\{\int_z^\infty \Phi(y)\,dy + \frac{2}{z}\left(1+\frac{u^2}{2(1+u)}\right)\Phi'(z)\right\},$$
$$z = \left(\frac{u}{\chi}\right)^{2/3}. \quad (48)$$

This expression is a quantum generalization of the well-known classical formula for the spectral distribution of radiation of an ultrarelativistic charged particle in a magnetic field (cf. [7], Eq. (74, 11)). The expression is rigorously valid for radiation by an electron in a constant crossed field, but when conditions (50) (below) are satisfied, it is also valid for the emission of radiation by an electron in an arbitrary constant field, in particular, in a magnetic field, if $F_{\mu\nu}$ in the invariant variables, u, χ is understood to be the intensity of the corresponding field.

Integrating (48) with respect to u (again making use of two integrations by parts in the first term), we obtain

$$I(\chi) = -\frac{e^2m^2\chi^2}{8\pi\sqrt{\pi}}\int_0^\infty dz\,z\,\frac{4+5u+4u^2}{(1+u)^4}\Phi'(z), \quad u = \chi z^{3/2}. \quad (49)$$

From (47), (49) it is easy to derive expansions of the quantities $F(\chi)$ and $I(\chi)$ with respect to the parameter χ, which were obtained in [3].

We note that the term $u^2/2(1+u)$ in the second term of the expressions (45), (46), and (48) is a consequence of the electron spin, so that the expressions for the probability and intensity of radiation by a scalar particle differ from the corresponding expressions (45), (46), and (48) by the absence of this factor. In order to obtain the expressions for the probability and intensity of radiation by a scalar particle, one must subtract $3u^2$ from the numerators of Eqs. (47), (49).

We note further that the parameters χ, χ', κ have a quantum character, χ, $\chi' \sim \hbar$, $\kappa \sim \hbar^2$, so that $u = \kappa/\chi' = \kappa(\chi-\kappa)^{-1}$ involves the Planck constant. At the same time, τ does not contain $\hbar$. Therefore the transition to the classical expressions in Eqs. (45)–(49) can be effected by neglecting κ compared to χ, i.e., by replacing u/χ by κ/χ and neglecting u compared to 1.

In conclusion we discuss the relation of the differential probability of a process in a crossed field to the corresponding probability in an arbitrary constant field. In [3] (cf. also [10]) arguments were presented according to which the total probability $W(\chi, f, g)$ of a process produced by a particle of 4-momentum p_μ in an arbitrary constant field $F_{\mu\nu}$, a probability that depends on the invariants

$$\chi = \frac{e\sqrt{(F_{\mu\nu}p_\nu)^2}}{m^3}, \quad f = \frac{e^2F_{\mu\nu}^2}{m^4}, \quad g = \frac{e^2F_{\mu\nu}{}^*F_{\mu\nu}}{m^4},$$

can be approximated by the probability $W(\chi, 0, 0)$ of the same process in a crossed field, with $F_{\mu\nu}$ in χ to be interpreted as the corresponding field strength, provided the following conditions

$$|f|, |g| \ll 1, \quad \chi^2 \gg |f|, |g|, \tag{50}$$

are satisfied. For similar reasons, if the conditions (50) hold, the invariant expressions for the differential probabilities in a crossed field are also good approximations to the corresponding differential probabilities in an arbitrary constant field, in the effective range of the variables u, τ which gives the main contribution to the total probability, provided $F_{\mu\nu}$ is interpreted as the corresponding field strength tensor. An essential feature here is the fact that the differential probabilities are invariants.

The differential intensity of radiation is not an invariant (cf. [7], p. 215). However the differential intensity emitted by an electron in a constant crossed field

$$\frac{dI}{du\,d\tau} = \frac{e^2m^2}{2\pi^2}\frac{u(u/2\chi)^{1/3}}{(1+u)^3}\left\{-\Phi^2(y) + (1+\tau^2)\left(1+\frac{u^2}{2(1+u)}\right) \times [\Phi^2 + y^{-1}\Phi'^2]\right\} \tag{51}$$

is an invariant, since an integration has been carried out over one of the three variables (viz. ψ, cf. [3, 11]) determining the differential intensity. Therefore, if conditions (50) hold, the differential intensity of the radiation emitted by a particle in an arbitrary constant field, and taken with respect to the variables u and τ only, will also be an invariant quantity, and will be given by Eqs. (51), (48) in the effective region of those variables. Thus, it is easy to see that the differential intensity of radiation emitted by a particle in a magnetic field, integrated with respect to the azimuthal angle in the reference system in which the particle is on the average at rest (for a classical particle this intensity is determined by the well-known Schott formula, cf. [7], Eq. (74, 8)) becomes an invariant in the ultrarelativistic limit, i.e., for $\chi^2 \gg f$. Therefore, if one interprets $F_{\mu\nu}$ in the quantum formula (48) as the magnetic field, this formula will become in the classical ultrarelativistic limit identical to Eq. (74.11) in [7], a formula which determines the spectral distribution of radiation of a particle in a magnetic field.

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