QUASICLASSICAL APPROXIMATION FOR NONSTATIONARY PROBLEMS

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The quasiclassical formula for the barrier penetration is extended to include the case of time-dependent potentials. Calculation of the penetration reduces to a determination of the subbarrier "trajectory" which formally satisfies the equations of classical mechanics and the complex initial condition (the "time" t for subbarrier motion is imaginary). It is shown that the calculations can be considerably simplified by applying the general formulas to the problem of ionization of atoms in the field of a strong light wave. Formula (47) is obtained for the probability of ionization of a bound level with an orbital angular momentum l in the field of a wave with an arbitrary elliptic polarization. Some qualitative features of subbarrier motion of a particle in a rapidly varying field are considered in Sec. 4 ($\omega \gg \omega_t$ where ω is the barrier oscillation frequency and ω_t is the frequency for tunneling through the fixed barrier). In contrast to the case of a stationary field the tunneling probability for $\omega \gg \omega_t$ is mainly determined by that part of the subbarrier trajectory which is adjacent to the exit point of the particle from the barrier. In this case the effective width of the barrier decreases with increasing frequency ω, and consequently the penetration increases.

1. INTRODUCTION

THE probability that a particle will pass through a potential barrier is usually calculated in the quasiclassical approximation (under the condition that the penetrability of the barrier is small and the well known conditions under which the semiclassical approach is valid are satisfied). Problems with time-varying barriers, in which the barrier can change while the particle passes through it, have recently become important. An example of such a problem is that of ionization of an atom by the field of a strong light wave (laser pulse focused with the aid of a lens). The theory of this process was considered by a number of workers [1–5], who used rather complicated computation methods. Yet the "insinuation" of an electron through a sufficiently broad and smooth barrier has a quasiclassical character in both a constant and an alternating field, and the lack of a convenient method for calculating the tunneling probability is due only the fact that no quasiclassical approximation has been developed for alternating fields.

The purpose of the present paper is to extent the quasiclassical method to include the nonstationary case. In Sec. 2 we derive formulas (7) and (24) for the probability w of tunneling through a periodic potential barrier $V(\mathbf{x}, t)$ of arbitrary form and of frequency ω, and for the momentum spectrum of the emitted particles. The probability w is determined by a function $\widetilde{W}$ which is calculated along the complex subbarrier particle "trajectory". $\widetilde{W}$ is closely related here to the classical action (see (19)). In Sec. 3 we consider the application of these formulas to the problem of level ionization in an alternating electric field. We derive formula (47) for the most general case (level with orbital angular momentum l in the field of an elliptically polarized wave), a case not considered in earlier papers [4,5]. It turns out here that formula (24) gives not only the correct exponential for the ionization probability, but also the exact form of the pre-exponential factor. A qualitative description of the tunnel effect in a homogeneous field $V(x, t) = -Fx \cos \omega t$ is the subject of Sec. 4. It is shown that a transition to an imaginary "time" explains many features of the subbarrier motion in an alternating field, such as the narrowing of the barrier with increasing frequency ω, which was observed in [5]. It is shown that when $\omega \gg \omega_t$ the tunneling probability is determined essentially by a small section of the subbarrier trajectory near the point of emergence.

2. TUNNEL EFFECT IN THE QUASICLASSICAL APPROXIMATION

The penetrability of a time-varying barrier can be obtained in principle by solving the Schrödinger equation

$$i\partial\psi/\partial t = \{H_0(\mathbf{x}) + V(\mathbf{x}, t)\}\,\psi(\mathbf{x}, t) \tag{1}$$

with the specified initial condition

$$\psi(\mathbf{x}, t)|_{t=t_0} = \varphi_0(\mathbf{x})e^{-iE_0t_0},\ \ E_0 = -\varkappa^2/2,\ \ H_0\varphi_0 = E_0\varphi_0. \tag{2}$$

Here $H_0(\mathbf{x}) = -\frac{1}{2}\Delta + V_0(\mathbf{x})$ is the unperturbed Hamiltonian[1)], and $V(\mathbf{x}, t)$ is the oscillating potential causing the tunnel transition from the bound state $\varphi_0(\mathbf{x})$ to the continuous spectrum state. The following assumptions are made with respect to $V_0(\mathbf{x}, t)$:[2)] 1) $V(\mathbf{x}, t)$ varies periodically, $V(\mathbf{x}, t+T) = V(\mathbf{x}, t)$, where $T = 2\pi/\omega$; 2) the potential $V(\mathbf{x}, t)$ is a weak perturbation in the region $\kappa r \lesssim 1$ in which the wave function $\varphi_0(\mathbf{x})$ is essentially concentrated, $|V(\mathbf{x}, t)| \ll \kappa^2$ for $\kappa r \lesssim 1$; 3) the turning points of the potential $V_0(\mathbf{x}) + V(\mathbf{x}, t)$ lie in the region $\kappa r \gg 1$ (for all t).

Under these conditions, the tunneling probability is determined by the remote "tail" of the wave function $\psi(\mathbf{x}, t)$, and there is a broad region in which the motion is quasiclassical. The perturbation $V(\mathbf{x}, t)$ causes the bound level to be transformed into a quasistationary state, the average lifetime of which is much longer than the atomic times ($\sim\kappa^{-2}$). Let us transform (1) into an equation for the quasistationary state. To this end we arbitrarily divide the potential $V_0(\mathbf{x})$ into two parts[3)]:

$$V_0(\mathbf{x}) = V_{\text{sh}}(\mathbf{x}) + V_{\text{Coul}}(\mathbf{x}),$$

where $V_{\text{sh}}(\mathbf{x})$ is the short-range part of the potential and V_{Coul} is the Coulomb "tail." Introducing the Green's function $G(\mathbf{x}_2t_2;\ \mathbf{x}_1t_1)$, which describes the motion of the electron when $r > a$,

$$\left\{i\frac{\partial}{\partial t_2} + \frac{1}{2}\Delta_2 - V_{\text{Coul}}(\mathbf{x}_2) - V(\mathbf{x}_2, t_2)\right\}G(\mathbf{x}_2 t_2;\mathbf{x}_1 t_1) = i\delta(\mathbf{x}_2 - \mathbf{x}_1)\,\delta(t_2 - t_1), \tag{3}$$

we rewrite Eq. (1) with initial condition (2) in the form of an integral equation:

$$\psi(\mathbf{x}_2, t_2) = -i\int_{t_0}^{t_2} dt_1 \int d\mathbf{x}_1 G(\mathbf{x}_2t_2;\ \mathbf{x}_1t_1)\,V_{\text{sh}}\,(\mathbf{x}_1)\,\psi(\mathbf{x}_1, t_1) + \int d\mathbf{x} G(\mathbf{x}_2, t_2; \mathbf{x}\, t_0)\,\psi(\mathbf{x}, t_0). \tag{4}$$

The last term in (4) describes the smearing of the initial state and attenuates in proportion to $[\kappa^2(t-t_0)]^{-3/2}$.

The usual procedure of switching $V(\mathbf{x}, t)$ on adiabatically at $t_0 \to -\infty$ leads to the following equation for the quasistationary state

$$\psi(\mathbf{x}_2, t_2) = -i\int_{-\infty}^{t_2} dt_1 \int d\mathbf{x}_1 G(\mathbf{x}_2t_2;\mathbf{x}_1t_1)\,V_{\text{sh}}\,(\mathbf{x}_1)\,\psi(\mathbf{x}_1, t_1). \tag{5}$$

The factor $V_{\text{sh}}(\mathbf{x}_1)$ cuts off this integral at $\kappa|\mathbf{x}_1| \lesssim 1$, so that $\psi(\mathbf{x}_1, t_1)$ can be replaced by the wave function of the unperturbed atom[4)]

$$\psi(\mathbf{x}_2, t_2) = -i\int_{-\infty}^{t_2} dt_1 e^{-iE_0t_1} \int d\mathbf{x}_1 G(\mathbf{x}_2t_2;\ \mathbf{x}_1t_1)\,V_{\text{sh}}\,(\mathbf{x}_1)\,\varphi(\mathbf{x}_1). \tag{6}$$

With the aid of calculations similar to those made in [4] we obtain the flux of particles going off to infinity, and obtain the tunneling probability w in the form of a sum of the probabilities of the multiphoton processes:

$$w = \sum w_n,\ \ w_n = 2\pi\int d\mathbf{p}\,\delta(p^2/2 - p_n^2/2)\,|F(\mathbf{p})|^2, \tag{7}$$

where $p_n = \sqrt{2(n-\nu)\omega}$, and

$$F(\mathbf{p}_n) = i\lim_{t\to\infty}\frac{\psi(\mathbf{p}_n, t)}{t - t_0}. \tag{8}$$

The quantity ν in (7) yields the tunneling threshold (the minimum number of quanta that must be absorbed in order for an electron to become detached from the atom). To determine ν, we take account of the fact that the electron emerging from under the barrier is under the influence of the field

$$V(\mathbf{x}, t) = \sum_{n=1}^{\infty} V_n(\mathbf{x})\cos(n\omega t + \alpha_n)$$

(the atomic potential $V_0(\mathbf{x})$ can be neglected as $r \to \infty$). Treating the influence of $V(\mathbf{x}, t)$ by the

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1)We use the atomic system of units: $e = \hbar = m = 1$.

2)An example of a perturbation $V(x,t)$ satisfying the conditions (1) – (3) may be the potential (26) with $F \ll F_0 = \kappa^3$.

3)For example, we can put:

$$V_{\text{sh}}(\mathbf{x}) = e^{-r/a}V_0(\mathbf{x}),$$
$$V_{\text{Coul}}(\mathbf{x}) = (1 - e^{-r/a})V_0(\mathbf{x}) \approx -\varkappa_c r^{-1}(1 - e^{-r/a}).$$

Here $\kappa a \gg 1$; a plays the role of the continuity point and drops out from the final formulas. It is necessary to separate V_{Coul} because the Coulomb interaction distorts the asymptotic form of the wave function at arbitrarily small distances from the atom, and must be taken into account when subbarrier motion is considered.

4)A rigorous justification for this substitution is given in the appendix.

Kapitza method[6,7], we replace V(x, t) by an effective time-independent potential:

$$V_{\text{eff}}(x)=\frac{1}{4\omega^2}\sum_{n=1}^{\infty}\frac{f_n^2(x)}{n^2},\quad f_n(x)=-\frac{\partial V_n}{\partial x}. \qquad (9)$$

Since $V_{\text{eff}}(x)>0$, the electron can go off to infinity only if $V_{\text{eff}}(x)$ is bounded when $|x|\to\infty$. In this case

$$\nu=\frac{\omega_0}{\omega}\left[1+\frac{1}{2(\omega\varkappa)^2}\sum_{n=1}^{\infty}\frac{f_n^2}{n^2}\right],\quad f_n=\lim_{x\to\infty}f_n(x) \qquad (10)$$

(we have confined ourselves here for simplicity to one-dimensional motion). This leads to a limitation on the potential V(x, t):

$$|V(\mathbf{x},t)|<Cr \text{ for } r\to\infty. \qquad (11)$$

If V(x, t) contains terms that are linear in x_i as $r\to\infty$ (see, e.g., (26)), this will affect the values of ν and p_n. The increase of ν is proportional to the kinetic energy, averaged over the period, of an electron moving to infinity in a potential V(x, t).

The partial probability w_n of tunneling with absorption of n quanta of energy $\hbar\omega$ is determined, in accordance with (8) by the quantities $F(p_n)$. It is possible to derive for them simple quasiclassical formulas. The wave function $\psi(p_n, t)$ corresponding to a state with a definite average momentum p_n at infinity, satisfies an equation similar to (6), in which $G(x_2t_2;\ x_1t_1)$ must be replaced by the Green's function in the mixed (p, x) representation:

$$G(\mathbf{p}_2t_2;\mathbf{x}_1t_1)=\frac{1}{(2\pi)^{3/2}}\int e^{-i\mathbf{p}_2\mathbf{x}_2}G(\mathbf{x}_2t_2;\mathbf{x}_1t_1)\,d\mathbf{x}_2. \qquad (12)$$

Using for $G(x_2t_2;\ x_1t_1)$ the quasiclassical approximation[8–10]

$$G(\mathbf{x}_2t_2;\mathbf{x}_1t_1)\sim\frac{\theta(t_2-t_1)}{[2\pi i(t_2-t_1)]^{3/2}}e^{iS(\mathbf{x}_2t_2;\,\mathbf{x}_1t_1)}, \qquad (13)$$

$$S(\mathbf{x}_2t_2;\mathbf{x}_1t_1)=\int_{t_1}^{t_2}\left(\frac{1}{2}\dot{\mathbf{x}}^2-V_{\text{Coul}}(\mathbf{x})-V(\mathbf{x},t)\right)dt \qquad (14)$$

and calculating the integral in (12) by the saddle-point method, we obtain the quasiclassical asymptotic expression for $G(p_2t_2;\ x_1t_1)$:

$$G(\mathbf{p}_2t_2;\mathbf{x}_1t_1)\sim\frac{1}{(2\pi)^{3/2}}e^{iW(\mathbf{p}_2t_2;\mathbf{x}_1t_1)}, \qquad (15)$$

where

$$W(\mathbf{p}_2t_2;\mathbf{x}_1t_1)=S(\mathbf{p}_2t_2;\mathbf{x}_1t_1)-\mathbf{p}_2\mathbf{x}_2.$$

The action S in (15) is calculated along the classical trajectory defined by the conditions $x(t_1)=x_1$ and $p(t_2)=p_2$. The quantities x_2 and p_1 are not independent variables and are determined from (16). By varying W at fixed values of t_1 and t_2 we obtain[5)] $\delta W=-x_2\delta p_2-p_1\delta x_1$, whence

$$\partial W/\partial\mathbf{p}_2=-\mathbf{x}_2,\qquad \partial W/\partial\mathbf{x}_1=-\mathbf{p}_1. \qquad (16)$$

For the total derivative dW/dt, taken along the trajectory, we get from (15): $dW/dt_1=dS/dt_1=-L(t_1)$; on the other hand,

$$\frac{dW}{dt_1}=\frac{\partial W}{\partial t_1}+\frac{\partial W}{\partial\mathbf{x}_1}\dot{\mathbf{x}}_1=\frac{\partial W}{\partial t_1}-\mathbf{p}_1\dot{\mathbf{x}}_1.$$

Comparing these expressions, we get

$$\partial W/\partial t_1=\mathbf{p}_1\dot{\mathbf{x}}_1-L(t_1)=H(t_1). \qquad (17)$$

Substituting formula (15) for G in (6), we arrive at an integral containing a rapidly oscillating exponential:

$$\psi(\mathbf{p}_2,t_2)=\frac{e^{-iE_0t_2}}{(2\pi)^{3/2}i}\int_{-\infty}^{t_2}dt_1\int d\mathbf{x}_1e^{i\widetilde{W}(\mathbf{p}_2t_2;\,\mathbf{x}_1t_1)}V_{\text{sh}}(\mathbf{x}_1)\varphi_0(\mathbf{x}_1), \qquad (18)$$

where

$$\widetilde{W}(\mathbf{p}_2t_2;\mathbf{x}_1t_1)=\widetilde{S}(\mathbf{p}_2t_2;\mathbf{x}_1t_1)-\mathbf{p}_2\mathbf{x}_2=\int_{t_1}^{t_2}[L(t)+E_0]\,dt-\mathbf{p}_2\mathbf{x}_2 \qquad (19)$$

($\widetilde{S}$ is the so-called reduced action[7]. The main contribution to the integral (18) is made by the saddle point. The saddle-point conditions with respect to the variables t_1 and x_1 have, with allowance for (16) and (17), the form

$$H(t_1^0)=E_0=-\varkappa^2/2,\quad \mathbf{p}(t_1^0)=0. \qquad (20)$$

Among all the paths that contribute, according to Feynman[8,9], to $\psi(p_2, t_2)$, the only ones that "survive" in the quasiclassical case are those lying in the vicinity of the classical trajectory. The specific feature of our problem is that there exists no real trajectory satisfying Newton's equations, the initial conditions (20), and the condition $p(t_2)=p_2$, since the passage of the particle through the potential barrier is impossible in classical mechanics. This causes the "initial instant" t_1^0 to go off to the complex plane.[5] Nonetheless, the formal apparatus of classical mechanics continues to operate.

The saddle point $x_1^{(0)}=x(t_1^0)$ lies in the region $\kappa r\gg 1$ (see[4]), making it possible to substitute in (18) $\varphi_0(x)$ in the form

$$\varphi_0(\mathbf{r})=C_{\varkappa l}\varkappa^{3/2}(\varkappa r)^{\lambda-1}e^{-\varkappa r}Y_{lm}(\mathbf{r}/r),\ \lambda=\varkappa_c/\varkappa. \qquad (21)$$

The factor $(\kappa r)^\lambda$ is the contribution of the Coulomb "tail" to the action $\widetilde{S}$ at the point r. Indeed,

5)With allowance for the well known formula $\delta S=p_2\delta x_2-p_1\delta x_1$ (see [7]).

$$\tilde{S}(r)=\int_{t_0}^{t}\dot{x}^2(t)\,dt=i\varkappa\int_{r_0}^{r}\left(1-\frac{r_0}{r}\right)^{1/2}dr=i\varkappa\Big(\sqrt{r(r-r_0)}$$

$$-r_0\operatorname{Arch}\sqrt{\frac{r}{r_0}}\Big)\approx i(\varkappa r-\lambda\ln\varkappa r+\text{const})\text{ for }r\gg r_0. \quad (22)$$

Here

$$r_0=2\lambda/\varkappa,\quad \varphi_0(r)\sim r^{-1}\exp[i\tilde{S}(r)]\ \text{ for }\ \varkappa r\gg 1.$$

We shall combine the term $(\kappa r-\lambda\ln\kappa r)$ in the argument of the exponential with the corresponding terms in the function $\widetilde{W}$, after which the action S in (14) should be calculated with the exact atomic potential $V_0(x)$, and the continuity parameter a drops out. As a result we get

$$\lim_{t_2\to\infty}\frac{\psi(\mathbf{p}_2,t_2)}{t_2-t_0}=-\frac{\omega}{2\pi}C_{\varkappa l}\left[\frac{\varkappa}{i(\partial^2W/\partial t^2)_0}\right]^{1/2}$$

$$\times Y_{lm}(\mathbf{n}_0)\exp\{i[\widetilde{W}(\mathbf{p}_2t_2;\mathbf{x}_1^{(0)}t_1^{(0)})-E_0t_2]\} \quad (23)$$

(only the zeroth harmonic of (18), which increases linearly with t_2, is of importance in the calculation of this limit).

All the saddle points lying in the strip $0\le \operatorname{Re} t \le T$ ($T=2\pi/\omega$) contribute to (23). Let us consider those saddle points with the smallest values of Im $\widetilde{W}$ and denote their number by g (thus, g = 2 for the field (26)). Assuming that we can neglect the interference between the individual terms of (23) when we calculate the tunneling probability (see expression (53) for $|F_n(p)|^2$ in [4]), we get:

$$|F(\mathbf{p})|^2=g\frac{\varkappa}{(2\pi)^2}|C_{\varkappa l}|^2\left|\frac{\partial^2W}{\partial(\omega t)^2}\right|^{-1}|Y_{lm}(\mathbf{n}_0)|^2$$

$$\times\exp\{-2\operatorname{Im}\widetilde{W}(\mathbf{p}t_2;\mathbf{x}_1^{(1)}t_1^{(1)})\}, \quad (24)$$

where $\mathbf{n}_0$ is a (complex) unit vector specifying the direction of the classical trajectory at the turning point (20).

Formulas 7 and (24) determine the tunneling probability w_n and the momentum spectrum of the emerging particles. We emphasize that (24) contains only quantities pertaining to the classical trajectory of the particle, and the value of $|F(p)|^2$ depends only on the subbarrier section of the trajectory. A highly illustrative description of the subbarrier motion is obtained by going over to imaginary time (see Sec. 4). For subbarrier motion in a constant field we have $p_2=0$ and $\widetilde{W}=\widetilde{S}$; in addition, by virtue of the energy conservation law, we have $\widetilde{S}=\int\dot{x}^2dt$. Replacing in (7) the summation over n by integration, we obtain the penetrability of the static barrier

$$D\sim\exp\left\{-2\left|\int_{\mathbf{x}_1}^{\mathbf{x}_2}\mathbf{p}\,d\mathbf{x}\right|\right\},\quad \mathbf{p}^2=2(E-V(x))<0. \quad (25)$$

Here $\mathbf{x}_1$ and $\mathbf{x}_2$ are the classical turning points, and the integral is taken over the extremal trajectory that minimizes Im $\widetilde{S}(\mathbf{x}_2,\mathbf{x}_1)$. In the one-dimensional case, the question of finding the extremal trajectory drops out, and (25) goes over into the well known formula [11] for the coefficient of transmission through the barrier.

In the adiabatic case ($\omega\ll\omega_t$), the extremal trajectory $\mathbf{x}(t)$ is close to the corresponding trajectory $\mathbf{x}_0(t)$ in a constant field. Therefore we can easily get with the aid of (19) and (24) the general form of the adiabatic correction to the tunneling probability (see [12]).

3. IONIZATION OF ATOMS BY AN ELECTRIC FIELD

To calculate with the aid of (7) and (24), it is necessary to specify more concretely the perturbation $V(\mathbf{x},t)$ that causes the tunneling. We put

$$V(\mathbf{x},t)=-\mathbf{F}(t)\mathbf{x},\quad \mathbf{F}(t)=\{F\cos\omega t,\varepsilon F\sin\omega t,0\}, \quad (26)$$

which corresponds to the problem in which the atom is ionized by the field of a light wave with ellipticity ϵ $(-1\le\epsilon\le1)$. Using the equation of motion $\ddot{\mathbf{x}}=-\nabla V_0(\mathbf{x})+\mathbf{F}(t)$ and the condition (20), we transform formula (19) into

$$\widetilde{W}(\mathbf{p},0;\mathbf{x}_1,t_1)=-\int_{t_1}^{0}\left\{\frac{1}{2}(\dot{\mathbf{x}}^2+\varkappa^2)+U_0(\mathbf{x})\right\}dt. \quad (27)$$

where $U_0(\mathbf{x})=V_0-x_i\partial V_0/\partial x_i$. In (27) we put $t_2=0$, so that Im $\widetilde{W}$ does not change after the particle emerges from under the barrier. Let us find the most probable value of the momentum $\mathbf{p}=\mathbf{p}_0$ on emerging from under the barrier, and the form of $\widetilde{W}(\mathbf{p})$ near $\mathbf{p}=\mathbf{p}_0$. Going over to the dimensionless variables[6)]

$$\boldsymbol{\xi}=\frac{\omega}{\varkappa}\mathbf{x},\quad \tau=i\omega t,\quad \mathbf{k}=(\mathbf{p}-\mathbf{p}_0)/\varkappa,$$

we write out the expansion

$$\boldsymbol{\xi}(\tau)=\boldsymbol{\xi}_0(\tau)+\boldsymbol{\xi}_1(\tau)+\boldsymbol{\xi}_2(\tau)+\ldots,\quad \boldsymbol{\xi}_n\sim k^n. \quad (28)$$

Here $\xi_0(\tau)$ is the extremal trajectory that minimizes Im $\widetilde{W}$, and $\xi_n(\tau)$ are the corrections for it (k is assumed to be a small parameter). Using the equations of motion and formula (27), we obtain a chain of equations for the determination of $\xi_n(\tau)$. Cutting this chain off at n = 2, we arrive at the following results: [12]

6)The "time" to for the subbarrier trajectory is pure imaginary, and therefore τ takes on real values.

1) The most probable momentum p_0 is given by the condition

$$\mathrm{Re}\,(\xi_0\dot{\xi}_1)_{\tau=0} = 0 \quad \text{or} \quad \mathrm{Im}\,\dot{\xi}_0(0) = 0. \tag{29}$$

2) The expansion of $\widetilde{W}(\mathbf{p})$ in the vicinity of the minimum point $\mathbf{p} = \mathbf{p}_0$ is

$$\widetilde{W}(\mathbf{p}) = i\frac{\omega_0}{\omega}[f(\mathbf{p}_0) + a_{ij}k_ik_j + \ldots], \tag{30}$$

where

$$f(\mathbf{p}_0) = \int\limits_{-\tau_0}^{0}\left\{1 - \dot{\xi}^2 + \frac{2}{\varkappa^2}U_0\left(\frac{\varkappa}{\omega}\xi\right)\right\}d\tau, \tag{31}$$

(the tensor a_{ij} does not depend on k) (31a)

$$a_{ij}k_ik_j = -\xi_1\dot{\xi}_1|_{\tau=0}$$

Owing to the factor $\omega_0/\omega \gg 1$, the momentum distribution is narrow, so that the retention in (30) of only terms quadratic in k_i is justified.

(3) $$\left(\frac{\partial^2 W}{\partial t^2}\right)_{t=t_0} = (\dot{\mathbf{x}}_0\ddot{\mathbf{x}}_0 - \dot{\mathbf{F}}\mathbf{x}_0)_{t=t_0}. \tag{32}$$

Thus, to determine the ionization probability it is sufficient to find the extremal trajectory $\xi_0(\tau)$ and the quantities a_{ij} connected with the correction $\xi_1(\tau)$. They satisfy the following equations:

$$\ddot{\xi}_0(\tau) = -\frac{1}{\gamma}\mathbf{f}(\tau) - \nabla U(\xi), \quad \dot{\xi}_0(-\tau_0) = 0,$$

$$\dot{\xi}_0(0) = -i\frac{\mathbf{p}_0}{\varkappa}, \tag{33}$$

$$\ddot{\xi}_i^{(1)} = -\frac{\partial^2 U}{\partial\xi_i^{(0)}\partial\xi_j^{(0)}}\xi_j^{(1)}, \quad \dot{\xi}_1(0) = -i\mathbf{k}, \tag{34}$$

where

$$\mathbf{f}(\tau) = \{\mathrm{ch}\,\tau, -i\varepsilon\,\mathrm{sh}\,\tau, 0\},\ U(\xi) = -\varkappa^{-2}V_0(\xi/\omega).$$

The trajectory $\xi_0(\tau)$ can be obtained for an arbitrary atomic potential $V_0(r)$ only by numerical calculation. We confine ourselves here to the simplest case of a δ-potential ($V_0(r) = 0$ with $r > 0$), which was already considered earlier[3-4]. In this case the condition (20) is somewhat modified:

$$\mathbf{x}(t_0) = 0,\ \mathbf{p}^2(t_0) = -\varkappa^2 \ \text{or}\ \dot{\xi}^2(-\tau_0) = 1. \tag{35}$$

The extremal trajectory $\xi_0(\tau)$ is determined from (33) and (26):[7)]

[7)]It satisfies the equation $\ddot{\xi}_0 = -f(\tau)/\gamma$. If the i-th component of the force $f_i(\tau)$ is real, then we get from the conditions (29), (33), and (35) that $p_i^{(0)} = 0$; on the other hand if $f_i(\tau)$ is pure imaginary, then $\xi_i^{(0)}(0) = 0$. For the field (26) of elliptically polarized light is follows therefore that the momentum p_0 is directed along the y axis, with $\xi_y^{(0)}(0) = 0$. The condition leads directly to expression (45) for p_0.

$$\xi_0(\tau) = \left\{\frac{1}{\gamma}(\mathrm{ch}\,\tau_0 - \mathrm{ch}\,\tau),\quad i\frac{\varepsilon}{\gamma}\left(\frac{\mathrm{sh}\,\tau_0}{\tau_0} - \frac{\mathrm{sh}\,\tau}{\tau}\right)\tau,\quad 0\right\}, \tag{36}$$

where $\tau_0 = \tau_0(\gamma, \epsilon)$ is the positive root of the equation

$$\mathrm{sh}^2\,\tau_0 - \varepsilon^2\left(\mathrm{ch}\,\tau_0 - \frac{\mathrm{sh}\,\tau_0}{\tau_0}\right)^2 = \gamma^2. \tag{37}$$

The solution of Eq. (34) for $\xi^{(1)}(\tau)$ is

$$\xi_i^{(1)}(\tau) = -i(a_{ij}k_j + k_i\tau),$$

and the initial condition at $\tau = -\tau_0$ assumes in the case of a δ-potential the form

$$[\xi_i^{(1)} - \lambda_{ij}\dot{\xi}_j^{(1)}]_{\tau=-\tau_0} = 0, \quad \lambda_{ij} = [\dot{\xi}_i^{(0)}\dot{\xi}_j^{(0)}(\dot{\xi}_0\ddot{\xi}_0)^{-1}]_{\tau=-\tau_0}. \tag{38}$$

From this we get $a_{ij} = \lambda_{ij} + \tau_0\delta_{ij}$. Formula (32) for the pre-exponential factor simplifies, when account is taken of the condition (35), to

$$\left(\frac{\partial^2 W}{\partial t^2}\right)_{t=t_0} = \dot{\mathbf{F}}\mathbf{x}|_{t=t_0} = i\varkappa F\frac{\mathrm{sh}\,2\tau_0}{2\gamma}\left(1 - \varepsilon^2 + \varepsilon^2\frac{\mathrm{th}\,\tau_0}{\tau_0}\right). \tag{39}$$

Substituting expression (36) for $\xi_0(\tau)$, we arrive at the following formula for the momentum spectrum of the electrons emitted upon ionization of a bound level with orbital angular momentum l:

$$|F(\mathbf{p})|^2_{p=p_n} = 4\pi D(\gamma, \varepsilon)\,|Y_{lm}(\mathbf{n}_0)|^2\exp\left\{-\frac{2\omega_0}{\omega}\left[f(\gamma, \varepsilon) + \sum_{i=1}^{3}c_i\left(\frac{p_i - p_i^{(0)}}{\varkappa}\right)^2\right]\right\}, \tag{40}$$

where

$$f(\gamma, \varepsilon) = \left(1 + \frac{1+\varepsilon^2}{2\gamma^2}\right)\tau_0 - \left(1 - \varepsilon^2 + 2\varepsilon^2\frac{\mathrm{th}\,\tau_0}{\tau_0}\right)\frac{\mathrm{sh}\,2\tau_0}{4\gamma^2}; \tag{41}$$

$$c_x = \tau_0\frac{(1-\varepsilon^2)(\tau_0 - \mathrm{th}\,\tau_0)}{\Delta},$$

$$c_y = \tau_0\left\{1 + \frac{\varepsilon^2(1 - \mathrm{th}\,\tau_0/\tau_0)^2}{\Delta\,\mathrm{th}\,\tau_0}\right\}, \tag{42}$$

$$c_z = \tau_0, \quad \Delta = (1-\varepsilon^2)\tau_0 + \varepsilon^2\,\mathrm{th}\,\tau_0;$$

$$D(\gamma, \varepsilon) = \frac{|C_{\varkappa l}|^2}{8\pi^3}\frac{\omega\gamma^2}{\varkappa\Delta}\frac{2\tau_0}{\mathrm{sh}\,2\tau_0}. \tag{43}$$

The unit vector $\mathbf{n}_0$ ($\mathbf{n}_0^2 = 1$) gives the direction of the initial velocity. Since $\dot{\xi}_y^{(0)}$ is imaginary, this vector is complex. Putting $\mathbf{n}_0 = \mathbf{n}_0(\theta, \varphi)$ and $\theta = i\psi$, we get from (36)

$$\mathrm{th}\,\psi = (|\dot{\xi}_y|/\dot{\xi}_x)_{\tau=-\tau_0} = \varepsilon(\mathrm{cth}\,\tau_0 - 1/\tau_0). \tag{44}$$

The most probable momentum of the emitted electrons differs from zero when $0 < |\epsilon| < 1$ and is directed along the y axis:

$$\mathbf{p}_0 = \{0, \pm k_0, 0\}, \quad k_0 = \varkappa \frac{\varepsilon}{\gamma} \frac{\operatorname{sh}\tau_0}{\tau_0} \tag{45}$$

(see Fig. 2 of [5]). This causes the most probable number of absorbed photons $n_0(\gamma, \epsilon)$ (with $\epsilon \neq 0$) to exceed the ionization threshold $\nu = (\omega_0/\omega)[1 + (1+\epsilon^2)/2\gamma^2]$:

$$\frac{n_0}{\nu} - 1 = \frac{\varepsilon^2}{1+(1+\varepsilon^2)/2\gamma^2}\left(\frac{\operatorname{sh}\tau_0}{\gamma\tau_0}\right)^2 = \begin{cases} \dfrac{\varepsilon^2}{(1+1/2\gamma^2)(\operatorname{Arsh}\gamma)^2} & \text{for} \quad |\varepsilon| \ll 1 \\ (2\tau_0 \operatorname{cth}\tau_0 - 1)^{-1} & \text{for } |\varepsilon| = 1 \end{cases} \tag{46}$$

(see Fig. 1).

Greatest interest attaches to the average ionization probability for unpolarized atoms. The corresponding averaging is effected with the aid of the formula $Y_{lm}^*(\mathbf{n}) \sim Y_{l,-m}(\mathbf{n}^*)$ and the addition theorem for spherical functions:

$$\frac{1}{2l+1}\sum_{m=-l}^{l} |Y_{lm}(\mathbf{n}_0)|^2 = \frac{1}{4\pi} P_l(\operatorname{ch} 2\psi),$$

$$\psi = \operatorname{Arth}\left\{\varepsilon\left(\operatorname{cth}\tau_0 - \frac{1}{\tau_0}\right)\right\}.$$

The parameter ψ increases monotonically with increasing γ; when $\gamma \gg 1$ we have

$$\operatorname{ch} 2\psi \approx \frac{1+\varepsilon^2}{1-\varepsilon^2+2\varepsilon^2/\ln 2\gamma}.$$

When $l = 0$, formulas (40)—(43) coincide with the corresponding expressions (23)—(28) from [5], where, however, less convenient variables were used.[8] As seen from the foregoing, all the quantities in the formulas for the ionization probability can be expressed in simple fashion in terms of τ_0. The dependence of τ_0 on the ellipticity ϵ of the light, obtained by numerically solving Eq. (37), is shown in Fig. 2 ($\tau_0 = \sinh^{-1}\gamma$ when $\epsilon = 0$). We note that the variable τ_0 has a simple physical meaning: τ_0/ω is the total time of motion of the particle under the barrier.

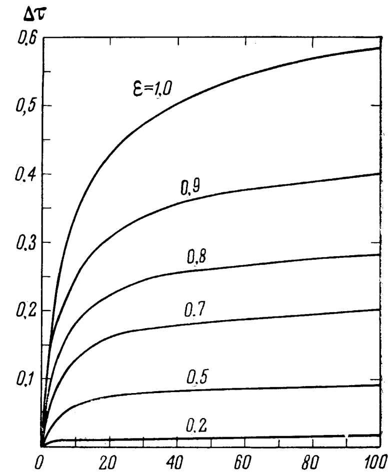

FIG. 2. The dependence of the time of subbarrier motion on the ellipticity ϵ. The ordinates represent the difference $\Delta t = \tau_0(\gamma,\epsilon) - \tau_0(\gamma,0)$.

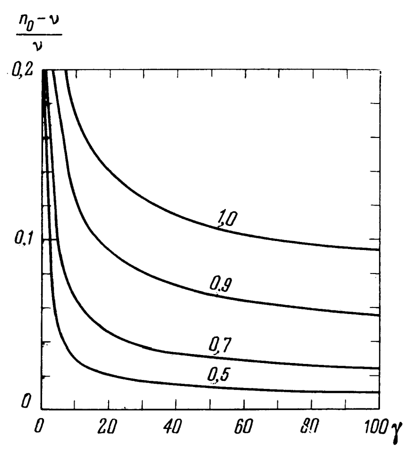

FIG. 1. $(n_0 - \nu)/\nu$ as a function of γ. The numbers of the curves indicate the values of the ellipticity ϵ of the light.

Substituting (40)—(43) into (7) and integrating in the (p_x, p_z) plane, we obtain the partial probability of ionization with absorption of n photons:

$$w_n = \omega \frac{|C_{\varkappa l}|^2}{\pi}\left(\frac{\omega}{2\omega_0}\operatorname{th}\tau_0\right)^{1/2} \frac{\tau_0}{\Delta}\frac{P_l(\operatorname{ch}2\psi)}{\operatorname{ch}^2\psi} R_n \times \exp\left\{-\frac{2\omega_0}{\omega} f(\gamma,\varepsilon)\right\}, \tag{47}$$

where

$$R_n = \left[\frac{(n-\nu)\operatorname{th}\tau_0}{2}\right]^{1/2} J_n,$$

$$J_n = \int_{-1}^{1} dx \exp\{-[a(1-x^2) + b(x-x_0)^2]\} I_0(c(1-x^2)). \tag{48}$$

The dimensionless constant $C_{\kappa l}$ is defined in (21), $I_0(z)$ is a Bessel function of imaginary argument, and

$$a = (n-\nu)\left[1 + \frac{(1-\varepsilon^2)(\tau_0 - \operatorname{th}\tau_0)}{\Delta}\right]\tau_0,$$

$$b = 2(n-\nu)\left[1 + \frac{\varepsilon^2(1-\operatorname{th}\tau_0/\tau_0)^2}{\Delta \operatorname{th}\tau_0}\right]\tau_0,$$

$$c = (n-\nu)\tau_0\Delta^{-1}\operatorname{th}\tau_0, \quad x_0 = [(n_0-\nu)/(n-\nu)]^{1/2}. \tag{49}$$

In the case when $\epsilon = 0$ (linear polarization), the integral J_n can be determined exactly:

[8] The parameter s_0 used in the formulas of [5] is given in terms of τ_0 by $s_0 = \epsilon(1 - \tanh\tau_0/\tau_0)$.

$$R_n = e^{-\alpha(n-\nu)}w(\sqrt{\beta(n-\nu)}), \quad \alpha = 2(\tau_0 - \operatorname{th}\tau_0), \quad \beta = 2\operatorname{th}\tau_0, \quad \tau_0 = \operatorname{Arsh}\gamma, \tag{50}$$

where $w(x) = \exp(-x^2)\int_0^x [\exp(t^2)]\,dt$ (a plot of $w(x)$ is given in [4]). Formula (47) then takes a simpler form

$$w_n = \omega\frac{|C_{\varkappa l}|^2}{\pi}\left(\frac{\omega}{2\omega_0}\operatorname{th}\tau_0\right)^{1/2} w(\sqrt{\beta(n-\nu)}) \times \exp\left\{-\left[\frac{2\omega_0}{\omega}f(\gamma) + \alpha(n-\nu)\right]\right\}, \tag{51}$$

which coincides with formula (61) of [4]. We note that expression (51) for w_n is more convenient when the frequency ω is fixed and $\gamma \gg 1$ ($\tanh \tau_0 \approx 1$). When $\epsilon \neq 0$ and $\gamma \sim 30$—50 the determination of the pre-exponential factor R_n calls for numerical calculations[9)]. The dependence of w_n on the orbital angular momentum l is given by the factor $|C_{\kappa l}|^2 P_l(\cosh 2\alpha)$, which increases when $\epsilon \to 1$.

4. SOME FEATURES OF SUBBARRIER MOTION IN AN ALTERNATING FIELD

The probability of tunneling through a nonstationary barrier is determined by the function $\widetilde{W}$ calculated along the classical-particle trajectory (see formulas (19) and (24)). In this case the "time" t' during which the subbarrier motion effected is pure imaginary for the extremal trajectory that minimizes Im $\widetilde{W}$.[5] We shall show that a consistent transition to a real variable $t = it'$ in the equations for the subbarrier motion leads to a clear picture of the passage of the particle through the barrier. We confine ourselves below to the case of a homogeneous field, $V(x, t') = -F(t')x$, but the results are more general.

We start with linear polarization. The extremal trajectory is one-dimensional and corresponds to a momentum $p_x = 0$ at emergence:

$$x(t) = F\omega^{-2}(\operatorname{ch}\omega t_0 - \operatorname{ch}\omega t), \quad -t_0 \leqslant t \leqslant 0, \quad x(-t_0) = 0, \quad \dot{x}(-t_0) = \varkappa, \quad \omega t_0 = \tau_0 = \operatorname{Arsh}\gamma. \tag{52}$$

We see from the equation of motion $\ddot{x} = -F\cosh\omega t$ that the oscillating external field $F(t') = F\cos\omega t'$ is transformed in the case of the subbarrier motion into a decelerating force. When $\gamma \gg 1$ the decelerating field increases exponentially on going deeper into the barrier ($F(-t_0)/F(0) = \sqrt{1+\gamma^2}$). The growth of $F(t)$ with increasing frequency ω leads to a reduction of the effective width of the barrier $x(0)$:

$$\varkappa x(0) = \frac{F_0}{2F}\left(\frac{1+\sqrt{1+\gamma^2}}{2}\right)^{-1} = \frac{2\omega_0}{\omega}\frac{\gamma}{1+\sqrt{1+\gamma^2}} \tag{53}$$

In actual experiments, [13,14] the laser frequency ω is fixed, and $\gamma \gg 1$; under these conditions, the width of the barriers in units of κ^{-1} (the atomic radius) is simply $2\omega_0/\omega$ and does not depend on the field intensity F.

Figure 3 gives an idea of the buildup of the reduced action $\widetilde{S}$ during the course of the subbarrier motion (for the extremal trajectory (36) the difference between $\widetilde{W}$ and $\widetilde{S}$ vanishes, since $p\cdot x|_{t=0} = 0$). The difference between low fre-

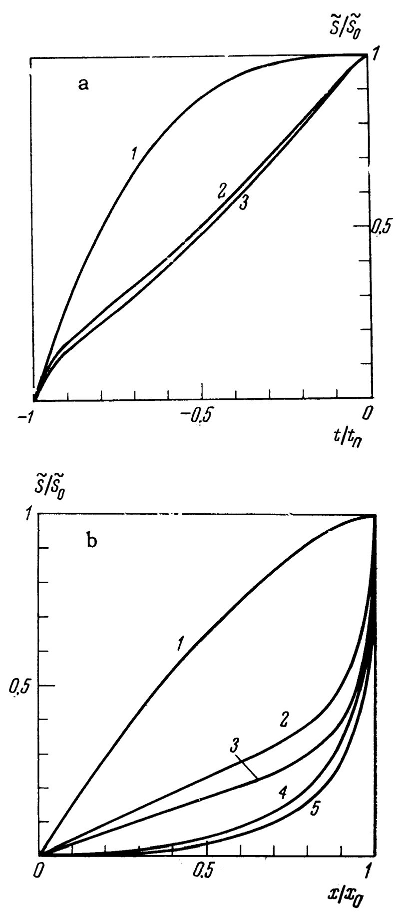

FIG. 3. Variation of the reduced action $\widetilde{S}$ during the course of subbarrier motion: a) as a function of t/t_0; b) as a function of x/x_0. The ordinates are the values of $\widetilde{S}(t, -t_0)/\widetilde{S}(0, -t_0)$. At the instant $t = 0$ the particle goes out from under the barrier, and Im S no longer varies. Curves 1, 2, and 3 pertain to the case of linear polarization, and 4 and 5 to circular polarization. The adiabaticity parameter γ has the values $\gamma = 0$ (curve 1), $\gamma = 30$ (curves 2 and 4), and $\gamma = 100$ (curves 3 and 5).

9)We note in this connection that the asymptotic (for $\gamma \gg 1$) formulas (32) and (33) for w_n given in [5] are still insufficiently accurate when $\gamma \sim 30$.

quencies ($\gamma \lesssim 1$) and high ones ($\gamma \gg 1$) can be seen most clearly in Fig. 3b, where the greater part of the action $\widetilde{S}$ builds up when $\gamma \gg 1$ on the final section of the subbarrier trajectory. Of paramount importance in the calculation of the tunneling probability at $\gamma \gg 1$ is therefore an accurate calculation of the trajectory in the vicinity of the emergence point $x(0)$, where the particle moves with low velocity.

This property of the subbarrier motion in a rapidly alternating field can be explained as follows: On changing over to the variables $\xi = \omega x/\kappa$ and $\tau = \omega t$, the action $\widetilde{S}$ takes the form

$$\widetilde{S}(\tau, -\tau_0) = i\frac{2\omega_0}{\omega}\int_{-\tau_0}^{\tau}\left(\dot{\xi}^2 + \frac{1}{2} - H\right)d\tau, \tag{54}$$

where

$$H(\xi, \dot{\xi}, \tau) = \frac{1}{2}\dot{\xi}^2 + U(\xi) + \frac{\xi \operatorname{ch}\tau}{\gamma}, \quad H(-\tau_0) = \frac{1}{2},$$

$$U(\xi) = -\frac{1}{\varkappa^2}V_0(x). \tag{55}$$

With the aid of the equation of motion ($\ddot{\xi} = -\partial U/\partial\xi - \gamma^{-1}\cosh\tau$) we get

$$\frac{\operatorname{Im}\widetilde{S}(\xi)}{\operatorname{Im}\widetilde{S}(\xi_0)} = 1 - a(\gamma)\sqrt{1 - \frac{\xi}{\xi_0}} + \ldots, \tag{56}$$

Therefore $(1/2 - H) \geq 0$, and it follows from (54) that $\operatorname{Im}\widetilde{S}(\tau, -\tau_0)$ is a monotonically increasing function of τ. Since $\xi(\tau) = \xi(0) - \tau^2/2\gamma$ when $\tau \to 0$, $\operatorname{Im}\widetilde{S}$ has a root singularity as a function of ξ at the final point:

$$\frac{dH}{d\tau} = \frac{\xi \operatorname{sh}\tau}{\gamma} < 0 \quad (-\tau_0 \leqslant \tau < 0). \tag{57}$$

where

$$a(\gamma) = \frac{2}{f(\gamma)}\operatorname{sh}\frac{\tau_0}{2}\left(\operatorname{th}\frac{\tau_0}{2}\right)^2 = \begin{cases} 3/8\gamma & \text{as } \gamma \to 0 \\ \sqrt{2\gamma}/\ln\gamma & \text{for } \gamma \gg 1 \end{cases} \tag{58}$$

The coefficient $a(\gamma)$ increases together with γ, and this explains the behavior of the curves in Fig. 3b.

We proceed to the case of elliptic polarization. The extremal trajectory is determined by (36); ξ_y is imaginary because the corresponding component of the force $\mathbf{f}$ is imaginary:

$$\frac{d^2\boldsymbol{\xi}}{d\tau^2} = -\frac{1}{\gamma}\mathbf{f}(\tau), \quad \mathbf{f}(\tau) = \{\operatorname{ch}\tau, \ -i\operatorname{sh}\tau, 0\}. \tag{59}$$

At the instant of emergence from under the barrier we have $\xi_x(0) = \gamma^{-1}(\cosh\tau_0 - 1)$ and $\xi_y = \xi_z = 0$; the width of the barrier $\xi_x(0)$ increases also with increasing ellipticity ϵ (see formula (63) of [5]), leading to a decrease in the ionization probability. To explain this phenomenon, we note that the motion along the x axis obeys the same law as in the case of linear polarization, but the initial velocity $\dot{\xi}_x(-\tau_0)$ increases (since $\dot{\xi}^2(-\tau_0) = 1$ and $\dot{\xi}_y^2(-\tau_0) < 0$). This leads to a corresponding increase of the time τ_0 needed to stop the particle.

The dependence of the subbarrier motion on the ellipticity of the light ϵ can be investigated with the aid of Eqs. (36) and (37). In the main, the influence of ϵ leads to the appearance of a "transverse" coordinate ξ_y; although $\xi_y = 0$ at the ends of the trajectory, we have here in the intermediate region $|\xi_y| \sim \epsilon\xi_x$. Figure 3b shows that when $\epsilon \neq 0$ the final section of the trajectory merely assumes a more important role in the buildup of the action $\widetilde{S}$.[10)]

This property, which is characteristic of subbarrier motion in the antiadiabatic case $\gamma \gg 1$, points to a way of taking Coulomb interaction into account (the Coulomb correction for the case $\gamma \lesssim 1$ was obtained earlier [15]). Replacing the Coulomb term $\sigma/\gamma\xi^2$ in the exact equation [15]

$$\ddot{\xi} = \frac{1}{\gamma}\left(\frac{\sigma}{\xi^2} - \operatorname{ch}\tau\right), \quad \sigma = \left(\frac{\gamma}{\gamma_c}\right)^2 = \lambda\frac{\omega}{2\omega_0}\gamma \tag{60}$$

by the constant force $\sigma/\gamma\xi_0^2$, we obtain an equation that can be solved analytically. This question will be considered in greater detail in another paper.

We have assumed so far that the particle emerges from under the barrier immediately upon reaching the turning point. In quantum mechanics, however, multiple reflections from the barrier boundary are also significant. This can be seen already in the simplest example:

$$V(x) = \begin{cases} V_0 & \text{for } 0 < x < a \\ 0 & \text{for } x < 0 \text{ and } x > a \end{cases} \tag{61}$$

A particle with momentum k ($k < K = \sqrt{2mV_0}$) incident on the barrier from the left can emerge at the point $x = a$ after $(2n + 1)$ "to and fro" passages. The amplitude of such a process is

$$A_{2n+1} = b(k, k')\,[a(k', k)]^{2n}\,b(k', k)\,e^{-(2n+1)\operatorname{Im}\widetilde{S}_0}, \tag{62}$$

where $\widetilde{S}_0 = \int_0^a p\,dx = k'a$, $k' = i\kappa = i\sqrt{K^2 - k^2}$ is the momentum of the particle under the barrier, and a and b are coefficients that take into account the reflection and refraction of the wave at the

10)The expansion (57) for the action S near the turning point $\xi_0 = \xi_x(0)$ retains the same form also when $\epsilon \neq 0$, and in this case $a(\gamma, \epsilon) \sim [\exp(\tau_0/2)]\tau_0^{-1}$, i.e., the coefficient a increases together with $|\epsilon|$.

points where the potential changes jumpwise.[11] Summing the contributions of all the paths, we obtain the amplitude of the emerging wave:

$$A=\sum_{n=0}^{\infty}A_{2n+1}=\frac{2ik\varkappa}{(k^2-\varkappa^2)\operatorname{sh}\varkappa a+2ik\varkappa\operatorname{ch}\varkappa a}. \qquad (63)$$

The wave function of a particle passing through the barrier is of the form $A\exp[ik(x-a)]$; the penetrability D of the barrier is equal to

$$D=|A|^2=\frac{4k^2\varkappa^2}{(k^2+\varkappa^2)^2\operatorname{sh}^2\varkappa a+4k^2\varkappa^2}. \qquad (64)$$

Formulas (63) and (64) coincide with the result of the accurate solution of the Schrödinger equation (see[11], p. 104). In the quasiclassical case $\operatorname{Im}\tilde{S}_0=\kappa a\gg 1$, and only the first term, corresponding to a single passage, "survives" in the sum (63). A similar situation obtains for an arbitrary time-constant barrier.

To assess the role of multiple reflections in an alternating field, we replace the short-range potential $V_0(x)$ by a reflecting wall at the point $x=0$. The subbarrier trajectory consists of n pieces $\xi_k(\tau)$, $1\le k\le n$ (see Fig. 4). The growth of the decelerating force $F(\tau)=-F\cosh\tau$ with increasing τ causes the amplitudes of the maxima to decrease rapidly with increasing n when $\gamma\gg 1$. The functions $\xi_k(\tau)$ satisfy the equations

$$\ddot{\xi}_k(\tau)=-\frac{\operatorname{ch}\tau}{\gamma}\ (-\tau_k<\tau<-\tau_{k-1}),\qquad \dot{\xi}_n(-\tau_n)=1,$$

$$\dot{\xi}_1(0)=0,\qquad \xi_k(-\tau_k)=\xi_k(-\tau_{k-1})=0,$$

$$\dot{\xi}_k(-\tau_k)=-\dot{\xi}_{k+1}(-\tau_k) \qquad (65)$$

(the last condition corresponds to elastic reflection from the wall). The solution of these equations is of the form

$$\xi_1(\tau)=\frac{1}{\gamma}(\operatorname{ch}\tau_1-\operatorname{ch}\tau),$$

$$\xi_k(\tau)=\frac{1}{\gamma}\left[\frac{-(\tau+\tau_{k-1})\operatorname{ch}\tau_k+(\tau+\tau_k)\operatorname{ch}\tau_{k-1}}{\tau_k-\tau_{k-1}}-\operatorname{ch}\tau\right],\quad 2\leqslant k\leqslant n, \qquad (66)$$

the values of τ_k being given by the system of equations

$$\frac{\operatorname{ch}\tau_2-\operatorname{ch}\tau_1}{\tau_2-\tau_1}=2\operatorname{sh}\tau_1,$$

$$\frac{\operatorname{ch}\tau_k-\operatorname{ch}\tau_{k-1}}{\tau_k-\tau_{k-1}}=2\operatorname{sh}\tau_{k-1}-\frac{\operatorname{ch}\tau_{k-1}-\operatorname{ch}\tau_{k-2}}{\tau_{k-1}-\tau_{k-2}}\quad(3\leqslant k\leqslant n-1), \qquad (67)$$

$$\frac{\operatorname{ch}\tau_n-\operatorname{ch}\tau_{n-1}}{\tau_n-\tau_{n-1}}=\operatorname{sh}\tau_n-\gamma.$$

The tunneling probability is $w_n\sim\exp(-2\operatorname{Im}\tilde{S}_n)$, where

$$\tilde{S}_n=i\frac{\omega_0}{\omega}f_n(\gamma),\qquad f_n(\gamma)=\left(1+\frac{1}{2\gamma^2}\right)\tau_n-\frac{\operatorname{sh}2\tau_n}{4\gamma^2}+\frac{1}{\gamma^2}\sum_{k=2}^{n}\frac{(\operatorname{ch}\tau_k-\operatorname{ch}\tau_{k-1})^2}{\tau_k-\tau_{k-1}}. \qquad (68)$$

To determine whether the contribution of the trajectories with $n>1$ can be discarded when $\gamma\gg 1$, we solved the equations in (67) numerically for $\gamma=100$. The result is shown in Fig. 6c. When $\omega_0/\omega\sim 10$ we have $w_2/w_1\sim 10^{-7}$, i.e., also in the case of an alternating field satisfying the conditions $\omega\ll\omega_0$ and $F\ll F_0$, it is sufficient to take into account a very simple subbarrier trajectory without reflections.

In conclusion, we make one remark concerning the replacement of the field of the light wave by a homogeneous electric field $F(t')=F\cos\omega t'$. As proposed in[4,5], such a replacement is valid if $\lambda\gg\kappa^{-1}$ ($\lambda=2\pi/k$ is the wavelength of the light). It is now clear that this condition is too weak: the external field must be homogeneous not only within the confines of the atom, but at much larger distances on the order of the width of the barrier. The correct condition follows from (53):

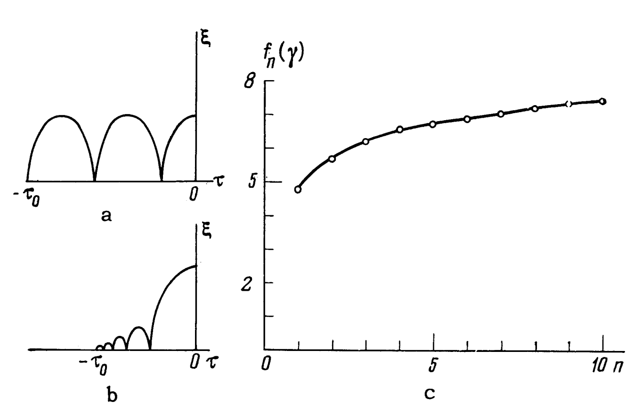

FIG. 4. Form of the subbarrier trajectory with n reflections: a) $\gamma\to 0$, b) $\gamma\gg 1$, c) the function $f_n(\gamma)$, which determines the penetrability of the barrier, for $\gamma=100$.

[11]When a plane wave is incident on an infinite step, a reflected and a refracted wave are produced:

$$\psi(x)=e^{ik_1x}+a(k_1,k_2)e^{-ik_1x}\quad\text{for } x<0,$$
$$\psi(x)=b(k_1,k_2)e^{ik_2x}\quad\text{for } x>0.$$

Here k_1 and k_2 are the wave vectors on the left and on the right of the interface $x=0$, and

$$a(k_1,k_2)=\frac{k_1-k_2}{k_1+k_2},\qquad b(k_1,k_2)=\frac{2k_1}{k_1+k_2}.$$

[12]It is seen from (55) that the sign of the potential $V_0(x)$ is reversed on going to the imaginary "time" τ.

$$kx_0 = 2\pi\frac{x_0}{\lambda} = \sqrt{\frac{2I}{mc^2}}\frac{\gamma}{1+\sqrt{1+\gamma^2}} \ll 1, \qquad (69)$$

where $I = \kappa^2/2$ is the ionization potential and m is the electron mass. For atoms $(2I/mc^2)^{1/2} \sim 10^{-2}$ and condition (69) is satisfied.

APPENDIX

To justify the approximation made on going over from (5) and (6) (a similar approximation was used in essence in [4,5]), we determine the correction to $\psi_0(x, t) = \varphi_0(x)\exp(i\kappa^2 t/2)$ by perturbation theory. We confine ourselves here to the particular case $V(x, t) = -Fz\cos\omega t$. Putting $\psi = \psi_0 + \psi_1 + \ldots$ we get the following equation for the correction ψ_1:

$$\left(i\frac{\partial}{\partial t} - H_0\right)\psi_1 = V(\mathbf{x}, t)\psi_0 = -Fr\cos\theta\cos\omega t\,\psi_0(\mathbf{x}, t). \qquad (A.1)$$

In the simplest case $l = 0$, (s-level), the variables in (A.1) separate, and the solution takes the form

$$\psi(\mathbf{x}, t) = \psi_0(\mathbf{x}, t)\left[1 + \frac{F}{2F_0}\cos\theta\,\frac{f_+(\xi)e^{-i\omega t} + f_-(\xi)e^{i\omega t}}{2}\right], \qquad (A.2)$$

where $F_0 = \kappa^3$ (field inside the atom), $\xi = \kappa r$, and the functions $f_\pm(\xi)$ satisfy the equation

$$\frac{d^2 f_\pm}{d\xi^2} + 2\alpha(\xi)\frac{df_\pm}{d\xi} + \left(\pm\eta - \frac{2}{\xi^2}\right)f_\pm = -4\xi. \qquad (A.3)$$

Here $\eta = \omega/\omega_0$ and $\alpha(\xi) = d\ln(r\varphi_0)/d(\kappa r)$; $\alpha(\xi) = -1$ for the level in the δ-potential, and $\alpha(\xi) = -1 + \xi^{-1}$ for the ground state of the hydrogen atom. Equation (A.3) can be readily solved if $\eta = 0$ (case of constant field):

$$f_\pm(\xi) = \begin{cases} \xi^2 & \text{for } \delta\text{-potential,} \\ \xi^2 + 2\xi & \text{for hydrogen atom.} \end{cases} \qquad (A.4)$$

The difference between $\varphi(r)$ and the wave function $\varphi_0(r)$ of the free atom becomes appreciable when $r \sim r_* = \kappa^{-1}\sqrt{F_0/F}$. This conclusion does not depend on the particular form of the potential $V_0(r)$; it can be shown that

$$\frac{\varphi(r)}{\varphi_0(r)} = 1 + \frac{F}{2F_0}(\kappa r)^2\cos\theta + O\left(\frac{F}{F_0}\kappa r, \left(\frac{F}{F_0}\right)^2\right) \qquad (A.5)$$

when $\kappa^{-1} \ll r \ll r_*$, independently of $V_0(r)$ and of the orbital angular momentum l. We shall therefore confine ourselves to the case of the δ-potential in our treatment of an alternating field.

The exact solution of (A.3) takes the form

$$f_\pm(\xi) = \frac{2}{n_\pm^2(1-n_\pm/2)^2}\left[(1 - n_\pm + \xi^{-1})(e^{n_\pm\xi} - 1) - n_\pm\left(1 + \left(1 - \frac{n_\pm}{2}\right)\xi\right)\right], \qquad (A.6)$$

where $n_\pm = 1 - \sqrt{1 \pm \eta}$. When $\omega \ll \omega_0$, the expressions for $f_\pm(\xi)$ simplify to:

$$f_\pm(\xi) \approx \frac{2}{n_\pm^2}[e^{n_\pm\xi} - (1 + n_\pm\xi)] = \xi^2\sum_{k=0}^{\infty}\frac{2}{(k+2)!}(n_\pm\xi)^k. \qquad (A.7)$$

From this we get

$$\frac{\psi(\mathbf{x}, t)}{\psi_0(\mathbf{x}, t)} = \begin{cases} 1 + \dfrac{F}{2F_0}\xi^2\cos\theta\cos\omega t + \ldots & \text{for } \xi \ll \eta^{-1}, \\ 1 + \dfrac{Fe^{\eta\xi}}{2F_0\eta^2}\cos\theta e^{-i\omega t} + \ldots & \text{for } \xi\eta \gtrsim 1. \end{cases} \qquad (A.8)$$

The range of values of R for which the substitution $\psi(x, t) \to \psi_0(x, t)$ is valid is given by the inequality $r \ll r_*$, where

$$\kappa r_* = \frac{2\omega_0}{\omega}\ln\left(1 + \frac{\gamma}{\gamma_*}\right) = \begin{cases} \sqrt{F_0/F} & \text{for } \gamma \ll \gamma_* \\ 2\omega_0/\omega & \text{for } \gamma \gtrsim \gamma_* \end{cases} \qquad (A.9)$$

(here $\gamma = \omega\kappa/F$ and $\gamma_* = \sqrt{F_0/F} \gg 1$). We note that r_* assumes a value on the order of the atomic radius (κ^{-1}) only when $\omega \sim \omega_0$, and therefore, for short-range potential, the substitution of ψ_0 for ψ in (5) is valid if $\omega \ll \omega_0$, as was proposed in [4,5]. It is also of interest to compare r_* with the dynamic width of the barrier r_0 defined by formula (53). In a constant field $r_0 \gg r_*$; when $\gamma = \gamma_*$ the values of r_0 and r_* become comparable in magnitude and become of equal order of magnitude when γ is increased further.

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