THE ASYMPTOTIC FORM OF THE ELASTIC SCATTERING AMPLITUDE AND THE RATIO OF ITS REAL AND IMAGINARY PARTS

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Inequalities for an asymptotically not pure imaginary amplitude f(E) are obtained which improve on the bound of Froissart. These imply under some weak assumptions that Re f(E) < 0 asymptotically. A relation between the total cross section $\sigma(E)$ and the ratio $\xi(E)$ of the imaginary and real parts of the amplitude established earlier with power accuracy, is extended to factors which vary slowly with energy. It is shown that the cross section becomes constant when $|\xi(E)| > (\ln E)^{1+\epsilon}$. The relation between the phase and the modulus of the elastic scattering amplitude is considered for arbitrary momentum transfers.

KHURI and Kinoshita[1] and afterwards the author[2] have established that there exists a close connection between the asymptotic forms of the forward elastic scattering amplitude f(E) [$f(E) = f_+(E) + f_-(E)$, where $f_+(E)$ is the scattering amplitude for a particle, and $f_-(E)$ for an antiparticle] and the ratio of its real and imaginary parts

$$\xi(E) = \frac{\operatorname{Im} f(E)}{\operatorname{Re} f(E)}. \tag{1}$$

In the present paper we continue the investigation of this connection, which is also considered for arbitrary momentum transfer $\sqrt{-t}$.

1. THE FROISSART INEQUALITY AND THE IMAGINARY PROPERTY OF THE AMPLITUDE

In this section we show that the Froissart inequality[3] 1)

$$|f(E)| \leqslant CE(\ln E)^2, \tag{2}$$

derived recently by Martin[4] in the axiomatic approach, can be considerably improved if f(E) is not pure imaginary asymptotically, i.e., if 2)

$$\operatorname{tg}\pi\alpha \leqslant |\xi(E)| \leqslant \operatorname{tg}\pi\nu, \qquad \pi/2 > \nu \geqslant 0, \qquad \pi/2 > \alpha \geqslant 0. \tag{3}$$

More specifically, we show that

1) for $\xi(E) < 0$ the integral

$$\int_E^\infty \frac{|f(E')|\,dE'}{E'^{1+2\nu+\varepsilon}}, \tag{4}$$

converges, where ϵ is, here and in the following, an infinitesimal positive quantity;

2) for $\xi(E) > 0$ the integral

$$\int_E^\infty \frac{|f(E')|\,dE'}{E'^{1-2\alpha+\varepsilon}}. \tag{5}$$

converges.

It follows immediately from the convergence of the integrals (4) and (5) that 3)

$$|f(E_i)| < CE_i^{2\nu+\varepsilon} \quad \text{for } \xi(E) < 0, \tag{6}$$

$$|f(E_i)| < CE_i^{-2\alpha+\varepsilon} \quad \text{for } \xi(E) > 0. \tag{7}$$

for an infinite sequence of energy values E_i ($E_i \to \infty$ for $i \to \infty$). Owing to the continuousness of f(E), the inequalities (6) and (7) will also hold in some neighborhood of the points E_i. It is easy to see that there are such points E_i located inside each interval from E to E + δE, where the constant δ can be chosen arbitrarily small.

It is essential to emphasize that in deriving formulas (4) and (5), no assumptions have been made on the behavior of f(E) except those which are always made in the axiomatic method. [In [1,2] it was assumed that there are no strong oscillations in

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1) For simplicity, all constants will be denoted by the same letter C.

2) Here and in the following, all relations hold at least for large E.

3) We emphasize that both formulas can only be obtained if $\nu \neq 1/2$. It is true that for $\xi(E) < 0$ the transition $\nu \to 1/2$ (the Froissart case) in (6) is continuous. But for $\xi(E) > 0$, even a small deviation from a pure imaginary amplitude $|\nu \neq 1/2|$ changes the formula discontinuously: even for $\alpha = 0$ the difference between formulas (7) and (2) is very large.

f(E).] These assumptions are considered in detail in Appendix II.

From the results obtained, the most interesting is the inequality (7), which implies that for $\xi(E) > 0$ there must exist an infinite sequence of energies E_i for which the total cross section $\delta(E) \sim \mathrm{Im}\, f(E)/E$ decreases more rapidly than $1/E$. However, it has been shown by Martin[5] under rather weak assumptions that

$$|f(E)| > C. \tag{8}$$

Relation (7) evidently is in contradiction with this inequality; and the experiments on strong interactions also practically exclude such a strong decrease of the cross section.

Finally we note that the inequality of Martin (8) can also be improved; viz., it can be shown that for $\xi(E) < 0$ the integral

$$\int_E^\infty \frac{f(E')\,dE'}{E'^{1+2\alpha-\varepsilon}} \tag{9}$$

diverges, i.e.,

$$|f(E_i)| > CE_i^{2\alpha-\varepsilon}. \tag{10}$$

For $\xi(E) > 0$ we obtain from (8), in accordance with (7), that $|f(E_i)| > CE_i^{2-2\alpha-\epsilon}$. This agrees with (2) only if $\alpha = 1/2$. [Indeed, to find these results it suffices to assume that $|f(E)| > CE^{-2\alpha+\epsilon}$.] For a pure imaginary amplitude, $|\alpha| = |\nu| = 1/2$, the inequality (10) yields

$$|f(E_i)| > CE_i^{1-\varepsilon}. \tag{10a}$$

Let us proceed to the proof of these results. For the proof of the convergence of the integrals (4) and (5) we construct the auxiliary function

$$\Phi(E) = e^{i\pi(1+\varepsilon/2)} \int_E^\infty \frac{f(E')\,dE'}{E'^{2+\varepsilon}}. \tag{11}$$

It is easy to see from (11) and (3) that

$$\xi_{\Phi(E)} = \mathrm{Im}\,\Phi(E)/\mathrm{Re}\,\Phi(E) \tag{12}$$

satisfies the condition

$$\mathrm{ctg}\,\pi(\alpha-\varepsilon_1) > |\xi_{\Phi(E)}| > \mathrm{ctg}\,\pi(\nu+\varepsilon_1) = \mathrm{tg}\,\pi(1/2-\nu-\varepsilon_1), \tag{13}$$

where ϵ_i is infinitesimal.

The function $\Phi(E)$ is analytic everywhere in the upper half-plane except for a semicircle with a finite radius about the origin. Using the inequality (2) and the Phragmén-Lindelöf theorem (cf., e.g., [6, 7]) we find that $\Phi(E) \to 0$ for $E \to \infty$ everywhere in the upper half-plane. It can be shown that $\Phi(E) \neq 0$ since[1] $f(E) \neq 0$ and $\mathrm{Im}\,(f(E)E^{-1} \geq 0$. Using the crossing symmetry of the function f(E) (cf. e.g., [1])

$$f^*(-E+i0) = f(E+i0), \tag{14}$$

it is easy to see that $\Phi(E)$ also has this property:

$$\Phi^*(-E+i0) = \Phi(E+i0). \tag{15}$$

We now show that there are no strong oscillations in $\Phi(E)$, i.e., that

$$\left|\frac{\Phi(E_2)}{\Phi(E_1)}\right| < N \quad \text{for} \quad E_2 > E_1, \tag{16}$$

where N is some constant number.[4)] Indeed, using (13) and the condition $\mathrm{Im}\, f(E) > 0$, we easily obtain

$$\left|\frac{\Phi(E_2)}{\Phi(E_1)}\right| < 1. \tag{17}$$

The above-enumerated properties of the function $\Phi(E)$ [formulas (13), (15), and (16)] allow us to apply to them Meĭman's theorem[8, 1] according to which $\Phi(E)$ must satisfy the inequality

$$|\Phi(E)| \leqslant CE^{-\gamma}, \quad \gamma = 1/2[1/2 - \nu - \varepsilon_1]. \tag{18}$$

This is not sufficient for us. However, we can strengthen this inequality. To this end we consider a new auxiliary function

$$\Phi_1(E) = e^{i\pi(1-\gamma_1/2)} \int_E^\infty \frac{f(E')\,dE'}{E'^{2-\gamma_1}}, \quad \gamma_1 < \gamma. \tag{19}$$

Integrating by parts, we have

$$\int_E^{\widetilde{E}} \frac{f(E')\,dE'}{E'^{2-\gamma_1}} = -\Phi(\widetilde{E})\,\widetilde{E}^{\gamma_1+\varepsilon} + \Phi(E)E^{\gamma_1+\varepsilon} + (\gamma_1+\varepsilon) \times \int_E^{\widetilde{E}} \Phi(E')\,E'^{\gamma_1-1+\varepsilon}\,dE'. \tag{20}$$

Using the inequality (18) we find that $\Phi_1(E) \to 0$ for $E \to \infty$.

Applying now the same considerations to $\Phi_1(E)$ as we did to $\Phi(E)$ [cf. the derivation of (13), (15), and (16)] we can easily show that the Meĭman theorem is also applicable to $\Phi_1(E)$, so that we obtain an inequality for $\Phi_1(E)$ analogous to (18) for $\Phi(E)$. Constructing in a similar way a sequence of functions $\Phi_i(E)$ and treating them in a fashion analogous to that employed in [2] for the sequence of functions $\omega_i(E)$, we finally obtain the desired inequalities (4) and (5). The divergence of the integral (9) is shown in the same way.

Using the method of [2] we can easily prove the inequalities (6), (7), and (10) for all asymptotic en-

4)Thus we prove the absence of strong oscillations in $\Phi(E)$ independently of whether such oscillations occur in f(E) or not. This is why we have introduced $\Phi(E)$ instead of f(E). In [2] it was necessary to assume the absence of strong oscillations in f(E) (cf. below).

ergies, but we must here make the additional assumption that there are no strong oscillations in f(E) (cf. Appendix II). On the basis of the considerations of the present paper we may also say that the inequalities obtained in [2] hold without any assumptions about the absence of strong oscillations in f(E), if they are referred to an infinite sequence of values E_i.

2. SLOWLY VARYING FACTORS IN THE SCATTERING AMPLITUDE

It has been shown earlier [cf. formulas (35a) to (39b) of [2]] that if the limit of $\xi(E)$ exists, it satisfies the relation

$$\lim_{E\to\infty}\xi(E)=\lim_{E\to\infty}\frac{\operatorname{Im}f(E)}{\operatorname{Re}f(E)}=-\operatorname{tg}\frac{\pi\beta}{2}. \quad (21)$$

The amplitude f(E) has the form of a power multiplied by a slowly varying function[5] $\psi(E)$:

$$f(E)=\psi(E)E^{\beta}\left[1-i\operatorname{tg}\frac{\pi\beta}{2}\right] \text{ as } E\to\infty, \quad (22)$$

$$E^{-\varepsilon}\leqslant|\psi(E)|\leqslant E^{\varepsilon}. \quad (23)$$

[According to [2], $\beta\leq 1$, where $|\psi(E)| < (\ln E)^2$ for $\beta = 1$.]

We investigate the properties of $\psi(E)$ in this limit, starting from the requirement of crossing symmetry. As will be easily seen, this consideration holds for arbitrary β. For simplicity, however, we set $\beta = 1$. [This case is also of the most interest—here $\psi(E)$ yields immediately the cross section.] Then

$$f(E)=iE\psi(E), \quad \sigma(E)=4\pi\operatorname{Re}\psi(E). \quad (24)$$

Asymptotically we have, according to (22) and (23),

$$\tilde{\xi}(E)=\frac{\operatorname{Im}\psi(E)}{\operatorname{Re}\psi(E)}\to 0 \text{ as } E\to\infty. \quad (25)$$

By establishing a connection between Im $\psi(E)$ and Re $\psi(E)$, we obtain certain restrictions on the asymptotic form of the cross section depending on the behavior of Re f(E). Thus, for example, it turns out that $\sigma(E)$ is asymptotically constant if

$$|\operatorname{Re}f(E)|<E(\ln E)^{-(1+\varepsilon)}. \quad (26)$$

We show further that Re f(E) itself must satisfy an inequality stronger than the Froissart inequality for $|f(E)|$, viz.,

$$|\operatorname{Re}f(E_i)|<CE_i\ln E_i \quad (27)$$

(E_i is an infinite sequence of energies).

We base our considerations on the assumption that there are no strong oscillations in $\psi(E)$, i.e., we assume that[6]

$$|\psi(E_2)|E_2^{-\varepsilon}<|\psi(E_1)|E_1^{-\varepsilon},$$
$$|\psi(E_2)|E_2^{\varepsilon}>|\psi(E_1)|E_1^{\varepsilon}, \quad E_2>E_1. \quad (28)$$

According to (23), $|\psi(E)|E^{-\epsilon}\to 0$, and $|\psi(E)|E^{\delta}\to\infty$ as $E\to\infty$. With the assumption (28), we show that $\psi(E)$ has asymptotically one and the same value in the entire upper half-plane, i.e., that

$$\frac{\psi(Ee^{i\varphi})}{\psi(E)}\to 1 \quad \text{as } E\to\infty; \quad \pi\geqslant\varphi\geqslant 0. \quad (29)$$

From assumption (28) we easily find that

$$\left|\frac{d\ln|\psi(E)|}{dE}\right|\equiv\left|\frac{d\operatorname{Re}\ln\psi(E)}{dE}\right|<\frac{\varepsilon}{E}. \quad (30)$$

Let us also assume that the phase $\varphi(E)$ of the function $\psi(E)$ satisfies the analogous condition[7]

$$\left|\frac{d\varphi(E)}{dE}\right|\equiv\left|\frac{d\operatorname{Im}\ln\psi(E)}{dE}\right|<\frac{\varepsilon}{E}. \quad (31)$$

This is a very weak condition which says that $\varphi(E) + \epsilon\ln E$ increases monotonically and $\varphi(E) - \epsilon\ln E$ decreases monotonically. However, according to (23), $\varphi(E)$ has the form $\varphi(E) = \pi n + \varphi'(E)$, where n is some constant integer (it can be shown that n = 0), and $\varphi'(E)\to 0$. Hence, only very strong (and therefore, physically not probable) oscillations of $\varphi(E)$ can lead to a violation of condition (31).

It follows from (30) and (31) that

$$\left|\frac{d\ln\psi(E)}{dE}\right|<\frac{\varepsilon}{E}. \quad (32)$$

Since according to (14), $\psi(E)$ has crossing symmetry,

$$\psi^*(-E+i0)=\psi(E+i0), \quad (33)$$

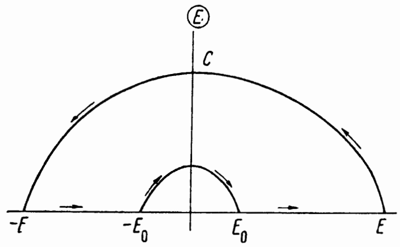
we have for $E < 0$

$$\left|\frac{d\ln\psi(E)}{dE}\right|<\frac{\varepsilon}{|E|}. \quad (34)$$

[5]Relation (22) was obtained in [2] under the assumption that there are no strong oscillations in f(E). However, it cannot be asserted that this is a necessary condition. The class of functions f(E) satisfying (22) may be wider than that considered in [2] (cf. also Appendix II).

[6]However, a number of our results can also be obtained by more general arguments (cf. Appendices I and II).

[7]This requirement is evidently superfluous, but we shall introduce it in order not to complicate the proof.



Applying the Phragmén-Lindelöf theorem to the function $E d \ln \psi(E)/dE$ and assuming that

$$\left| \frac{d \ln \psi(E)}{dE} \right| < e^{\varepsilon |E|}, \qquad (35)$$

we find that in the entire complex plane

$$\left| \frac{d \ln \psi(E)}{dE} \right| < \frac{\varepsilon}{|E|}. \qquad (36)$$

Then, taking account of the properties of the analytic function $\ln \psi(E)$ [and also using (36)],

$$\left| \frac{d \ln \psi(Ee^{i\varphi})}{d\varphi} \right| = \left| E \frac{d \ln \psi(Ee^{i\varphi})}{dE} \right| < \varepsilon; \quad \pi \geqslant \varphi \geqslant 0, \qquad (37)$$

we easily obtain

$$\left| \ln \frac{\psi(Ee^{i\varphi})}{\psi(E)} \right| = \left| \int_0 \frac{d \ln \psi(Ee^{i\varphi'})}{d\varphi'} d\varphi' \right| < \varepsilon\varphi, \qquad (38)$$

i.e., the desired result (29). This property of $\psi(E)$ allows us to establish a connection between $\operatorname{Im} \psi(E)$ and $\operatorname{Re} \psi(E)$, as was our aim.

Let us consider the integral of $\psi(E)E^{-1}$ along the contour C (cf. the figure). Since $\psi(E)E^{-1}$ has no singularities inside the contour, we have

$$\int_C \frac{\psi(E')\,dE'}{E'} = 0. \qquad (39)$$

Using (31) and the mean value theorem, we obtain

$$\int_0^\pi \operatorname{Re} \psi(Ee^{i\varphi})\,d\varphi = \pi \operatorname{Re} \psi(Ee^{i\tilde{\varphi}}) = -2 \int_{E_0}^{E} \frac{\operatorname{Im} \psi(E')}{E'} dE' + C, \qquad (40)$$

where

$$\pi \geqslant \tilde{\varphi} \geqslant 0; \quad C = \int_0^\pi \operatorname{Re} \psi(E_0 e^{i\varphi})\,d\varphi.$$

Using (29), we obtain finally

$$\pi K(E) \operatorname{Re} \psi(E) = -2 \int_{E_0}^{E} \frac{\operatorname{Im} \psi(E')}{E'} dE' + C;$$
$$K(E) \to 1 \quad \text{as} \quad E \to \infty. \qquad (41)$$

For $\beta = 1$, (41) can be rewritten in the form [cf. (24) and (26)]

$$\sigma(E) = \frac{1}{2\pi^2 K(E)} \int_{E_0}^{E} \frac{\operatorname{Re} f(E')\,dE'}{E'^2} + C. \qquad (41a)$$

This equation is the main result of this section. From it we can derive a number of consequences. First of all it is seen that $\sigma(E) \to$ const if the integral

$$\int_{E_0}^{E} \frac{\operatorname{Re} f(E')\,dE'}{E'^2}$$

converges, which is evidently the case if

$$|\operatorname{Re} f(E)| < CE (\ln E)^{-(1+\varepsilon)},$$

or even

$$|\operatorname{Re} f(E)| < CE (\ln E)^{-1} (\ln \ln E)^{-1} \ldots (\ln \ldots \ln E)^{-(1+\varepsilon)}. \qquad (42)$$

Thus we can make assertions about the asymptotic form of the cross section by investigating experimentally the behavior of $\operatorname{Re} f(E)$. Furthermore, it is easy to see that the inequality (27) must be satisfied for the validity of (2).

By applying considerations analogous to the ones underlying the derivation of (40) to the function $\ln \psi(E)$, we obtain an equation which connects the modulus and the phase of $\psi(E)$:

$$|\psi(E)| = CK'(E) \exp\left(-\frac{2}{\pi} \int_{E_0}^{E} \frac{\operatorname{Im} \ln \psi(E')\,dE'}{E'} \right)$$
$$\cong C \exp\left(-\frac{2}{\pi} \int_{E_0}^{E} \frac{\tilde{\xi}(E')\,dE'}{E'} \right), \qquad (43)$$

where $K'(E) \to 1$ for $E \to \infty$, in analogy to $K(E)$ in (41). To arrive at the last equation we have used [cf. (25)]

$$\operatorname{Im} \ln \psi(E') \equiv \operatorname{arctg} \tilde{\xi}(E') \to \tilde{\xi}(E').$$

For $\beta = 1$ [cf. (21)] this equation connects the cross section with the quantity $\xi(E)$ measured in experiment:

$$\sigma(E) \cong C \exp\left(\frac{2}{\pi} \int_{E_0}^{E} \frac{dE'}{\xi(E')E'} \right). \qquad (43a)$$

By estimating the integral of (43a), we arrive at the following conclusions:

1) The behavior of $\sigma(E)$ depends essentially on the sign of $\xi(E)$:

$$\sigma(E) \geqslant \text{const} \quad \text{for} \quad \xi(E) > 0, \qquad (44)$$

$$\sigma(E) \leqslant \text{const} \quad \text{for} \quad \xi(E) < 0. \qquad (45)$$

For a more specific behavior of the function $\xi(E)$ we have

2) $\sigma(E) \to$ const if the integral (43a) converges, which is evidently the case when $|\xi(E)| > (\ln E)^{1+\epsilon}$ or even when

$$|\xi(E)| > \ln E \ln \ln E \ldots (\ln \ldots \ln E)^{1+\varepsilon}. \qquad (46)$$

3) $(\ln E)^{2b/\pi-\varepsilon} < \sigma(E) < (\ln E)^{2b/\pi+\varepsilon}$ for $\xi(E) \to \ln E/b$. (47)

4) $(\ln\ldots\ln E)^{2b/\pi-\varepsilon} < \sigma(E) < (\ln\ldots\ln E)^{2b/\pi+\varepsilon}$ (48)

for $\xi(E) \to b^{-1}\ln E\ldots(\ln\ldots\ln E)$.

5) For positive ξ(E) the general inequality[8)]

$$\xi(E_i) > \pi^{-1}\ln E_i. \tag{49}$$

holds. Otherwise, σ(E) does not satisfy the Froissart inequality. We note that the existing experimental data on pp and $\pi^{\pm}$p scattering seem to indicate that $\xi(E) < 0$, at least in the region $2 < E < 30$ BeV. The way in which $|\xi(E)|^{-1}$ drops off is difficult to determine with present experimental accuracies.

3. SCATTERING AT $t \neq 0$

In this section we consider the elastic scattering amplitude A(s, t) for nonvanishing t [s is equal to the square of the energy of the colliding particles in the c.m.s.; for t = 0 the amplitude A(s, 0) reduces to f(E)] and investigate how the asymptotic form of |A(s, t)| in s (for fixed t) depends on the asymptotic phase of the amplitude A(s, t). We shall again use crossing symmetry (cf. e.g., [6]):

$$A^*(-s-d+i0,t) = A(s+i0,t), \quad d = 2\mu^2+2m^2-t \tag{50}$$

(μ and m are the masses of the colliding particles). It is convenient to consider A(s, t) as a function of the variables s' and t:

$$A(s,t) = \tilde{A}(s',t), \quad s' \equiv s - d/2. \tag{51}$$

Then condition (50) yields

$$\tilde{A}^*(-s'+i0,t) = A(s'+i0,t) \tag{52}$$

(the prime on s will be omitted in the following).

We assume that A(s, t) is limited from below by the inequality[9)]

$$|\tilde{A}(s,t)| > C\exp\{-s^{2-\varepsilon}\}, \tag{53}$$

which is a very weak restriction. [From the double dispersion relations of Martin[5] we obtain a much stronger restriction than (53), viz.,

$$|\tilde{A}(s,t)| > C\exp\{-b(t)\sqrt{s}\ln s\}, \tag{54}$$

where b(t) is some function of t.]

We show that there exists a close connection between the asymptotic phase and modulus of A(s, t). Namely, if the modulus of the phase increases comparatively slowly then A(s, t) (at least for an infinite number of values s_i) can also only decrease relatively slowly—somewhat more slowly than indicated by the inequality (53) [or even the inequality (54)]; on the other hand, a rapid increase of the modulus of the phase leads in general to a rapid decrease of |A(s, t)|. We show in particular, that if the phase of A(s, t) is bounded, then |A(s, t)| can decrease no more rapidly than some power of s.

For the proof we write A(s, t) in the form

$$\tilde{A}(s,t) = e^{a(s,t)}e^{i\pi n}. \tag{55}$$

From the crossing symmetry of the function A(s, t) we find that with an appropriate choice of n

$$a^*(-s+i0,t) = a(s+i0,t). \tag{56}$$

The quantity Im a(s, t) will, for simplicity, be called the phase of A(s, t).

Let us first consider the case where Im a(s, t) increases, if at all, relatively slowly:

$$|\mathrm{Im}\, a(s,t)| < s^{\varepsilon} \tag{57}$$

and let us show that in this case

$$|\tilde{A}(s,t)| > Ce^{-s^{\varepsilon}}, \tag{58}$$

if there are no strong oscillations in A(s, t). Moreover, the phase and modulus of A(s, t) are connected by the relation

$$|\tilde{A}(s,t)| = C\exp\left(-\frac{2}{\pi K(s,t)}\int_{s_0}^{s}\frac{\mathrm{Im}\, a(s',t)\,ds'}{s'}\right), \tag{59}$$

where K(s, t) → 1 for s → 1; s_0 is a constant.

The inequality (58), i.e., the inequality |Re a(s, t)| < s^{ϵ} will be proved by contradiction: if Re a(s, t) > s^{ϵ_1} for some $\epsilon_1 > 0$, then a(s, t) must be purely real asymptotically, owing to (57). However, by the inequality (53) and using the method of [2] (cf. also Sec. 1 of the present paper) we easily obtain the inequality $|a(s_i, t)| < Cs_i^{\epsilon}$, which is in contradiction with our initial assumption. Hence the inequality

$$\mathrm{Re}\, a(s_i,t) < Cs_i^{\varepsilon} \tag{60}$$

must hold (s_i is, as always, an infinite sequence of values of s). In the absence of strong oscillations (60) leads to the desired inequality (58).

To obtain (59) it suffices to use the fact that it is equivalent to the equation

$$K(s,t)\,\mathrm{Re}\, a(s,t) = -\frac{2}{\pi}\int_{s_0}^{s}\frac{\mathrm{Im}\, a(s',t)\,ds'}{s'}, \tag{61}$$

which is analogous to eq. (14) for the function ψ(E) and is derived in exactly the same way. Regarding a(s, t) we must of course make the same assumptions as for ψ(E) [cf. Sec. 2, formulas (28) and (30)].

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8)Formulas (44) to (49) may be obtained from the more general equation (A. II. 4) (cf. Appendix II).

9)In this section, all constants are in general functions of t.

It follows from (59) that if Im a(s, t) is bounded, then

$$C_2 < \operatorname{Im} a(s,t) < C_1, \tag{62}$$

$$s^{(-2C_2+\varepsilon)/\pi} > |\tilde{A}(s,t)| > Cs^{(-2C_1-\varepsilon)/\pi}. \tag{63}$$

Hence in this case $|A(s, t)|$ can decrease no more rapidly than by a power law.

Let us now consider the case where the modulus $|\operatorname{Im} a(s, t)|$ increases rapidly. Here it can be shown that the inequality

$$|\operatorname{Im} a(s,t)| > s^{\rho}, \quad \rho \neq 2n+1, \quad n \text{ — integer} \tag{64}$$

leads to the inequality

$$|\operatorname{Re} a(s_i,t)| > \varepsilon s_i^{\rho}. \tag{65}$$

The case $\rho = 2n + 1$ is excluded; here nothing can be said about the quantity Re a(s, t). The proof of (65) is carried out in the same way as the proof of (60).

We can also prove the opposite inequalities, for example, if $|A(s, t)|$ decreases no more rapidly than by a power law, then either the phase of A(s, t) is bounded or its modulus increases at most $\sim s$.

Finally, we consider the case where the phase Im a(s, t) approaches a limit for $s \to \infty$:

$$\operatorname{Im} a(s,t) \to -\pi\beta(t)/2 \quad \text{as} \quad s \to \infty. \tag{66}$$

We then find easily from (65) (here $C_1 = C_2$) that A(s, t) has a "Regge" form, as it were:

$$A(s,t) = \psi(s,t)\, s^{\beta(t)}\left[1 - i\,\mathrm{tg}\,\frac{\pi\beta(t)}{2}\right], \tag{67}$$

where $\psi(s, t)$ is a slowly varying function of s:

$$Cs^{-\varepsilon} < |\psi(s,t)| < Cs^{\varepsilon}. \tag{68}$$

The functions $\psi(s, t)$ can be treated in the same way as in Sec. 2. In particular, we find that

$$\psi(s,t) \to R(t) \tag{69}$$

for

$$\left|\frac{\operatorname{Im}\psi(s,t)}{\operatorname{Re}\psi(s,t)}\right| < \frac{C}{(\ln E)^{1+\varepsilon}}. \tag{70}$$

In this case [i.e., when (70) is fulfilled] the asymptotic form of A(s, t) is therefore determined by poles in the l plane.

In conclusion I express my deep gratitude to E. L. Feĭnberg for his constant interest in this work and valuable remarks.

APPENDIX I

We show that the inequalities (46) to (49) are fulfilled for an infinite sequence of points E_i without assumption (28). Let us derive, for example, the inequality (46), i.e., let us show that $\sigma(E) \to$ const if the integral

$$\int_{E_0}^{E} \frac{\operatorname{Im}\ln\psi(E')}{E'}\,dE'.$$

converges.

Indeed, we have [cf. (40) and (43)]

$$\operatorname{Re}\ln\psi(Ee^{i\tilde{\varphi}}) = -\frac{2}{\pi}\int_{E_0}^{E}\frac{\operatorname{Im}\ln\psi(E')\,dE'}{E'} + C, \tag{AI.1}$$

where $\tilde{\varphi}(E)$ is determined by the mean value theorem (40).[10] According to our assumption about the behavior of Im ln ψ(E) we have

$$\int_{E_0}^{E}\frac{\operatorname{Im}\ln\psi(E')}{E'}\,dE' \to \tilde{C} \quad \text{as} \quad E \to \infty. \tag{AI.2}$$

We show that then there exists an infinite sequence of energies E_i ($E_i \to \infty$ for $i \to \infty$) for which

$$\operatorname{Re}\ln\psi(E_i) \to C + \tilde{C}, \text{ i.e., } |\psi(E')| \to \exp(C + \tilde{C}). \tag{AI.3}$$

For the proof of (AI.3) we show that none of the following two inequalities is fulfilled:

$$|\psi(E)| > \exp(C + \tilde{C} + \varepsilon), \tag{AI.4}$$

$$|\psi(E)| < \exp(C + \tilde{C} - \varepsilon). \tag{AI.5}$$

Assume, for example, that (AI.4) is satisfied. We construct $\psi'(E) = \psi(E)^{-1}$. According to (AI.4) and (31), $\psi'(E)$ is bounded on the real axis. By the Phragmén-Lindelöf theorem[6] it is bounded in the whole upper half-plane. However, according to another theorem of Phragmén and Lindelöf,[6] $\psi'(E)$ can be bounded only if the manifolds of limit values of $\psi'(E)$ for $E \to \infty$ on the curve L ($L = E\,e^{i\tilde{\varphi}}$) and on the real axis either intersect or surround one another. This is not true in our case because of (AI.2) and (AI.4). Hence the inequality (AI.4) cannot be satisfied.

In an analogous manner one proves the impossibility of satisfying (AI.5) [the same considerations must here be applied to $\psi(E)$]. Hence there must exist an infinite sequence of E_i for which the double inequality

$$\exp(C + \tilde{C} - \varepsilon) < |\psi(E_i)| < \exp(C + \tilde{C} + \varepsilon), \tag{AI.6}$$

holds, i.e., for which (AI.3) is satisfied. Hence $\sigma(E_i) \to \exp(C + \tilde{C})$ for $\beta = 1$ [cf. (21)].

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[10] Here we assume for simplicity that $\tilde{\varphi}(E)$ is continuous in E. However, the results remain practically unchanged if this is not the case.

If we assume from the outset that $|\psi(E)|$ has a (fine or infinite) limit, then we find that $|\psi(E) \to \exp(C + \tilde{C})$. Therefore we have $\sigma(E) \to \exp(C + \tilde{C})$ for $\beta = 1$. The remaining inequalities, i.e., the inequalities (47) to (49) for E_i, are obtained by the same method.

APPENDIX II

The only additional assumption made by us regarding the scattering amplitude f(E) is the assumption that there exist no strong oscillations in f(E). Therefore the question of the character of the admissible oscillations in f(E) is extremely important for our considerations. It was recently brought to our attention by N. N. Meĭman that in [2] it was in effect assumed that there are no strong oscillations in each of the infinite number of auxiliary functions $\omega_i(E)$ used in the derivation, which depreciates the value of the proof. However, when this condition is reformulated in a certain way, the significance of the results obtained is reestablished. We can say that the following assertion is proved in essence in [2] [cf. (4)]: if λ is the lowest number such that $|f(E)| < CE^{\lambda+\epsilon}$ and the oscillations of f(E) are restricted by the condition

$$\left|\frac{f(E_2)}{f(E_1)}\right| < C\left(\frac{E_2}{E_1}\right)^{\lambda+\varepsilon}, \quad E_2 > E_1, \tag{AII.1}$$

then $\lambda \leq 2 - 2\alpha$. With this single restriction on the oscillations, the required restrictions on all functions $\omega_i(E)$ are automatically obeyed. The other results of [2] can also be reformulated in an analogous fashion.

However, it can be shown that actually the results of [2] and formulas (6), (7), and (10) of the present paper remain valid even under a much weaker condition on the oscillations. Indeed, it can be shown that the Meĭman inequality (on which the discussion in [2] was based) is satisfied for any of the auxiliary functions $\omega_i(E)$ if only a number N can be found (which may be arbitrarily large) such that $|f(E)E^N|$ ($|f(E)E^{-N}|$) increases (decreases) monotonically.

Indeed, applying to f(E) considerations analogous to the ones applied to $\psi(E)$ in deriving (40), we easily find that the following double inequality is fulfilled for $E \to \infty$:[11)]

$$M^{-1} < \left|\frac{f(Ee^{i\varphi})}{f(E)}\right| < M, \tag{AII.2}$$

where M is some number. An analogous relation will also hold for any of the auxiliary functions $\omega_i(E)$.

However, it follows from the results of [9] (cf. p. 708) that one has, independently of the character of the oscillations of the function $\omega_i(E)$,

$$|\omega_i(Ee^{i\gamma})| \leqslant CE^{\varkappa_i/2}, \quad \text{if} \quad \left|\frac{\operatorname{Im}\omega_i(E)}{\operatorname{Re}\omega_i(E)}\right| \geqslant \operatorname{tg}\pi\varkappa_i \tag{AII.3}$$

on some curve L ($L \equiv E\,e^{i\gamma}$) in the complex plane. The inequalities (AII.2) and (AII.3) then yield the required Meĭman inequality.

Using (AII.3), we also easily find a relation for $\psi(E)$:

$$\operatorname{Re}\ln\psi(E) = -\frac{2}{\pi}\int_{E_0}^{E}\frac{\operatorname{Im}\ln\psi(E')}{E'}dE' + g(E), \tag{AII.4}$$

where g(E) is a bounded function. This equation, just as (43), leads to the inequalities (46) to (49).

[11)]Moreover, we must subject the phase to a condition corresponding to condition (31).

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