CONTRIBUTION TO THE THEORY OF FIELD EMISSION FROM METALS. II

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Total-energy and normal-energy distribution functions of field emitted electrons are calculated (for T = 0 and T > 0) as integral characteristics of the dispersion law and are investigated for various cases. The temperature dependence of the field emission current is considered and it is shown that in some cases it can be decreasing.

In the preceding paper[1] we calculated the transparency coefficient of the potential barrier for electrons in a metal, and the cold-emission current, for an arbitrary dispersion law. The expression for the field-emission current is

$$j_z = -\frac{2e}{h^3}\int \Phi(E) f(E)\, dE, \qquad \Phi(E) = \int_{\Sigma(E)} D(E, \mathbf{P})\, d^2\mathbf{P} \quad (1)$$

(the z axis is directed outward from the metal perpendicular to its surface). Here f(E) is the Fermi distribution function, $\Sigma(E)$ part of the projection of the equal-energy surface on the $p_x p_y$ plane, contained in the central Brillouin zone of the plane lattice (representing the projection of the reciprocal lattice on the $p_x p_y$ plane); **P** is the component of the quasimomentum (p) tangential to the surface of the metal; D(E, **P**) is the effective transparency coefficient (which takes into account all the Bloch waves with given E and **P** incident on the surface). Apart from the pre-exponential factor, which is of the order of unity, D depends only on the energy V (denoted $E^{(2)}$ in[1]) of the electron motion (outside the metal) along the z axis, equal to $V = E - P^2/(2m_0)$:

$$D(E, \mathbf{P}) \approx e^{-\xi(V)}, \quad \xi(V) = \frac{4}{3}\sqrt{2}\left(\frac{I}{-V}\right)^{1/2}\frac{\theta(v)}{v^2}, \quad v = \frac{e^{3/2}F^{1/2}}{-V}, \quad (2)$$

where $I = m_0 e^4/\hbar^2 = 27.2$ eV is the atomic energy unit, m_0 and e are the mass and charge of the electron, F is the intensity of the external electric field, and θ is a function that decreases from 1 to 0 when the argument increases from 0 to 1 (a table of its values is given in[2], Appendix 1); the energy is reckoned from the energy of the electron outside the metal in the absence of an external field. It is assumed that $\xi(V) \gg 1$.

It is obvious from (1) and (2) that at zero temperature the order of magnitude of j_z is determined principally by the maximum value of the transparency coefficient at $E \le -w$ ($-w$ is the Fermi level and w is the work function), namely $D_{Max} \approx \exp\{-\xi(V_{max})\}$. The main result of[1] is as follows: The equality $V_{Max} = -w$, which holds for free electrons, is satisfied in the case of a complicated dispersion law only when the Fermi surface crosses the p_z axis; otherwise $V_{Max} < -w$, i.e., the effective work function $W = -V_{Max}$ is larger than the true work function, and from the difference $W - w$ we can estimate the distance from the Fermi surface to the p_z axis[1)].

In this paper we estimate the energy distribution of the emitted electrons and the temperature dependence of the field-emission current; these quantities, naturally, contain a larger amount of information on the dispersion law than the effective work function.

To measure the energy distribution of the electrons one customarily uses the decelerating potential method (see, for example,[2], Sec. 14), which ensures a fairly high resolution (Young and Muller[3] give a value 0.02—0.03 eV). Since this method makes it possible to obtain the distribution both with respect to the total energies E (in the case of a pointlike emitter and a spherical collector[3]), and with respect to the energies V of the motion normal to the surface of the metal (in the case of flat electrodes), we shall consider both cases.

1. DISTRIBUTION OF THE EMITTED ELECTRONS WITH RESPECT TO THE TOTAL ENERGIES

It is seen from (1) that the distribution function of the current j_z with respect to the total energies

[1)]If the quantity V_{Max} is connected with an anomalously small group, then the effective work function determining the exponent in the expression for j_z is larger than $-V_{Max}$ (see[1], footnotes [5)6)]).

of the electrons is proportional to $\Phi(E) f(E)$. As shown in[1] (formulas (13)—(18)), we have

$$\Phi(E) \approx S(E) \exp\{-\xi(E-\Delta(E))\},$$
$$\Delta(E) = P^2_{min}(E)/(2m_0), \qquad (3)$$

where for large groups S(E) are slowly varying functions, and for small groups $S(E) \sim \pm(E - E_g)$ (the upper sign pertains to the electrons and the lower to holes; E_g is the value of the energy at which the groups are respectively created or vanish), and $P_{min}(E)$ is the minimum value of P in the region $\Sigma(E)$. Expression (3) is valid if

$$\xi(E-\Delta(E)) \gg 1 \quad \text{or} \quad (E_{min} - E) \times [1 - \Delta'(E_{min})]/\varepsilon(E_{min}) \gg 1; \qquad (4)$$

here E_{min} is the minimum energy at which the electron can leave the metal over the potential barrier[2]:

$$E_{min} - \Delta(E_{min}) = -e^{3/2}F^{1/2}, \text{ i.e., } \xi(E_{min} - \Delta(E_{min})) = 0; \qquad (5)$$

$$\varepsilon(E) = -\frac{1}{\xi'(E-\Delta(E))} = \frac{\alpha(v_1)e^3F}{2\sqrt{2I}[-E+\Delta(E)]} = \frac{2}{3}\alpha(v_1)\theta(v_1)\frac{-E+\Delta(E)}{\xi(E-\Delta(E))}, \qquad (6)$$

$$v_1 = e^{3/2}F^{1/2}/[-E+\Delta(E)], \quad \alpha(v_1) = [\theta(v_1) - (2/3)v_1\theta'(v_1)]^{-1}$$

($\alpha(v_1)$ decreases from 1 to 0.9004 when v_1 increases from 0 to 1);

$$\varepsilon(E_{min}) = \alpha(1)(e^{3/2}F^{1/2})^{3/2}/(2\sqrt{2I}). \qquad (7)$$

In the derivation of inequality (4) and in a number of similar cases which will be encountered later on, we assume that the corresponding expansions are valid.

When T = 0, there exist three types of distribution, depending on the value of the energy (denoted $\tilde{E}_M$) at which $\max\limits_{E \le -w} \Phi(E)$ is reached (see Fig. 1). Each type corresponds to a definite case considered in[1] (Sec. 2): $\tilde{E}_M = -w$ to case 1, $\tilde{E}_M = \tilde{E}_m$ ($\Phi'(E_m) = 0$) to cases 2, 3b), and 3d), and $\tilde{E}_M = E_2$ to case 3a). It follows therefore that it is possible to distinguish between these cases experimentally (primarily between case 1 on the one side and cases 2 and 3 on the other), which is essential for the interpretation of the effective work function in terms of the dispersion law.

[2]For more details concerning this quantity see [4], Sec. 1. For simplicity, we do not consider the case b) of [4], when the function $\Delta(E)$ has a discontinuity at the point $E = E_{min}$; the generalization to this case is trivial.

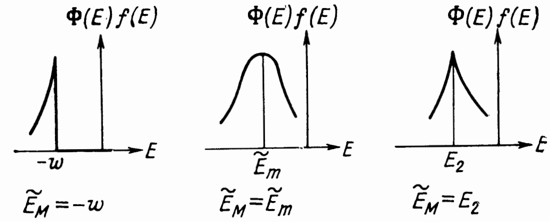

FIG. 1. Typical curves showing the distribution of emitted electrons with respect to the total energies at T = 0.

The change of the distribution with increasing temperature has the following character: If $\tilde{E}_M = -w$ (case 1), the distribution shifts towards higher energies and the total current is appreciably increased; if $\tilde{E}_M = \tilde{E}_m$ or $\tilde{E}_M = E_2$ (cases 2 and 3), no significant change takes place in the distribution, and the total current is somewhat decreased (until the temperature becomes so high that higher bands or higher groups of the same band begin to take part in the emission).

It is of interest to trace the variation of the position of the maximum of the distribution function as a function of the temperature. When T > 0 the sharp (peaked) maximum corresponding to a jump of the derivative of the distribution function, provided a "gap" with a center at the origin occurs in $\Sigma(E)$ at this value of the energy. In any other case, the maximum is smooth and its position is determined in the usual fashion from the equation $d[\Phi(E)f(E)]/dE = 0$, i.e.,

$$\omega(E) = \{T[1+\exp\{-(E+w)/T\}]\}^{-1} \text{ or } E = -w - T\ln\{[\omega(E)T]^{-1} - 1\}, \qquad (8)$$

where $\omega(E) = \Phi'(E)/\Phi(E) = S'(E)/S(E) + [1-\Delta'(E)]/\epsilon(E)$. Equation (8) and all that follow is based on expression (3), i.e., it is valid in the energy region bounded by the inequality (4); it is assumed with this that in any case $\xi(-W) \gg 1$. It is seen from Eq. (8) that for any temperature its root E satisfies the inequality $\omega(E) \ge 0$, i.e., $E \le \tilde{E}_m$ or $E \le E_2$ (the latter in the presence of a sharp maximum of the function $\Phi(E)$ at the point $E = E_2$).

To investigate the function $\tilde{E}_m(T)$, which is specified in implicit form by Eq. (8), we introduce the auxiliary variable $u = (E + w)/T$ (we neglect the temperature dependence of the work function). The connection between u and E is given by

$$u/(1+e^{-u}) = (E+w)\omega(E), \qquad (9a)$$

and between u and T by

$$u = -\ln\{[\omega(-w+uT)T]^{-1} - 1\}; \qquad (9b)$$

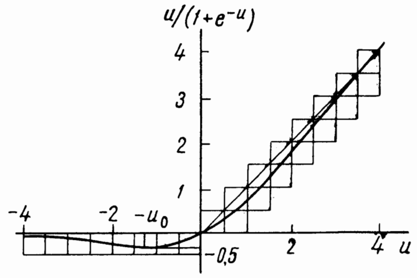

FIG. 2. Plot of the function $u/(1 + e^{-u})$.

actually it is more convenient to use in lieu of (9b) one of the relations

$$T = (E + w)/u = [\omega(E)(1 + e^{-u})]^{-1}. \qquad (9c)$$

Figure 2 shows the plot of the function $u/(1 + e^{-u})$. When $u = -u_0$ it has a minimum equal to $-(u_0 - 1)$, where $u_0 = 1.278$ is the root of the equation $e^{-u} - u + 1 = 0$. Plots of the function $\xi = (E + w)\omega(E)$ in the region of interest are shown schematically in Fig. 3[3)]. These plots were constructed with allowance for the fact that the transition from the positive to the negative values of $\omega(E)$ with increasing E occur either at the point $E = \tilde{E}_m$, where $\omega(E)$ vanishes (variants a and c), or at the point $E = E_2$ (in the case when a gap appears in $\Sigma(E)$, where $\omega(E)$ experiences a jump (variants b and d)). Variants a and b pertain to case 1 of[1] ($\tilde{E}_M = -w$, $\omega(-w) > 0$), variant c pertains to the cases 2, 3b), and 3d) ($\tilde{E}_M = \tilde{E}_m$), and variant d to case 3a) ($\tilde{E}_M = E_2$). Let us examine these variants.

Variant a. Plots of the functions u(T) and $E = \tilde{E}_m(T)$ are shown schematically in Fig. 4. The positions of the characteristic points of the plots are determined by the relations

$$T_0 = T_1/2, \quad T_1 = 1/\omega(-w), \qquad (10)$$

$$T_m = (1 - 1/u_0)/\omega(\tilde{E}_{m\,min}) = (-w - \tilde{E}_{m\,min})/u_0,$$

and if

$$a \equiv \frac{d}{dE}\left[\frac{1}{\omega(E)}\right]_{E=-w} \ll \frac{1}{u_0 - 1},$$

we can put $1/\omega(\tilde{E}_{m\,min}) \approx T_1$. If $E_{min} < \tilde{E}_m$, then the tails of the curves become meaningless in view of the inequality (4).

3)We confine ourselves to consideration of only that band and that group in which emission takes place at T = 0. Actually, as already mentioned, for sufficiently high temperatures, higher groups may get involved in the emission, as a result of which additional maxima of the distribution functions are produced. In particular, such a situation must take place in cases 2 and 3 of [1], which correspond to variants c and d below.

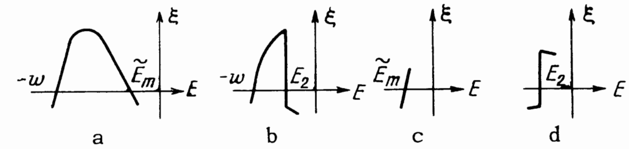

FIG. 3. Different variants of the energy dependence of the quantity $\xi = (E + w)\omega(E)$.

In the limiting cases, the function T(E) or T(u) can be expressed explicitly. If $u \gg 1$ (the maximum of the distribution function is situated in the Boltzmann tail of the Fermi function), i.e., $(E + w)\omega(E) \gg 1$, then, as follows from (9c),

$$T \approx 1/\omega(E). \qquad (11)$$

As seen from Fig. 4, the intervals of the values of T and E, where $u \gg 1$, are bounded from both sides: from the right—either by the inequality

$$\tilde{E}_m - E \gg [-(\tilde{E}_m + w)\omega'(\tilde{E}_m)]^{-1}$$
$$(\omega'(\tilde{E}_m) < 0), \text{ i.e., } T \ll \tilde{E}_m + w,$$

or by the inequality (4), and from the left by the inequality

$$E + w \gg T_1, \text{ i.e., (if } a \ll 1)$$
$$(T - T_1)/T_1 \gg a;$$

if these inequalities are incompatible, then there exists no region where $u \gg 1$.

If $|E + w| \equiv |u|T \ll T_1/a$ then, expanding $\omega^{-1}(-w + uT)$ in (9b) in powers of uT, we get

$$T \approx T_1/(1 + e^{-u} - au). \qquad (12)$$

The condition $|u|T \ll T_1/a$, which is equivalent to $(1 + e^{-u})/|u| \gg a$, is satisfied in the vicinities of the points T = 0 and $T = T_0$. When $a \ll (u_0 - 1)^{-1}$ these vicinities overlap, i.e., formula (12) is valid in the entire region $u < 0$ ($T < T_0$). On the positive side of u, the region of validity of (12) is bounded by the condition $u \ll 1/a$, i.e.,

$$(T - T_0)/T_0 \ll 1 \text{ for } a > 1, \ (T - T_1)/T_1 \ll 1 \text{ for } a \ll 1.$$

For sufficiently small a, the regions where formulas (12) and (11) are valid become superimposed,

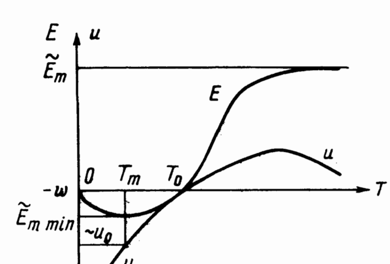

FIG. 4. Schematic plots of the functions $u = u(T)$ and $E = \tilde{E}_m(T)$ for variant a.

and in the common section the T(E) dependence is linear: $T - T_1 \approx a(E + w)$.

Finally, if $e^u \ll 1/a$, then we can put in (9b) $\omega(-w + uT) \approx \omega(-w)$ and we obtain in lieu of (12) an even simpler formula $T \approx T_1/(1 + e^{-u})$ or [4]

$$E \approx -w - T\ln[(T_1 - T)/T]. \qquad (13)$$

The condition $e^{-u} \gg a$ is equivalent to $(T_1 - T)/T_1 \gg a$, i.e.,

$$T \ll T_1/a \text{ for } a > 1, \quad (T_1 - T)/T_1 \gg a \text{ for } a \ll 1.$$

With increasing temperature, the current of the above-the-barrier electrons assumes an ever increasing role, and ultimately the emission becomes thermionic. The field emission (i.e., essentially tunnel emission) remains under the condition

$$\Phi(E)f(E) \gg \Phi(E_{min})f(E_{min}) \qquad (E = \tilde{E}_m(T))$$

or, taking (3), (5), and (9c) into account

$$\ln[S(E)/S(E_{min})] - \xi(E - \Delta(E)) + (E_{min} - E)\omega(E)[1 + e^{-u(E)}] \gg 1. \qquad (14)$$

We note that (14) is a stronger inequality than (4), with the possible exception of the case when

$$S'(E)/S(E) \gg [1 - \Delta'(E)]/\varepsilon(E).$$

If $E_{min} < \tilde{E}_m$, then, expanding the left side of the inequality (14) near the point $E = E_{min}$ at which it vanishes (for which purpose we extrapolate the u(E) dependence defined by (9a) down to $E = E_{min}$), we get [5]

$$E_{min} - E \gg e^{u(E_{min})}/\omega(E_{min}) \text{ for } u(E_{min}) \lesssim 1, \quad (14a)$$

$$(E_{min} - E)^2 \gg -2/\omega'(E_{min}) \text{ for } u(E_{min}) \gg 1. \quad (14b)$$

Of course, it is necessary that these inequalities be compatible with the performed expansion; in particular, (14b) must not contradict the condition $u(E) \gg 1$. If $\tilde{E}_m < E_{min}$, then the left side of the inequality (14) vanishes at a certain value of E which is smaller than $\tilde{E}_m$, and the expansion must be carried out in the vicinity of this value.

Variant b. If $E_{min} < E_2$, everything is the same as in variant a when $E_{min} < \tilde{E}_m$.

If $E_{min} > E_2$, then when $\tilde{E}_m(T) < E_2$, corresponding to $T < T_2$ (T_2 is determined from (8) or from (9) in which we must put $E = E_2 - 0$), everything is the same as in variant a. When $T > T_2$, the maximum of the distribution function becomes peaked, and regardless of the temperature, it remains at the point $E = E_2$. If

$$T_2 < (E_{min} - E_2)/\{\xi(E_2) - \ln[S(E_2)/S(E_{min})]\} \equiv T_3,$$

the condition for the subbarrier current to be negligible assumes in lieu of (14) the form

$$(T_3 - T)/T_3 \gg \{\xi(E_2) - \ln[S(E_2)/S(E_{min})]\}^{-1}.$$

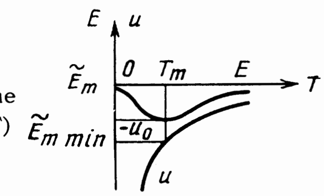

FIG. 5. Schematic plots of the functions $u = u(T)$ and $E = \tilde{E}_m(T)$ for variant c.

Variant c. Plots of the functions u(T) and $E = \tilde{E}_m(T)$ are shown schematically in Fig. 5. At all temperatures, the maximum is smooth, $u < 0$, $E \le \tilde{E}_m \le -w$. The values of T_m and $\tilde{E}_{m\,min}$ are determined by the same equations (10) as in variant a.

For sufficiently small values of $\tilde{E}_m - E$, linearizing $\omega(E)$ in (8) and recognizing that $\omega(\tilde{E}_m) = 0$, we get

$$E - \tilde{E}_m = \{\omega'(\tilde{E}_m)T \times [1 + \exp\{(-w - E)/T\}]\}^{-1}. \qquad (15)$$

If this relation is valid for $E = \tilde{E}_{m\,min}$ (meaning also for all values of $\tilde{E}_m(T)$), then

$$\omega'(\tilde{E}_m)(\tilde{E}_{m\,min} - \tilde{E}_m) \cdot \times (-w - \tilde{E}_{m\,min}) = u_0 - 1.$$

When $T \ll -w - \tilde{E}_m$, the quantity $\tilde{E}_m - E$ is exponentially small and from (15) we get for it the explicit expression

$$E - \tilde{E}_m = \exp\{(w + \tilde{E}_m)/T\}/\omega'(\tilde{E}_m)T.$$

Variant d. If $(E_2 + w)\omega(E_2 - 0) > -(u_0 - 1)$ (we recall that in our case $E_2 < -w$), then Eq. (8) (or else the equations (9)) has for $E = E_2 - 0$ the two roots $T_2^{(1)}$ and $T_2^{(2)}$. When $T_2^{(1)} < T < T_2^{(2)}$ the maximum of the distribution function is smooth, and the function $\tilde{E}_m(T)$ decreases from a value E_2 to a value $\tilde{E}_{m\,min}$ in the temperature interval $[T_2^{(1)}, T_m]$, and increases from $\tilde{E}_{m\,min}$ to E_2 in the interval $[T_m, T_2^{(2)}]$ (T_m and $\tilde{E}_{m\,min}$ are determined by Eqs. (10)). Outside the interval $[T_2^{(1)}, T_2^{(2)}]$, the maximum is sharp and is located at the point $E = E_2$.

If $(E_2 + w)\omega(E_2 - 0) < -(u_0 - 1)$, then Eqs. (8) and (9) have no roots at $E = E_2 - 0$, so that the maximum of the distribution function is sharp at all temperatures and is located at the point $E = E_2$.

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[4]Formula (13) for the model of free electrons is contained in the paper of Young [5]. The same paper gives plots of the distribution function for several values of T.

[5]Multipliers of the order of unity are not discarded in the inequalities pertaining to the exponents.

Let us specify the results for the case when $\Sigma(E)$ contains a large vicinity near the origin in the entire energy interval of interest $\tilde{E}_{m\,min} < E < E_{min}$ (this is the situation, in particular, in the free-electron model). In this case $\Delta(E) \equiv 0$,

$$S(E) = 2\pi m_0 \varepsilon(E) \text{ and}$$
$$\omega(E) = [1 + \varepsilon'(E)] / \varepsilon(E),$$
$$\varepsilon'(E) = \beta(v)\varepsilon(E)/(-2E)$$
$$= \alpha(v)\beta(v)\theta(v)/3\xi(E) \ll 1 \quad (16)$$

(see (6)); here $v = -e^{3/2}F^{1/2}/E$, $\beta(v) = 1 + 2v\alpha'(v)/\alpha(v)$; $\beta(v)$ decreases from 1 to 0.76 when v increases from 0 to 1. The order of magnitude of the characteristic temperatures is determined from (10) and (16): $T_1 \sim 10^{-2}$—10^{-1} eV $\sim 10^2$—10^3 degrees. The value of E_{min}, which is equal to $-e^{3/2}F^{1/2}$ (see (5)), is smaller than that of E_2 or $\tilde{E}_m$, so that variant a or b is obtained. The parameter a is equal to

$$a \approx \varepsilon'(-w) \ll 1.$$

Using (9a) and (2), we can readily show that

$$u(E_{mim}) > 2\sqrt{2I/w}(1 - v_0)/\alpha(1)v_0^{3/2}$$
$$> v_0^{1/2}\xi(-w) \gg 1, \quad v_0 = e^{3/2}F^{1/2}/w.$$

This means that the region of applicability of the asymptotic formula (11), $T \approx \epsilon(E)/[1 + \epsilon'(E)]$, is bounded from the left by the inequality $(T - T_1)/T_1 \gg a$ and from the right by the inequality (4), and the compatibility of these inequalities is ensured by the condition $\xi(-w) \gg 1$ if this inequality is sufficiently strong. The same condition allows us to rewrite the inequality (4) in the form

$$(T_{max} - T)/T_{max} \gg \varepsilon'(E_{min}) \approx \beta(1)T_{max}/2e^{3/2}F^{1/2},$$

where $T_{max} = \epsilon(E_{min})/[1 + \epsilon'(E_{min})]$ (see (7)). The inequality (14b) (which is stronger than (4)) takes the form

$$(-e^{3/2}F^{1/2} - E)^2 \gg 4e^{3/2}F^{1/2}T_{max}/\beta(1)$$

or

$$[(T_{max} - T)/T_{max}]^2 \gg \beta(1)T_{max}/e^{3/2}F^{1/2}.$$

This inequality serves as a criterion for neglecting the above-the-barrier current, if it is compatible with the condition $u(E) \gg 1$, i.e., if

$$(1 - v_0)^2 \gg \sqrt{2}\,\alpha(1)(w/I)^{1/2}v_0^{5/2},$$

which is the stronger requirement than $\xi(-w) \gg 1$. In any case, the above-the-barrier current is negligibly small when $T = T_0$, for at this temperature (14) is equivalent to the inequality $\xi(-w) \gg 1$.

Figure 6 shows a plot of $E = \tilde{E}_m(T)$ for the case under consideration with w = 4.5 eV and $F = 2 \times 10^7$ V/cm; at these values of the parameters

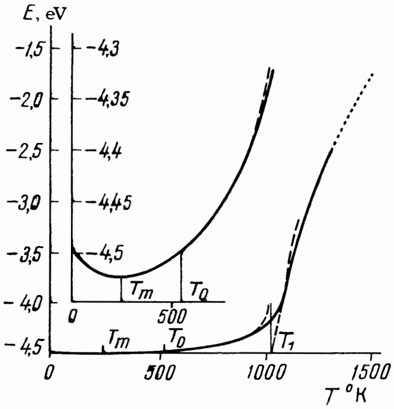

FIG. 6. Plot of the function $E = \tilde{E}_m(T)$ for the case when $\omega(E)$ is determined by formula (16); w = 4.5 eV, $F = 2 \times 10^7$ V/cm; $T_m = 224°$, $T_0 = 514°$, $T_1 = 1029° = 0.0887$ eV, $T_{max} = 1520° = 0.1310$ eV. The main curve was constructed from the exact equations (9a) and (9c); the section not satisfying the inequality (4) is shown dashed. The dashed lines correspond to the asymptotic formulas (11) – (13).

$E_{min} = -e^{3/2}F^{1/2} = 1.697$ eV and $\xi(-w) = 26.6$.

Without writing out the rather cumbersome formulas for the case of a small group with $\Delta(E) \equiv 0$, we note only that in this case it is expedient, in the derivation of the asymptotic formulas, to expand not the entire function $\omega(E) \pm 1/(E - E_g) + 1/\epsilon(E)$, but only the term $1/\epsilon(E)$.

2. DISTRIBUTION OF EMITTED ELECTRONS WITH RESPECT TO THE NORMAL MOTION ENERGIES

The distribution of the electrons with respect to the energies V of the motion along the z axis can be obtained in analogy with the distribution with respect to the total energies (see[1], derivation of formula (11)), by changing over from the integration variables p_x, p_y, p_z in the initial expression for j_z to the variables p_x, p_y, E. This yields

$$j_z = -\frac{2e}{h^3}\int dV \int_{\Omega(V)} D(V + P^2/(2m_0), \mathbf{P}) f(V + P^2/(2m_0))\, d^2\mathbf{P},$$

where $\Omega(V)$ is that part of the projection of the surface $\mathscr{E}(\mathbf{p}) - (p_x^2 + p_y^2)/(2m_0) = V = \text{const}$ on the $p_x p_y$ plane, which is contained in the central Brillouin zone of the plane lattice; the remaining notation is the same as before. Assuming

$$\xi(V) \gg 1, \text{ i.e. } -e^{3/2}F^{1/2} - V \gg \varepsilon(E_{min})$$

(cf. with (4)) and using relation (2), we get

$$j_z = -\frac{2e}{h^3}\int e^{-\xi(V)}\chi(V)\,dV, \quad (17)$$

$$\chi(V) = \int_{\Omega(V)} f(V + P^2/(2m_0))\, d^2\mathbf{P}; \quad (18)$$

thus, apart from a pre-exponential factor of the order of unity, the distribution function of the current j_z with respect to V is proportional to $e^{-\xi(V)}\chi(V)$. Owing to the factor $e^{-\xi(V)}$, it decreases rapidly with decreasing V, so that the region $-W - V \gg \epsilon(E_M)$ is of no interest (E_M is determined by the relation $E_M - \Delta(E_M) = -W$).

Let us consider the function $\chi(V)$, which depends on the dispersion law. Calculation of the integral (18) in polar coordinates leads to the expression

$$\chi(V) = m_0 \int \chi_V(\sqrt{2m_0 u}) f(V + u)\, du, \qquad (18a)$$

where $\chi_V(P)$ is the total length (in radians) of those parts of the circle of radius P with center at the origin, which lie within $\Omega(V)$ (cf. the quantity $\varphi_E(P)$ in[1], formula (14)). As can be readily seen, these are the same parts of the circle which lie within the confines of the figure $\Sigma(V + P^2/(2m_0))$ (inasmuch as the cylinder P = const intersects the surfaces V = const and E = const ($E = V + P^2/(2m_0)$) along the same line).

We can draw a number of consequences from the connection established in this fashion between the figure $\Omega(V)$ and the figures $\Sigma(E)$. Thus, the equation of the boundary of $\Omega(V)$ can be obtained from the equation for the boundary of $\Sigma(E)$ by substituting for E the expression $V + (p_x^2 + p_y^2)/2m_0$. This denotes, in particular, that if $\Sigma(E)$ is bounded by the curves

$$(p_x - p_x^0)^2/2m_1 \pm (p_y - p_y^0)^2/2m_2 = \pm(E - E^0),$$

then the boundaries of $\Omega(V)$ are also second-order curves (certain particular cases are considered in the Appendix). Further, $\chi_V(P) = \varphi_{V + P^2/(2m_0)}(P)$, from which it follows, in turn, that the minimum value of P in the region $\Omega(V)$ (which we denote by $\mathcal{P}_{min}(V)$) coincides with $P_{min}(E(V))$, i.e., $\delta(V) = \Delta(E(V))$, where $\delta(V) = \mathcal{P}^2_{min}(V)/2m_0$, E(V) is the smallest root of the equation[6] $E - \Delta(E) = V$ located in the interval in which the function $E - \Delta(E)$ grows. It follows from these relations that

$$V + \delta(V) = E(V), \quad E_M = E(V_{Max} - 0). \qquad (19)$$

We note that the functions $\delta(V)$ and E(V) have discontinuities at the points $V = V_{max}$, where V_{max} is the maximum of the quantity $E - \Delta(E)$ (higher than that lying to the left), so that $E(V_{max} - 0)$ is E_m or E_2; at these points, the parts of the figure $\Omega(V)$ that are close to the origin vanish. The jumps of these functions at the discontinuity points are positive, i.e., $E(V_{max} + 0) > E(V_{max} - 0)$ or if (V_{max}, V_1) is a forbidden interval, we have $E(V_1 + 0) > E(V_{max} - 0)$ and analogously for $\delta(V)$.

As seen from formula (18), at zero temperature $\chi(V)$ is equal to the area of the part of the figure $\Omega(V)$ which lies inside the circle $(p_x^2 + p_y^2)/2m_0 \le -w - V$, which henceforth will be denoted $C(-w - V)$ (compare with the function $\Phi(E)$ for thermionic emission, formula (6) of[4])[7]. Consequently, the upper limit of the distribution $V_{max} = -W$ (see footnote[1]) is the largest value of V at which $\Omega(V)$ has a common point with the circle $C(-w - V)$ (compare with the definition of the quantity E_{min} in[4]), i.e., allowing for a possible discontinuity of the function $\delta(V)$ at the point $V = V_{Max}$, $V_{Max} + \delta(V_{Max} - 0) + w \le 0$ and either $V_{Max} + \delta(V_{Max} + 0) + w \ge 0$ or $V_1 + \delta(V_1 + 0) + w \ge 0$ (the latter if (V_{Max}, V_1) is a forbidden interval).

With the aid of (19) we can reduce these inequalities to the form

$$E(V_{Max} - 0) + w \equiv E_M + w \leqslant 0,$$
$$E(V_{Max} + 0) + w \geqslant 0 \text{ and } E(V_1 + 0) + w \geqslant 0. \qquad (20)$$

It follows therefore that either $V_{Max} \ne V_{max}$ (the function E(V) is continuous at the point $V = V_{Max}$) and $E_M = -w$ (case 1 of[1]), or $V_{Max} = V_{max}$ and E(V) is discontinuous at the point $V = V_{Max}$ and $E_M = E_m$ (case 2 in[1]), or else $E_M = E_2$ (case 3 in[1]). In case 1 either $\Omega(-w)$ contains the origin ($\delta(-w) = 0$) and $V_{Max} = -w$, i.e., $C(-w - V)$ vanishes when $V = V_{Max}$, or $\delta(-w) > 0$, $V_{Max} < -w$ and when $V = V_{Max}$ the figures $C(-w - V)$ and $\Omega(V)$ become tangent; in cases 2 and 3 when $V = V_{Max}$ the parts of $\Omega(V)$ situated inside $C(-w - V)$ vanish, whereas the circle $C(-w - V)$ no longer touches the other parts of $\Omega(V)$ (if they exist) (compare with the variants of the definition of E_{min}[4]).

Unlike the distribution function $\Phi(E)f(E)$ with respect to the total energies, which vanishes jumpwise at the point $E = -w$ when T = 0, the function $\chi(V)$, and with it the entire distribution function

[6]Besides the forbidden intervals, there can exist also certain intervals of V in which the equation $E - \Delta(E) = V$ has no roots of the required type and E(V) is determined in a different manner, but such intervals are of no interest to us in what follows. We note that $E_{min} = E(-e^{3/2}F^{1/2})$, as seen from formula (5) (with account of the remark [2]).

[7]Comparing (17) with formulas (5) and (6) of [4], we can readily verify that if the circle $C(-w - V)$ is contained in $\Omega(V)$ for all essential values of V ($-w - V \lesssim \epsilon(-w)$) and an analogous situation takes place for thermionic emission, then the distribution functions of the cold-emission current with respect to $-w - V$ and of thermionic current for $E + e^{3/2}F^{1/2}$ for $T = \epsilon(-w)$ coincide (apart from a normalization factor). For the free-electron model, this symmetry was noted by Young][5].

with respect to V, is continuous (in particular, $\chi(V_{Max}) = 0$) with the exception of special cases when the equation of the boundary $\Sigma(E)$ takes the form $P^2/(2m_0) = E - E^0$ and the figure $\Omega(V)$ vanishes jumpwise for a certain value of V (see the Appendix, variants 1b and 3b).

When $T > 0$ the integral (18) is similar in its structure to the expression (1), so that the investigation of $\chi(V($ is similar to the investigation of $\Phi(E)$ in[1] (the relations obtained below are valid, of course, also in the limit when $T = 0$). Formula (18a) is conveniently represented in the form

$$\chi(V) = m_0 \int^{U_{max}(V)} \chi_V(\sqrt{2m_0[\delta(V)+U]})f(E(V)+U)\,dU,$$

where $U_{max}(V) = \mathcal{P}^2_{max}(V)/(2m_0) - \delta(V)$. Owing to the decrease in the Fermi function, an appreciable contribution to the integral can be made only by the interval $0 < U < U_0$, where $U_0 + E(V) + w \sim T$ if $-w - E(V) \gg T$ and $U_0 \sim T$ in all other cases. Consequently, in order of magnitude $\chi(V)/f(E(V)$ is equal to the area of that part of the figure $\Omega(V)$ which lies inside the circle $C(\delta(V) + U_0)$.

If $\Omega(V)$ is so small that $f(E(V) + U_{max}(V)$ $\approx f(E(V))$, then $\chi(V)/f(E(V))$ is equal to the area of $\Omega(V)$. If $\chi_V(\sqrt{2m_0[\delta(V) + U]}) = \text{const}$ in the entire interval $0 < U < U_{max}(V)$, then

$$\chi(V) = m_0\chi_V T\left\{\ln\left[1+\exp\left(\frac{-w-E(V)}{T}\right)\right] - \ln\left[1+\exp\left(\frac{-w-E(V)-U_{max}(V)}{T}\right)\right]\right\} \quad (21)$$

(concrete examples are given in the Appendix). When $U_{max} > U_0$ we have

$$\chi(V) = m_0\chi_V T\ln\{1+\exp\{[-w-E(V)]/T\}; \quad (21a)$$

for this formula to be valid, it is actually sufficient to have $\chi_V(\sqrt{2m_0[\delta(V) + U]}) = \text{const}$ only in the interval $0 < U < U_0$ (for example, in order that the region $\Omega(V)$ include a sufficiently large vicinity of the origin.

If the boundary of $\Omega(V)$ in the section that is essential for $\chi(V)$ can be approximated by arcs of two symmetrical surfaces of radius $\mathcal{R}$ (for which it is necessary, in general, that the quantity $\mathcal{R}(\mathcal{R} + \mathcal{P}_{min})/m_0U_0$ be large), then when $\delta(V) \ll U_0$ the quantity $\chi(V)$ is expressed by formula (21a) with $\chi_V = 2\pi$, and when $\delta(V) \gg U_0$ we have

$$\chi(V) = 4m_0\sqrt{\mathcal{R}(V)/\{\delta(V)[\mathcal{R}(V)+\mathcal{P}_{min}(V)]\}} \times T^{3/2}F_{1/2}([-w-E(V)]/T), \quad (22)$$

where

$$F_{1/2}(v) = \int_0^\infty \frac{s^{1/2}\,ds}{1+e^{s-v}}$$

is a function (introduced in[6]) whose plot is shown in Fig. 7. The asymptotic formulas for $F_{1/2}(V)$ are as follows:

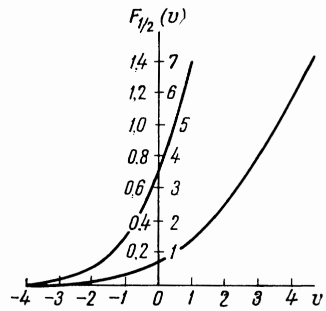

FIG. 7. Plot of the function $F_{1/2}(v)$.

$$F_{1/2}(v) \approx \begin{cases} (\sqrt{\pi}/2)e^v & (-v \gg 1) \\ {}^2/_3 v^{3/2}(1+\pi^2/8v^2) & (v \gg 1) \end{cases}$$

$$F_{1/2}(0) = \left(1 - \frac{1}{\sqrt{2}}\right)\Gamma\left(\frac{3}{2}\right)\zeta\left(\frac{3}{2}\right) = 0.678.$$

We note that

$$\mathcal{R}\,/\,(\mathcal{R}+\mathcal{P}_{min}) = R(1-\Delta')\,/\,(R+P_{min})$$

(the argument V is implied in the left side of the inequality, and the argument E(V) in the right side), where R(E) is the radius of curvature of the boundary of the figure $\Sigma(E)$ at the point $P = P_{min}(E)$. The radii are assumed positive when the curve is convex in the direction of the origin, and if $\mathcal{R} < 0$ then $-\mathcal{R} > \mathcal{P}_{min}$. The case when $-\mathcal{R} = \mathcal{P}_{min}$, i.e., when the internal boundary of $\Omega(V)$ is a circle with center at the origin, is described by the formulas of the Appendix, variants 1c ($V < E_g$) and 3a ($V > E_2$).

Let us separate the temperature-dependent term of the function $\chi(V)$, putting $\chi(V)|_{T=0} = \chi_0(V)$. If $w + E(V) + U_{max}(V) \le 0$, for which it is necessary to have $V \le V_{Max}$ (see (20)), then $\chi(V) - \chi_0(V) \le 0$, and if $w + E(V) \ge 0$, i.e., $V \ge V_{Max}$ (see (20)), then $\chi(V) - \chi_0(V) \ge 0$, $\chi_0(V) = 0$; the equality $\chi(V) - \chi_0(V) = 0$ holds for those values of V, at which $\Omega(V)$ does not exist and $\chi(V) = \chi_0(V) = 0$. The figure $\Omega(V)$ does not exist in the forbidden intervals of V, and if we disregard the higher groups (their contribution is exponentially small for large $[w + E(V_{max} + 0)]/T$ or $[w + E(V_1 + 0)]/T$), then they do not exist for all $V \ge V_{max}$.

The expression for $\chi(V) - \chi_0(V)$ simplifies in the limiting cases $w + E(V) \gg T$ and $-w - E(V) - U_{max}(V) \gg T$, owing to the transformation of the Fermi distribution function into a Boltzmann distribution function (for electrons and holes respectively). If $w + E(V) < 0$, i.e., $V \le V_{Max}$ (see (20)), and $w + E(V) + U_{max}(V) > 0$, then $\chi(V) - \chi_0(V)$ can have any sign.

In the limiting case when

$-w-E(V)\gg T$ and $w+E(V)+U_{max}(V)\gg T$,

we get

$$\chi(V)-\chi_0(V)\approx m_0\frac{\pi^2}{6}T^2\frac{d}{du}\chi_V(\sqrt{2m_0u})\,|_{u=-w-V},\qquad(23)$$

if the function $\chi_V(\sqrt{2m_0u})$ is linearizable in the interval $|u+w+V|\lesssim T$ (see, for example,[7], formula (57.1)); on the other hand, if χ_V is constant in that interval, then $\chi(V)-\chi_0(V)$ is determined by the exponentially small terms which have been left out from (23) (if χ_V is constant for all values of the argument at which $\chi_V\neq 0$, then these terms can be obtained from formula (21)). In the immediate vicinity of the point $V=V_{Max}$, formula (23) is not applicable, for by virtue of the first of the inequalities (20) and the obvious relation[8)] $U_{max}(V_{max}-0)=0$, either the first condition (case 1 of[1]) or the second (cases 2 and 3) condition of its validity is not satisfied.

Figure 8 shows schematically the function $\chi(V)$ near the point $V=V_{Max}$ in typical cases without allowance for the influence of the higher groups. In case of[1], when $V_{Max}=-w-\Delta(-w)<V_{max}$, $\chi(V)$ increases with increasing temperature; with this, formula (21a) is applicable if $\Delta(-w)=0$ (Fig. 8a), and formula (22) is applicable if $\Delta(-w)>0$ (Fig. 8b) (the particular case when $\mathcal{R}=-\mathcal{P}_{min}$ is discussed in the Appendix, variant 3a, $\eta>0$). In cases 2 and 3 of[1], when $V_{Max}=V_{max}>-w-\Delta(-w)$, $\chi(V)$ decreases with increasing temperature (Fig. 8c); cases 3a and 3d, with an effective mass that is isotropic in the projection on the p_xp_y plane, are considered in the Appendix, variant 3a, $\eta>0$, and variant 2, $\zeta<0$.

In concluding this section let us consider the question of the position of the maximum of the distribution function. When $T=0$ the distribution function near $V=V_{Max}$ is proportional in typical cases to $(V_{Max}-V)^se^{-\xi(V)}$; for the cases shown

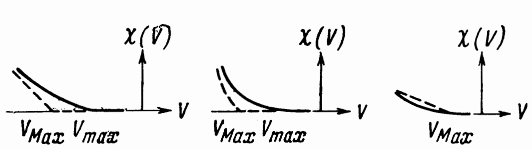

FIG. 8. Schematic plots of the function $\chi(V)$ in the vicinity of the point $V=V_{max}$. The dashed lines are shown for the plots at $T=0$.

[8)]This relation is incorrect only in the general case, when at the point $V=V_{max}$ the figure $\Omega(V)$ vanishes abruptly (see the Appendix, variant 3b, $\eta>0$), but in this case formula (23) is nontheless applicable, since χ_V is constant.

in Figs. 8a, c, and 8b we have $s=1$ and $s=3/2$, respectively. If this dependence extends sufficiently far, then the maximum is reached at $V\approx V_{Max}-s\epsilon(E_M)$. In view of the complexity of the expression for the distribution function with respect to V, it is impossible to trace in general form the variation of the position of its maximum $V_m(T)$ with the increasing temperature. We therefore confine ourselves to the case when formula (21a) with $E(V)\equiv V$ is valid in the entire interval $V_m(0)<V<-e^{3/2}F^{1/2}$ of interest to us (this is the situation, in particular, in the free-electron model).

The investigation of the function $V_m(T)$ is similar in this case to that of $E_m(T)$ in the preceding section. The initial equations determining $V_m(T)$ are

$$(V+w)/\varepsilon(V)=u/(1+e^u)\ln(1+e^{-u}),$$
$$T/\varepsilon(-w+uT)=1/(1+e^u)\ln(1+e^{-u}),$$

and one of them can be replaced by any one of the relations

$$T=(V+w)/u=\varepsilon(V)/(1+e^u)\ln(1+e^{-u}).$$

We see therefore that $V_m(T)$ is a monotonically increasing function. When $T=T_1^{(0)}/(2\ln 2)$, where $T_1^{(0)}=\epsilon(-w)$, $v=0$ and $V=-w$. If $V+w\gg T_1^{(0)}$, i.e., $(T-T_1^{(0)}/T_1^0\gg a^{(0)}$, where $a^{(0)}=\epsilon'(-w)\ll 1$, then $u\gg 1$ and $T\approx\epsilon(V)$ (compare with formulas (11) and (16)). If $|V+w|\ll T_1^{(0)}/a^{(0)}$, i.e., $T<T_1^{(0)}$ or $(T-T_1^{(0)}/T_1^{(0)}\ll 1$, then

$$T\approx T_1^{(0)}/[(1+e^u)\ln(1+e^{-u})-a^{(0)}u].$$

The regions where the two asymptotic formulas are valid overlap, and in the common section we have $T-T_1^{(0)}=a^{(0)}(V+w)$. If $T\ll T_1^{(0)}$, then $-u\gg 1$ and

$$V\approx -w-(1-a^{(0)})T_1^{(0)}[1-\exp\{-(1-a^{(0)})T_1^{(0)}/T\}].$$

3. TEMPERATURE DEPENDENCE OF THE FIELD EMISSION CURRENT

If the temperature is not too high ($T\ll 1/|\omega(-w)|$), and the temperature-dependent term is small compared with the value of the current at $T=0$ ($|j_z-j_{z0}|\ll j_{z0}$), then this term can be calculated in general form from Eq. (1), using formula (57.1) of[7]:

$$j_z-j_{z0}\approx-\frac{2e}{h^3}\frac{\pi^2}{6}T^2\Phi'(-w)$$
$$=-\frac{2e}{h^3}\frac{\pi^2}{6}T^2\Phi(-w)\,\omega(-w).\qquad(24)$$

This expression is valid if the function $\Phi(E)$ is linearizable in the interval $|E+w|\lesssim T$; in the opposite case, when the value of the chemical potential $-w$ is close to the point $E=\tilde{E}_m$, where $\Phi'(E)$ van-

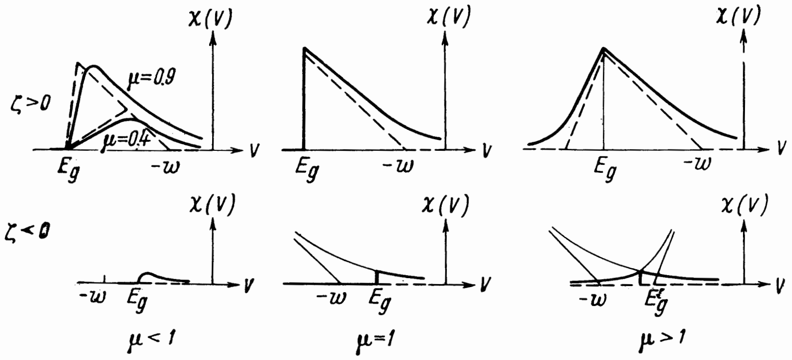

FIG. 9. Schematic diagrams of the function $\chi(V)$ for variant 1 of the Appendix. The dashed lines show the plots for $T = 0$.

ishes, it is necessary to retain that term of (57.1) which is proportional to T^4.

In case 1 of[1] we have $\Phi'(-w) > 0$ and j_z increases with increasing temperature. In cases 2 and 3, $\Phi'(-w)$ can be negative, i.e., the current can decrease with increasing T; incidentally, when $-w - E_M \gg \epsilon$. The temperature term is exponentially small:

$$\frac{j_z - j_{z0}}{j_{z0}} \sim \frac{\Phi(-w)}{\Phi(E_M)} \sim \exp\{-[\xi(-w) - \xi(E_M)]\}.$$

If formula (21a) with $E(V) \equiv V$ is valid in the entire region where the distribution function with respect to V is essentially different from zero (i.e., at $V = W < (1/T - 1/T_1^{(0)})^{-1}$, $-w - V < T_1^{(0)}$), then we can calculate from (17) the temperature dependence of the current in a broader temperature interval $(T_1^{(0)} - T)/T_1^{(0)} \gg \sqrt{a^{(0)}/2}$, in which the function $\xi(V)$ is linearizable (and remains large) until the distribution function falls off. Using the formula

$$\int_{-\infty}^{\infty} e^{U/T_1} \ln(1 + e^{-U/T})\, dU = \pi T_1/\sin(\pi T/T_1) \qquad (T < T_1),$$

we obtain

$$\frac{j_z}{j_{z0}} = \frac{\pi T}{T_1^{(0)}} \Big/ \sin\frac{\pi T}{T_1^{(0)}}; \qquad (25)$$

this result was first obtained by Murphy and Good[8] for the free-electron model.

Similar calculations can be carried out for the several cases considered in the Appendix. They yield the following results:

variants 1a, b:

$$j_z = -\frac{2e}{h^3} 2\pi m_0 T_1^{(0)2} e^{-\xi(-w)} \frac{\pi T}{T_1^{(0)}} \left[\frac{1}{\sin(\pi T/T_1^{(0)})} - \frac{\exp(-\mu\zeta/T_1^{(0)})}{\sin[\pi T(1-\mu)/T_1^{(0)}]}\right],$$

if $(T_1^{(0)} - T)/T_1^{(0)} \gg \sqrt{a^{(0)}/2}$ $(1-\mu)\zeta \gg T_1^{(0)}$ and the corresponding formula of the Appendix for $\chi(V)$ is valid when

$$\mu\zeta + V + w \leqslant \{1/[(1-\mu)T] - 1/T_1^{(0)}\}^{-1};$$

variant 2:

$$j_z = -\frac{2e}{h^3} \cdot 2\pi m_0 T_1^{(0)2} e^{-\xi(-w)} \frac{\pi T}{T_1^{(0)}} \left[\frac{1}{\sin(\pi T/T_1^{(0)})} - \frac{\exp(-\mu\zeta/T_1^{(0)})}{\sin(\pi T/T_\mu)}\right]$$

if $(T_\mu - T)/T_\mu \gg \sqrt{a^{(0)}/2}$, $(T_\mu - T)/T_\mu \gg T/\zeta$ $(T_\mu = T_1^{(0)}/(1+\mu))$ and the corresponding formula of the Appendix for $\chi(V)$ is valid when $-w - V - \mu\zeta < T_1^{(0)}$ (it is assumed that $\mu\zeta < T_1^{(0)}$; in the opposite case we return to expression (25) and to the corresponding validity conditions).

The author is grateful to M. Ya. Azbel' and G. E. Zil'berman for a discussion of the present work.

APPENDIX[9)]

Let us make the formula (21) more concrete for the case when the boundaries of $\Omega(V)$ are circles with center at the origin.

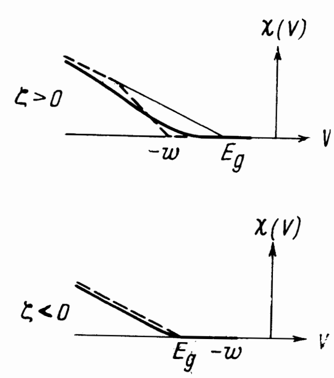

FIG. 10. Schematic plots of the function $\chi(V)$ for the variant 2 of the Appendix. The dashed lines show plots for $T = 0$.

9)Stratton [10] considered a complicated dispersion law, which, however, is not of the most general form, but is one in which the figure Σ (E) encloses the origin. In this case a deviation from that obtained in the free-electron model occurs only for small groups. Stratton's concrete results pertain to the case considered in our Appendix (variants 1 and 2); some of the formulas contain inaccuracies.

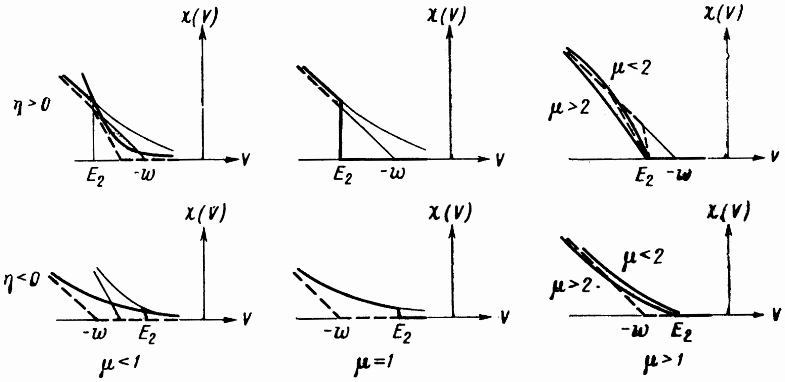

FIG. 11. Schematic plots of the function $\chi(V)$ for variant 3 of the Appendix. The dashed lines shown the plots for T = 0.

1. $\Sigma(E)$ – circle $P^2/(2m) \leq E - E_g$ (electron group; $E_g = E_1$). We introduce the notation $\mu = m/m_0$ and $\zeta = -w - E_g$. We have

a) $\mu < 1$:

$$\chi(V) = 0 \quad (V \leqslant E_g),$$

$$\chi(V) = 2\pi m_0 T \Big\{\ln[1 + \exp\{(-w - V)/T\}] - \ln\Big[1 + \exp\Big\{\Big(\zeta - \frac{V - E_g}{1-\mu}\Big)\Big/T\Big\}\Big]\Big\}$$
$$(V \geqslant E_g);$$

b) $\mu = 1$ (free-electron model):

$$\chi(V) = 0 \quad (V < E_g),$$
$$\chi(V) = 2\pi m_0 T \ln[1 + \exp\{(-w - V)/T\}];$$

graphs of the distribution function, plotted on the basis of this formula for a broader interval of temperatures and fields, are contained in the paper by Dolan and Dyke[9] (see also[2], Sec. 8);

c) $\mu > 1$:

$$\chi(V) = 2\pi m_0 T \times \ln\Big[1 + \exp\Big\{\Big(\zeta - \frac{E_g - V}{\mu - 1}\Big)\Big/T\Big\}\Big] \quad (V \leqslant E_g),$$

$$\chi(V) = 2\pi m_0 T \ln[1 + \exp\{(-w - V)/T\}]$$
$$(V \geqslant E_g).$$

Plots of the function $\chi(V)$ are shown in Fig. 9. For positive ζ we get case 1 of[1], and negative ζ correspond to higher groups.

2. $\Sigma(E)$ – circle $P^2/(2m) \leq E_g - E$ (hole group; $E_g = E_2$). We put $E_g + w = \zeta$. With this

$$\chi(V) = 2\pi m_0 T \Big\{\ln[1 + \exp\{(-w - V)/T\}] - \ln\Big[1 + \exp\Big\{\Big(-\zeta + \frac{E_g - V}{1+\mu}\Big)\Big/T\Big\}\Big]\Big\} \quad (V \leqslant E_g),$$

$$\chi(V) = 0 \quad (V \geqslant E_g).$$

The plots are shown in Fig. 10. When $\zeta > 0$ we get case 1, when $\zeta < 0$ we get case 3d of[1].

3. $\Sigma(E)$ – region $P^2/(2m) \geq E - E_2$ (region outside circle). We put $-w - E_2 = \eta$. We get

a) $\mu < 1$:

$$\chi(V) = 2\pi m_0 T \ln[1 + \exp\{(-w - V)/T\}] \quad (V \leqslant E_2),$$

$$\chi(V) = 2\pi m_0 T \ln\Big[1 + \exp\Big\{\Big(\eta - \frac{V - E_2}{1-\mu}\Big)\Big/T\Big\}\Big] \times (V \geqslant E_2);$$

b) $\mu = 1$:

$$\chi(V) = 2\pi m_0 T \ln[1 + \exp\{(-w - V)/T\}] \quad (V < E_2),$$
$$\chi(V) = 0 \quad (V \geqslant E_2);$$

c) $\mu > 1$:

$$\chi(V) = 2\pi m_0 T \Big\{\ln[1 + \exp\{(-w - V)/T\}] - \ln\Big[1 + \exp\Big\{\Big(\eta - \frac{E_2 - V}{\mu - 1}\Big)\Big/T\Big\}\Big]\Big\} \quad (V \leqslant E_2),$$

$$\chi(V) = 0 \quad (V \geqslant E_2).$$

The plots are shown in Fig. 11. When $\eta > 0$, $\mu < 0$ and when $\eta < 0$ we get case 1 of[1]; when $\eta > 0$ and $\mu = 1$, the value $\max_{E \leq -w} [E - \Delta(E)]$ is attained in the entire interval $E_2 \leq E \leq -w$ (this unlikely case is not considered separately in[1]; when $\eta > 0$ and $\mu > 1$ we get case 3a of[1].

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