TRANSPORT PHENOMENA IN A COLLISIONLESS PLASMA IN A TOROIDAL MAGNETIC SYSTEM

A. A. GALEEV and R. Z. SAGDEEV

Institute for Nuclear Physics, Siberian Branch, Academy of Sciences, SSSR

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Transport phenomena associated with toroidal particle drift in a collisionless plasma are investigated. The presence of so-called "trapped" particles leads to a sharp rise in the transport coefficients, the magnification factor being approximately $(R/r)^{3/2}$ where R and r are respectively the major and minor radii of the torus.

1. INTRODUCTION

RECENT experiments with a thermally ionized cesium plasma in a toroidal magnetic system have shown that there is a range of parameters for which the lifetime of the plasma (completely isolated from the walls) is determined by classical transport processes due to Coulomb collisions.[1] In toroidal systems the displacement of the plasma as a consequence of drift motion in the toroidal magnetic field can lead to an appreciable increase in the transport coefficients as compared with the usual case of plasma diffusion across a magnetic field. This effect was first noted by Budker.[2] Pfirsch and Schlüter [3] have carried out a quantitative analysis for the ambipolar diffusion case (cf. also [4–6] where the results of Pfirsch and Schlüter have been reproduced by other methods). Shafranov has determined the thermal conductivity coefficient.[7]

The work cited above has verified and refined the proposition stated in [2] and the numerical value of the ambipolar diffusion coefficient has been found to agree with the experimental value.[1]

It is shown in the present paper that in a low-density plasma in a toroidal system the transport effects due to binary collisions can be enhanced by a large factor (the factor is of order $(R/r)^{3/2}$) over what would be expected from a direct extrapolation of the results of the work in [3–7] to the case of low-density plasmas. As will be shown below, this effect is due to an effective increase in the frequency of Coulomb collisions due to the presence of trapped particles.

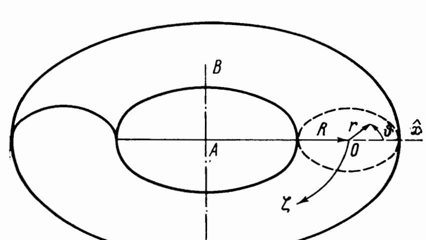

FIG. 1. General diagram of the torus.

2. TOROIDAL DRIFT OF INDIVIDUAL PARTICLES

We shall consider the simplest possible model of a toroidal magnetic field (cf. Fig. 1).

Here, the plane (r, ϑ) contains the toroidal axis AB while the distance along the perimeter is given by means of the angular coordinate ζ. The primary magnetic field is the field produced by a straight current flowing along the axis AB

$$\mathbf{H} \approx H_0\left(1 - \frac{r}{R}\cos\vartheta\right)\mathbf{e}_\zeta, \quad \varepsilon \equiv \frac{r}{R} \ll 1. \tag{1}$$

The supplementary field

$$\Delta\mathbf{H} = -\frac{ir}{2\pi R}H_0\mathbf{e}_\vartheta, \quad \Theta \equiv \frac{ir}{2\pi R} \ll 1 \tag{2}$$

provides the rotational transform. The magnitude of the rotational transform $i(r)$ is a function of the coordinate r only. The change in the absolute magnitude of the primary magnetic field H due to the additional field can be neglected if $\Theta^2 \ll \epsilon$, as is assumed here. For the configuration we have chosen the magnetic surfaces are specified by the equation $r = \text{const}$.

The pressure of the plasma confined in the field is assumed to be small:

$$\beta \equiv 4\pi n(T_i + T_e)/H^2 \ll 1,$$

so that the perturbation of the magnetic field due to the plasma can be neglected. In addition, the Larmor radius of the particles is assumed to be so small that the distortion of the particle distribution function arising from the toroidal drift is also small and thus does not lead to the production of significant electric fields along the magnetic surface. This assumption is valid if[8]

$$\frac{r_{ci}}{\Theta n}\frac{dn}{dr} \ll 1, \tag{3}$$

where r_{ci} is the ion Larmor radius and $n(r)$ is the particle density in the plasma.

The drift approximation is used to describe the motion of the particles. Because of the toroidal symmetry of the system we need only consider the projection of the particle trajectories on the (r, ϑ) plane. The equations for the guiding center of a particle with charge e and mass m in a radial electric field specified by potential $\Phi(r)$ and a magnetic field given by (1) assume the familiar form:

$$\frac{dr}{dt} = -\frac{\mu H_0/m + v_\parallel^2}{\omega_c R}\sin\vartheta, \quad \mu \equiv \frac{mv_\perp^2}{2H_\zeta}, \tag{4}$$

$$r\frac{d\vartheta}{dt} = -\frac{\mu H_0/m + v_\parallel^2}{\omega_c R}\cos\vartheta + \frac{c}{H_\zeta}\frac{d\Phi}{dr} - \Theta v_\parallel, \tag{5}$$

where $v_\perp$ and $v_\parallel$ are the velocity components perpendicular and parallel to the magnetic field and $\omega_c = eH_0/mc$ is the cyclotron frequency. On the right side of (5), in addition to the diamagnetic, centrifugal, and electric drifts, we must also take account of the rotation of the particles around the primary magnetic field H_ζ due to the rotational transform. Using the conservation of particle energy E and the conservation of the adiabatic invariant $\mu = mv_\perp^2/2H_\zeta$ we can find the longitudinal velocity of a particle $v_\parallel$ with given E and μ:

$$v_\parallel = \sigma\left\{\frac{2}{m}[E - e\Phi(r) - \mu H_\zeta(r, \vartheta)]\right\}^{1/2}. \tag{6}$$

Substituting this expression in (4) and (5) we can find another constant of the motion:

$$J = \omega_c\int_0^r \Theta dr + v_\parallel(1 + \varepsilon\cos\vartheta). \tag{7}$$

In fields of more complicated geometry this quantity plays the role of a longitudinal adiabatic invariant.[9]

In the limit given by (3) the deviation of particles from the magnetic surface is very small and the quantity J can be expanded in terms of this deviation. We take the origin of coordinates to be the point $(r_0, 0)$ and expand $J(r, \vartheta)$ to terms of second order inclusively in the radial deviation of the particles, thereby obtaining the trajectory equation in the form[8]

$$r - r_0 \cong \{\Delta v \pm [(\Delta v)^2 + 2r_0\omega_c v_g(\cos\vartheta - 1)]^{1/2}\}/\omega_c\Theta, \tag{8}$$

where

$$\Delta v(r_0, 0) = v_\parallel(r_0, 0) - v_E(r_0)/\Theta,$$

$$v_E \equiv \frac{c}{H}\frac{d\Phi}{dr},$$

$$v_g = \frac{\mu H_0/m + v_E^2/\Theta^2}{\omega_c R}.$$

It is then obvious that a particle with velocity $(\Delta v)^2 < 4r_0\omega_c v_g$ will be trapped in a toroidal magnetic field with a rotational transform. This trapping effect arises because the line of force of the magnetic field go from the inner portion of the toroidal tube to an outer portion and vice versa as a result of the rotational transform. For this reason the magnitude of the magnetic field varies along the line of force (the field is larger at the inner region of the torus and smaller at the outer region). The weakly trapped particles exhibit the largest displacement from the magnetic surface

$$\Delta r_t(\vartheta = 0) = 4[\mu H_0/m + v_E^2\Theta^{-2}]^{1/2}/\omega_c\Theta$$

(cf. Fig. 2). Untrapped particles which are almost trapped exhibit half of this displacement $\Delta r_u = 0.5\,\Delta r_t$. The displacement of untrapped particles with velocity $\Delta v \sim v$ is found to be small (of order $\epsilon^{1/2}$) compared with the displacement of trapped particles.

The particle motion in time is described by the equation of motion for the ϑ coordinate. Neglecting the toroidal drift, we can write this equation in dimensionless variables:[10]

$$r\frac{d\vartheta}{dt} = -\sigma_*\Theta\left[\left(v^2 + \frac{v_E^2}{\Theta^2}\right)\varepsilon\right]^{1/2}[2\varkappa^2 - 1 + \cos\vartheta]^{1/2}, \tag{9}$$

where

$$v^2 = 2(E - e\Phi(r))/m, \quad \sigma_* = \operatorname{sign}\Delta v(r, \vartheta),$$

$$2\varkappa^2 = [\Delta v(r_0, 0)]^2/(v^2 + v_E^2\Theta^{-2})\,\varepsilon.$$

It then follows that the motion of the trapped particles can be described in terms of elliptic functions with modulus $\kappa^2 < 1$. The period of oscillation of the trapped particles along the closed trajectory is then given by

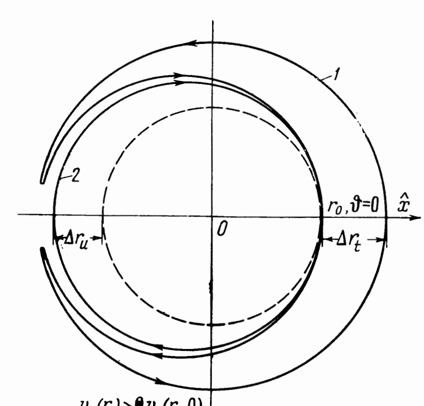

FIG. 2. Trajectory of trapped particles (1) and transiting particles (2).

$$\tau = \frac{4r}{[(\Theta^2 v^2 + v_E^2)\varepsilon]^{1/2}} \int_0^{\vartheta_0} \frac{d\vartheta}{[2(\varkappa^2 - \sin^2(\vartheta/2))]^{1/2}}$$

$$= \frac{4\sqrt{2} r K(\varkappa)}{\Theta[\varepsilon(v^2 + v_E^2\Theta^{-2})]^{1/2}} \quad (10)$$

where $K(\kappa)$ is a complete elliptic integral of the first kind while ϑ_0 is a zero of the expression in the radical.

Our problem is now to take account of the effect of collisions on the drift motion described above and to determine the net result on transport phenomena in the plasma.

3. TRANSPORT PHENOMENA IN A LOW-DENSITY MAXWELLIAN PLASMA

We shall use the Boltzmann kinetic equation in the drift approximation, writing the collision terms in the Landau form

$$\frac{\partial f_j}{\partial t} + [\mathscr{H}, f_j] = \mathrm{St}\{f_j\}, \quad (11)$$

where

$$[\mathscr{H}, f_j] \equiv \Big\{ -\frac{\mu H_0/m_j + v_\parallel^2}{\omega_{cj} R} \sin\vartheta \frac{\partial}{\partial r} + \Big(-\Theta v_\parallel + \frac{c}{H_0}\frac{d\Phi}{dr} - \frac{\mu H_0/m_j + v_\parallel^2}{\omega_{cj} R}\cos\vartheta \Big)\frac{\partial}{\partial\vartheta} + \Theta \frac{\mu H_0/m_j + v_\parallel^2}{\omega_{cj} R}\sin\vartheta \frac{\partial}{\partial v_\parallel} \Big\} f_j,$$

$$\mathrm{St}\{f_j\} \equiv -\sum_{j'} \frac{\partial}{\partial v_\alpha} 2\pi\lambda \frac{e_j^2 e_{j'}^2}{m_j} \int d\mathbf{v}' \Big[\frac{\delta_{\alpha\beta}}{u} - \frac{u_\alpha u_\beta}{u^3}\Big] \times \Big(\frac{f_j(\mathbf{v})}{m_{j'}} \frac{\partial f_{j'}(\mathbf{v}')}{\partial v_\beta'} - \frac{f_{j'}(\mathbf{v}')}{m_j}\frac{\partial f_j(\mathbf{v})}{\partial v_\beta} \Big), \quad u_\alpha = v_\alpha - v_\alpha'.$$

We first consider the case in which collisions are not very rare so that a Maxwellian particle distribution can be established by virtue of collisions in a small region with dimensions given by $\Delta v \sim \sqrt{\epsilon}\, v_T$ ($v_{Tj} = \sqrt{2T_j/m_j}$ is the thermal velocity) within the period of gyration of the trapped particle in the closed trajectory τ. The time required to establish equilibrium within this region can be estimated from the expression for the collision term (11) and is of the order of the following quantities:

$$\tau_p = \varepsilon \nu_j^{-1}, \quad \nu_j = \frac{16\sqrt{\pi}\lambda e^4 n(r)}{3 m_j v_{Tj}^3}. \quad (12)$$

On the other hand we assume that collisions can be neglected for the majority of particles. Thus, our limitations on the collision frequency can be expressed by the inequality

$$\Theta\sqrt{v_{Tj}^2 + v_E^2\Theta^{-2}}/r > \nu_j > \varepsilon^{3/2}\Theta\sqrt{v_{Tj}^2 + v_E^2\Theta^{-2}}/r. \quad (13)$$

The solution of the kinetic equation can now be sought in the form of an expansion in the toroidal parameter. We write the distribution function in the form

$$f_j(\mu, v_\parallel; r, \vartheta) = f_j^{(0)}(\mu, v_\parallel, r) + f_j^{(1)}(\mu, v_\parallel; r, \vartheta), \quad (14)$$

where

$$f_j^{(0)} = \frac{n_j(r)}{\pi^{3/2} v_{Tj}^3} \exp\Big[-\frac{2\mu H/m_j + v_\parallel^2}{v_{Tj}^2} - \frac{2e_j\Phi(r)}{v_{Tj}^2} \Big]$$

is a local Maxwellian function while the correction $f_j^{(1)}$ takes account of toroidal effects. Linearizing the kinetic equation (11) we now write it in the form

$$(-\Theta v_\parallel + v_E)\frac{\partial f_j^{(1)}}{r\partial\vartheta} = -\nu_j f_j^{(1)} + \frac{\mu H_0/m_j + v_\parallel^2}{R}\sin\vartheta\Big[-\Theta\frac{\partial}{\partial v_\parallel} + \frac{1}{\omega_{cj}}\frac{\partial}{\partial r}\Big] f_j^{(0)}. \quad (15)$$

For reasons of simplicity we now write the collision term in the τ approximation. This procedure is justified for rare collisions; to take account of collisions themselves we need only use the correct contour around the singularity that arises. Further, writing the dependence on the angle ϑ in exponential form we write the solution of the equations:

$$f_j^{(1)} = \sum_{(\pm)} \Big\{ \Big(\frac{\mu}{m_j} H_0 + v_\parallel^2 \Big) \Big/ R \Big\} \Big\{ \Big[\Theta \frac{\partial}{\partial v_\parallel} - \frac{1}{\omega_{cj}}\frac{\partial}{\partial r} \Big] f_j^{(0)} \Big(\frac{v_E}{r} - \frac{\Theta v_\parallel}{r} \mp i\nu_j \Big)^{-1} \Big\} e^{\pm i\vartheta}. \quad (16)$$

Multiplying this expression by the radial drift velocity of the particles

$$-[(\mu H_0/m_j + v_\parallel^2)/\omega_{cj}R]\sin\vartheta$$

and integrating over velocity, we obtain the particle flux across the magnetic field:

$$\langle nv\rangle_j = -\frac{1}{m_j}\int_0^{2\pi} d\vartheta \int_0^\infty d\mu H_0 \int_{-\infty}^{+\infty} dv_\parallel f_j^{(1)} \frac{\mu H_0/m_j + v_\parallel^2}{\omega_{cj} R}\sin\vartheta$$

$$= -\frac{\pi}{m_j}\int_0^\infty d\mu H_0 \int_{-\infty}^{\infty} dv_\parallel \frac{(\mu H_0/m_j + v_\parallel^2)^2}{\omega_{cj} R^2}\pi\delta(v_E - \Theta v_\parallel) \times r\Big[-\Theta\frac{\partial}{\partial v_\parallel} + \frac{1}{\omega_{cj}}\frac{\partial}{\partial r}\Big] f_j^{(0)}.$$

After some simple calculations we find

$$\langle nv\rangle_j = -\frac{\pi\varepsilon^2 r_{cj}^2}{2r} v_{Tj} F_j\Big(\frac{v_E}{\Theta}\Big)\Big\{1 + \Big(1 + \frac{3v_E^2}{\Theta^2 v_{Tj}^2}\Big)\Big\}\frac{dn_j}{dr}, \quad (17)$$

where $F_j = \pi^{-1/2}\exp(-v_E^2/\Theta^2 v_{Tj}^2)$ is the relative density of trapped particles in velocity space.

It is interesting to note that the particle flux appears to be proportional to a kind of particle work in the field of a "magnetostatic curvature wave" of the magnetic field (this is to be compared with Landau damping in a collisionless plasma). In some sense this particle diffusion is a result of the effect of a frictional force on the "wave." Hence it is completely reasonable that the diffusion is not ambipolar. Defining the diffusion coefficient in the usual way

$$D_{\perp j} = -\langle nv\rangle_j / \nabla n,$$

and using (17) we can write

$$D_{\perp j} = \frac{\Delta r_{tj}^2}{\tau_j}\sqrt{\varepsilon} F_j, \tag{18}$$

where the factor $\sqrt{\epsilon}\,F_j$ takes account of the smallness of the number of trapped particles; an estimate of the deviation of these particles from the magnetic surface Δr_t has been given earlier while the displacement time is found to be of the order of the orbit time around the closed orbit. This result is completely reasonable since it is precisely within this time that any significant disturbance of the local Maxwellian distribution maintained by particle collisions can be achieved.

Because the plasma is neutral, an electric field must arise in such a way as to reduce the ion diffusion to the level of electron diffusion. To a high degree of accuracy the magnitude of this field can be determined from the condition

$$e\Phi(r) = T_i \ln n(r). \tag{19}$$

Under these conditions the diffusion becomes ambipolar while the diffusion coefficient is given by (17) computed for electrons taking account of (19):

$$D_\perp = \frac{2\sqrt{\pi}\,\varepsilon^2 r_{ce}}{|\Theta| r}\frac{cT_e}{eH_0} \qquad \varepsilon^{3/2}\frac{\Theta v_{Te}}{r} < \nu_e < \frac{\Theta v_{Te}}{r}. \tag{20}$$

As before, the thermal conductivity coefficient for the ions under these conditions is found to be much larger than for the electrons:

$$\chi_{\perp i} = \frac{3\sqrt{\pi}\,\varepsilon^2 r_{ci}}{2|\Theta| r}\frac{cT_i}{eH_0}, \qquad \varepsilon^{3/2}\frac{\Theta v_{Ti}}{r} < \nu_i < \frac{\Theta v_{Ti}}{r}. \tag{21}$$

4. EQUILIBRIUM AND TRANSPORT PHENOMENA FOR VERY RARE COLLISIONS

We now consider the case of very rare collisions; in this case the relaxation time for the trapped particle distribution due to collisions is much larger than the period associated with their motion:

$$\tau_p \sim \varepsilon \nu_j^{-1} \gg \tau_j = r/\Theta v_{Tj}\sqrt{\varepsilon}. \tag{22}$$

Under these conditions, as a first approximation we can neglect collisions and write the solution of the kinetic equation at the outset in the form of a function of the integrals of the motion:

$$f_t = f_t^{(0)}(\mu, E, J), \quad f_u = f_u^{(0)}(\mu, E, J, \sigma), \tag{23}$$

where $f_t^{(0)}$ and $f_u^{(0)}$ are the distribution functions for the trapped and untrapped particles. The derivatives of the distribution functions defined in this way with respect to the longitudinal velocity (and sometimes even the functions themselves) exhibit discontinuities at the surfaces that divide the phase volumes for the trapped and untrapped particles.[8] This result is not surprising since a change in the topology of the trajectories which is also discontinuous occurs at these surfaces (Fig. 2). However, in the presence of even a very small number of collisions close to the surface there will arise a transition layer which will appear to give rise to a continuous transition of the functions $f_t^{(0)}$ and $f_u^{(0)}$ into each other. In an earlier section we have found that the diffusion of particles across the magnetic field can be explained in terms of an interaction of the particles with a "magnetostatic curvature wave" of the magnetic field and this was found to be analogous to the collisionless damping of this wave in a Maxwellian plasma. Correspondingly, we now wish to compute the work of the particles in field of this wave for very rare collisions, in which case we must take account of the relaxation of the distribution of resonant particles under the influence of the wave. The analogous problem of damping of a plasma wave of finite amplitude has been investigated qualitatively in [11] and a rigorous quantitative solution has been given by Zakharov and Karpman.[12] The results of this work indicate that the damping factor is reduced in the rarefied plasma in proportion to the number of collisions. In the following we shall follow the method used in the work cited.

We solve the kinetic equation by the method of successive approximations, writing the particle distribution function in the form

$$f_j = f_j^{(0)}(E, \mu, J; \sigma) + f_j^{(1)}(E, \mu, J; \sigma; \vartheta) + \ldots \tag{24}$$

In addition, we linearize the collision term since deviations from the Maxwellian distribution are important only in a small region of velocities of the trapped particles. The kinetic equation then assumes the form

$$(-\Theta v_\parallel + v_E)\frac{\partial f_j^{(1)}}{r\partial\vartheta} + \Theta\,\frac{\mu H_0/m_j + v_E^2\Theta^{-2}}{R}\sin\vartheta\,\frac{\partial f_j^{(1)}}{\partial v_\parallel}$$

$$= \sum_j \frac{2\pi\lambda e_j^2 e_{j'}^2}{m_j}\frac{\partial}{\partial v_\alpha}\left\{\left(\eta_{j'} + \eta_{j'}' - \frac{\eta_{j'}}{2x_{j'}}\right)\right. \tag{25}$$

$$\left.\times\left[\frac{\delta_{\alpha\beta}}{v} - \frac{v_\alpha v_\beta}{v^3}\right] + \frac{v_\alpha v_\beta}{v^3}\frac{\eta_{j'}}{x_{j'}}\right\}\left(\frac{\partial f_j^{(0)}}{m_j\partial v_\beta} + \frac{2v_\beta}{m_{j'}v_{Tj'}^2}f_j^{(0)}\right),$$

where

$$\eta_{j'} \equiv \eta(x_{j'}) = \frac{2}{\sqrt{\pi}} \int_0^{x_{j'}} e^{-t} \sqrt{t}\, dt,$$

$$\eta'(x_j) = \frac{\partial \eta(x_j)}{\partial x_j}, \quad x_j = \frac{2\mu H_0}{m_j v_{Tj}^2}.$$

Here, we have neglected the derivative of the corrections to the distribution function with respect to the r coordinate because the following inequality is satisfied:

$$\sqrt{\varepsilon}\, \partial f_j^{(1)} / \partial r \ll \partial f_j^{(1)} / r\partial\vartheta.$$

Furthermore, in the first approximation the distribution function is most sensitive to changes in the longitudinal velocity and we can neglect all the other derivatives in (25). We also neglect all quadratic terms in the electric field because the assumption that the ion-Larmor radius is small (3) and the assumption of ambipolar diffusion (19) imply that

$$\frac{v_E}{\Theta v_{Ti}} = \frac{r_{ci}}{2\Theta n} \frac{dn}{dr} \ll 1. \qquad (26)$$

Finally, we shall rewrite the relations in the new variables μ, n^2 and ϑ making use of the following substitution of variables:

$$v_\parallel = \frac{v_E}{\Theta} + 2\sigma \left[\frac{\mu}{m} H_0 \varepsilon \left(\varkappa^2 - \sin^2 \frac{\vartheta}{2}\right)\right]^{1/2}. \qquad (27)$$

The kinetic equation now assumes the form

$$-\Theta v_{Tj} \sqrt{2x_j\varepsilon} \frac{\partial f_j^{(1)}}{r\partial\vartheta} = \varepsilon^{-1} \nu_j A(x_j) \frac{\partial^2}{\partial \varkappa^2} \left\{ \sigma \left(\varkappa^2 - \sin^2 \frac{\vartheta}{2} \right)^{1/2} \times \left(\frac{\partial f_j^{(0)}}{\partial \varkappa^2} + 2x_j \varepsilon f_j^{(0)} \right) + c_j \sqrt{2x_j\varepsilon}\, f_j^{(0)} \right\}, \qquad (28)$$

where

$$A_j(x_j) = \frac{3\sqrt{\pi}}{4} \sum_{j'} \left(\eta_{j'} + \eta'_{j'} - \frac{\eta_{j'}}{2x_{j'}} \right) x_j^{-3/2}, \quad c_j \equiv \frac{v_E}{\Theta v_{Tj}}.$$

Integrating both parts of (28) with respect to the angle ϑ over the limits $(0, 2\pi)$, from the periodicity conditions on all these physical quantities (with respect to the angular coordinate ϑ) we find an equation for the function $f_j^{(0)}$:

$$\frac{\partial}{\partial \varkappa^2} \left[\int_0^{2\pi} \left\{ \sigma \sqrt{\varkappa^2 - \sin^2 \frac{\vartheta}{2}} \left(\frac{\partial f_j^{(0)}}{\partial \varkappa^2} + 2x_j \varepsilon f_j^{(0)} \right) + c_j \sqrt{2x_j\varepsilon}\, f_j^{(0)} \right\} d\vartheta \right] = 0. \qquad (29)$$

We first consider the transiting particles. We will seek a solution which, appearing as a function of the constants of the motion $(J - 2\sigma\sqrt{\mu H_0 \epsilon / m\kappa / \Theta})$, μ, and κ^2, becomes a Maxwellian distribution in the limit $\kappa^2 \to \infty$. Making use of the work of Berk and Galeev [8] and the work of Zakharov and Karpman [12] we find the solution of interest here:

$$f_{u_j}^{(0)} = \frac{n_j(r)}{\pi^{3/2} v_{Tj}^3} \exp\left[-\frac{e\Phi(r)}{T_j} - x_j - c_j^2 - 2x_j \varepsilon \varkappa^2 - \frac{\pi\sigma\sqrt{2\varepsilon x_j}\, c_j}{2} \int_1^{\varkappa^2} \frac{dt}{t^{1/2} E(t^{-1/2})} \right] \times \left\{ 1 + \frac{\sigma \sqrt{2x_j\varepsilon}\, r_{cj}}{\Theta n(r)} \frac{dn(r)}{dr} \left(\sqrt{\varkappa^2 - \sin^2(\vartheta/2)} - \frac{1}{2} \int_1^{\varkappa^2} \frac{dt}{t^{1/2} E(t^{-1/2})} \right) \right\}, \qquad (30)$$

where $E(t^{-1/2})$ is a complete elliptic integral of the second kind while the expression in the curly brackets is essentially the first two terms of an expansion of the function of the third constant of the motion:

$$N\left\{ \omega_{cj} \int_0^r \Theta\, dr + 2\sigma \sqrt{\mu H_0 \varepsilon / m_j} \times \left(\sqrt{\varkappa^2 - \sin^2(\vartheta/2)} - \frac{1}{2} \int_1^{\varkappa^2} \frac{dt}{t^{1/2} E(t^{-1/2})} \right) \right\} \approx n(r) + \frac{2\sigma\sqrt{\mu H_0/m_j}}{\omega_{cj}\Theta} \frac{dn(r)}{dr} \times \left(\sqrt{\varkappa^2 - \sin^2(\vartheta/2)} - \frac{1}{2} \int_1^{\varkappa^2} \frac{dt}{t^{1/2} E(t^{-1/2})} \right), \qquad (31)$$

where

$$N\left(\omega_{cj} \int_0 \Theta\, dr \right) \equiv n(r).$$

The determination of the distribution function for the trapped particles is facilitated by the fact that it must be symmetric with respect to the sign of the longitudinal velocity σ. Consequently all those terms must vanish on the right side of the kinetic equation (28) which contain σ. For a solution of the following form ([(compare with (30)]:

$$f_{tj}^{(0)} = \frac{n(r)}{\pi^{3/2} v_{Tj}^3} e^{-x_j} F_j(\varkappa^2) \times \left\{ 1 + \frac{\sigma\sqrt{2x_j\varepsilon}\, r_{cj}}{\Theta n(r)} \frac{dn(r)}{dr} \sqrt{\varkappa^2 - \sin^2(\vartheta/2)} \right\} \qquad (32)$$

the last condition becomes simply

$$\partial F_j / \partial \varkappa^2 + 2x_j \varepsilon F_j(\varkappa^2) = 0. \qquad (33)$$

We can solve this equation together with the kinetic equation for the correction $f^{(1)}$, which is now simplified considerably:

$$-\Theta v_{Tj} \frac{\partial f_j^{(1)}}{r\partial\vartheta} = \nu_j A_j(x_j) \frac{\partial}{\partial \varkappa^2} \left(c_j + \frac{r_{cj}}{2\Theta n} \frac{dn(r)}{dr} \right) f_j^{(0)}.$$

As a result we find the complete solution:

$$f_{tj} = \frac{n(r)}{\pi^{3/2} v_{Tj}^3} \exp\{-c_j^2 - x_j - 2x_j \varepsilon \varkappa^2\}$$
$$\times \left\{1 + \frac{\sigma \sqrt{2x_j \varepsilon}\, r_{cj}}{\Theta n} \frac{dn(r)}{dr} \sqrt{\varkappa^2 - \sin^2(\vartheta/2)} + \frac{2\nu_j A_j r \vartheta}{v_{Tj}\Theta} x_j \varepsilon \left(c_j + \frac{r_{cj}}{2\Theta n} \frac{dn(r)}{dr}\right)\right\}. \quad (34)$$

Thus, taking proper account of even very weak collisions removes all the arbitrariness in the determination of the distribution function which always exist if one neglects collisions completely, this has been shown, for example, in the work of Berk and Galeev.[8] From a comparison of (30) and (34) it is evident that the particle distribution function in velocity $f_j^{(0)}$ is continuous at the point $\kappa^2 = 1$ whereas its derivatives are different at $\kappa^2 \to 1 \pm 0$ (cf. Fig. 3). Hence, in the vicinity of the point $\kappa^2 = 1$ there is a narrow transition region whose structure can be determined only from a solution of the complete equation in (28) rather than by perturbation theory.

Fortunately, as we shall show below, the transport coefficients of interest here do not depend on the fine structure of this transition region and the difference in the values of the derivatives on both sides of the region can be determined. In terms of the variables μ and κ^2 the element of phase volume can be written

$$2\pi H_0 \frac{d\mu}{m} dv_\parallel (\mu, \varkappa^2, \vartheta) \equiv 2\pi \sqrt{2\varepsilon\mu H_0/m}\, d\mu H_0 \frac{d\varkappa^2}{\{2[\varkappa^2 - \sin^2(\vartheta/2)]\}^{1/2}} \sum_\sigma . \quad (35)$$

Multiplying the particle distribution function by the particle velocity across the magnetic field $(\mu H_0/m\omega_c R \sin\vartheta)$ and integrating over phase volume, we can find the particle flux across the magnetic field:

$$\langle nv \rangle_j = -\frac{(2\varepsilon)^{3/2}}{\omega_{cj}} \int_0^{2\pi} d\vartheta \sum_\sigma \int_0^\infty \cdot \left(\frac{\mu}{m} H_0\right)^{3/2} d\frac{\mu}{m} H_0$$
$$\times \int_{\sin^2(\vartheta/2)}^\infty \frac{[f_j^{(0)} + f_j^{(1)}]\, d\varkappa^2}{2\sqrt{2}[\varkappa^2 - \sin^2(\vartheta/2)]^{1/2}} \frac{\sin\vartheta}{r}. \quad (36)$$

It is evident that in the absence of collisions the integrand is a total differential with respect to the angle ϑ and that the flux will vanish after integration. Thus, the frictional force acting on the particles is proportional to the collision frequency.

We now consider separately the contributions to the flux due to transiting and trapped particles as well as the contribution due to particles from

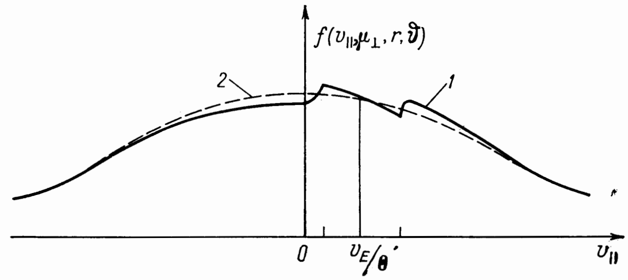

FIG. 3. General form for the particle distribution in a toroidal magnetic system for rare collisions (1) and frequent collisions (2).

the transition layer. We shall denote these particle classes respectively by $\langle nv \rangle^{(1,2,3)}$. The total flux is then given by the expression

$$\langle nv \rangle = \langle nv \rangle^{(1)} + \langle nv \rangle^{(2)} + \langle nv \rangle^{(3)}.$$

In considering the transiting particles $(\kappa^2 > 1)$ in (36) we can change the order of integration with respect to κ^2 and ϑ and then integrate this expression by parts. Using the kinetic equation for $f_j^{(1)}$ we can express the particle flux in terms of the right side of this equation:

$$\langle nv \rangle^{(1)} = \frac{\nu_j v_{Tj}^4}{2\Theta\omega_{cj}} \sum_\sigma \int_0^\infty A_j(x_j) x_j dx_j \int_0^{2\pi} d\vartheta \int_1^\infty d\varkappa^2 \sqrt{\varkappa^2 - \sin^2(\vartheta/2)}$$
$$\times \frac{\partial}{\partial\varkappa^2} \left\{\sigma \sqrt{\varkappa^2 - \sin^2(\vartheta/2)} \left(\frac{\partial f_{ju}^{(0)}}{\partial\varkappa^2} + 2x_j \varepsilon f_{ju}^{(0)}\right) + c_j \sqrt{2x_j\varepsilon} f_{ju}^{(0)}\right\}. \quad (37)$$

Having computed in explicit form the first derivative $\partial f_{ju}^{(0)}/\partial\kappa^2$ in accordance with (30) and having carried out the second integration by parts, we reduce this expression to the form

$$\langle nv \rangle^{(1)} = -\frac{\sqrt{2\varepsilon}\,\nu_j r_{cj}}{\pi^{3/2}\Theta} \int_0^\infty A(x_j) e^{-x_j} x_j^{3/2} dx_j$$
$$\times \left\{4 - \frac{\pi^2}{2} + 2\int_1^\infty \frac{dt}{t^{1/2}} \left[K(t^{-1/2}) - \frac{\pi^2}{4E(t^{-1/2})}\right]\right\}$$
$$\times \left(c_j + \frac{r_{cj}}{2\Theta n} \frac{dn}{dr}\right). \quad (38)$$

In similar fashion, substituting in the expressions for the flux (36) the distribution function for the trapped particles, we find

$$\langle nv \rangle^{(2)} = \frac{2\sqrt{2\varepsilon^3}\,\nu_j r_{cj}}{\Theta\pi^{3/2}} \int_0^\infty A(x_j) e^{-x_j} x_j^{5/2} dx_j \left(c_j + \frac{r_{cj}}{2\Theta n} \frac{dn}{dr}\right). \quad (39)$$

It is evident that this contribution can be neglected compared with the contribution of the transiting particles.

For the transition region we have

$$\langle nv \rangle^{(3)} = \frac{\nu_j v_{Tj}^4}{2\Theta\omega_c} \sum_\sigma \int_0^\infty A_j(x_j) x_j dx_j \int_0^{2\pi} d\vartheta \int_{1-\delta(\vartheta)}^{1+\delta(\vartheta)} d\varkappa^2$$

$$\times \sqrt{\varkappa^2 - \sin^2(\vartheta/2)}\frac{\partial}{\partial\varkappa^2}\left\{\sigma\sqrt{\varkappa^2-\sin^2(\vartheta/2)}\left(\frac{\partial f_j}{\partial \varkappa^2}+2x_j\varepsilon f_j\right)\right.$$
$$\left. + c_j\sqrt{2x_j\varepsilon f_j}\right\}, \quad 0<\delta(\vartheta)\ll 1.$$

Here, it is sufficient to consider only the term with the second derivative since only the first derivative $\partial f_j/\partial\kappa^2$ has different values at the boundaries of the transition layer; the values of the function itself can be considered to be the same to accuracy of order $\sim\delta \ll 1$. Then, the magnitude of the flux can be expressed in terms of the discontinuity of the derivative $\partial f_j/\partial\kappa^2$ inside the transition layer:

$$\langle nv\rangle^{(3)} = \frac{\nu_j v_{Tj}^4}{2\Theta\omega_{cj}}\int_0^\infty A(x_j)x_j dx_j \int_0^{2\pi} d\vartheta \sum_\sigma \sigma\left(\varkappa^2 - \sin^2\frac{\vartheta}{2}\right)$$
$$\times\left\{\frac{\partial f_{ju}^{(0)}}{\partial\varkappa^2} - \frac{\partial f_{jt}^{(0)}}{\partial\varkappa^2}\right\}\Bigg|_{\varkappa^2=1} \tag{40}$$
$$= -\left(\frac{\pi\varepsilon}{2}\right)^{1/2}\frac{r_{cj}}{\Theta}\int_0^\infty e^{-x_j}A(x_j)x_j^{3/2}dx_j n\left(c_j + \frac{r_{cj}}{2\Theta n}\frac{dn}{dr}\right).$$

Using the numerical estimate of the integral on the right side of (38) taken from [12] one can easily show that the contribution of the transiting particles is numerically smaller than the contribution due to the transition layer. Computing the integral on the right side of (40) in explicit form we finally obtain the final result

$$\langle nv\rangle \approx \langle nv\rangle^{(3)} = \alpha_j\frac{\nu_j r_{cj}^2\sqrt{\varepsilon}}{\Theta^2}\left(\frac{dn}{dr}+\frac{e_j}{T_j}n\Phi'(r)\right),$$
$$\alpha_j = \frac{3\pi}{8\sqrt{2}}\left[\delta_{je}+\frac{1}{2\sqrt{2}}+\ln(1+\sqrt{2})\right]. \tag{41}$$

It then follows that the diffusion of the particles becomes ambipolar for an electric field whose magnitude is determined (as before) by (19) and that the diffusion coefficient is larger by a factor $\epsilon^{-3/2}$ than that computed by Pfirsch and Schlüter:[3]

$$D_\perp \approx 3{,}6\frac{\nu_e r_{ce}^2}{\varepsilon^{3/2}}\frac{4\pi^2}{i^2}, \quad \nu_e \ll \frac{v_{Te}\Theta\varepsilon^{3/2}}{r}. \tag{42}$$

The ion thermal conductivity can be computed in completely analogous fashion:

$$\chi_{\perp i} \approx 0{,}4\frac{\nu_i r_{ci}^2}{\varepsilon^{3/2}}\frac{4\pi^2}{i^2}, \quad \nu_i \ll \frac{v_{Ti}\Theta\varepsilon^{3/2}}{r}. \tag{43}$$

5. CONCLUSION

Let us now compare our results with those obtained earlier. In the limit of very rare collisions, where the particles trapped in the region of weak magnetic field do not succeed in exhibiting a Maxwellian distribution, the diffusion coefficient is proportional to the collision frequency, being described by (42) and the line denoted by 1 in

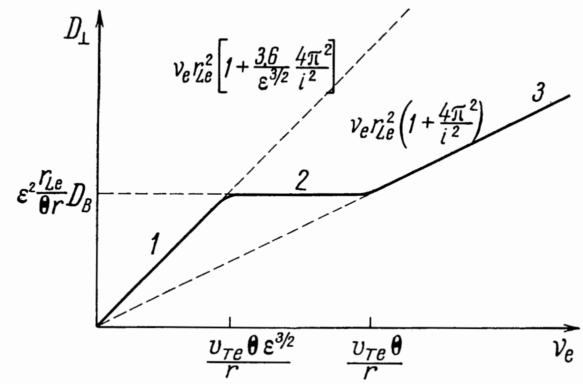

FIG. 4. The particle diffusion coefficient as a function of the electron-ion collision frequency.

Fig. 4. The physical meaning of this expression is clear. If, in the relation $D_\perp \sim (\Delta r_e)^2\nu$ we substitute $\Delta r_e \sim r_{ce}\sqrt{\epsilon}/\Theta$ (for the trapped particles) and $\nu \sim \nu_e\epsilon^{-1}$ (the effective collision frequency for the trapped particles is large) we obtain the result (42), which takes account of the fact that the fraction of trapped particles $\sim\sqrt{\epsilon}$.

It is also very interesting to consider the case in which collisions can establish a Maxwellian distribution for trapped particles in velocity while the main mass of particles are collisionless. In this situation the particle diffusion is determined only by the plasma parameters (20) and is independent of the collision frequency (the segment 2 in Fig. 4). Finally, for a highly collision dominated plasma, in which the hydrodynamic model can be used to describe the plasma, we obtain the result of Pfirsch and Schlüter. The behavior of the thermal conductivity due to the particles is completely analogous. It will be evident that all of the considerations given above apply so long as turbulence effects can be neglected.

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