WAVE PROPAGATION IN A RANDOM MEDIUM. THE CORRELATION GROUP METHOD

V. M. FINKEL'BERG

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Various approximations encountered in the calculations of the mean wave field strength in a medium with random inhomogeneities are considered by a general method of expansion into correlation groups. Higher correlations of the random characteristics of the medium, or correlations of the positions of random scatterers, are taken into account. The conditions of applicability of the approximations employed are found.

1. INTRODUCTION

THERE are many problems connected with the propagation of waves in media having random inhomogeneities. These include: the behavior of an electron in a disordered condensed body, particularly in a crystal with random impurities[1-4]; acoustic[5-6] and electromagnetic waves[7-9] in a medium with a fluctuating refractive index; ionization losses in an inhomogeneous medium[10]; effective dielectric constant of randomly inhomogeneous dielectrics[11,12] and in general the dielectric constant of a continuous medium[13]; elastic properties of polycrystals[14]; molecular diffusion in a turbulent medium[5], etc. Also related is the theory of multiple scattering of waves[15] with its numerous applications.

We confine ourselves henceforth to the case of a scalar wave field, using for this purpose the Schrödinger equation for the Green's function G:

$$(\Delta_{\mathbf{x}} + E - V(\mathbf{x}))G(\mathbf{x},\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y}), \qquad (1)$$

describing a particle having an energy E and a mass m ($\hbar^2/2m = 1$), placed in a random external field V(**x**). Such a description can be applied directly (disregarding spin) to an electron in a crystal with random impurities under conditions where the single-band approximation is valid[2-4]. Not being interested here in boundary effects, we shall regard the field V(**x**) as statistically homogeneous and isotropic.

In most investigations, one of two possible very simple assumptions is made regarding the random field V(**x**): it is either regarded as normally distributed[3-8], or it is assumed that the potential V is produced by independent scattering centers[2,15]. Under these assumptions it is possible, by using the diagram-summation method developed in quantum electrodynamics, to obtain an approximate solution of the problem and to estimate the region of its applicability. It is of interest to ascertain what changes are introduced in the available results by allowance for the deviation of the distribution of the field V(**x**) from the Gaussian law, or by allowance for the correlations between the positions of the scatterers. The attempts hitherto attempted to take into account, within the framework of the diagram technique, the higher correlations of the field V[16,17] are more tentative than systematic. In this paper we consider this question in the limiting cases of large and small wavelengths.

The approximations constructed assuming a normal distribution of the potential V or of the potential of the independent scatterers can be generalized, and it is possible to get rid of the already mentioned assumption both in the formal expressions and in the conditions for their applicability.

2. PERTURBATION THEORY FOR THE AVERAGE GREEN'S FUNCTIONS AND GROUP EXPANSIONS

Physical interest attaches not to the functional dependence of the solution G(V) of Eq. (1) on the field V(**x**) in itself, which naturally cannot be obtained in general form, but to the first and second moments of the Green's function:

$$\mathcal{G}(\mathbf{x},\mathbf{x}') = \langle G(\mathbf{x},\mathbf{x}')\rangle,$$
$$\mathcal{W}(\mathbf{x},\mathbf{x}';\mathbf{y},\mathbf{y}') = \langle W(\mathbf{x},\mathbf{x}';\mathbf{y},\mathbf{y}')\rangle. \qquad (2)$$

The brackets $\langle \ldots \rangle$ denote here averaging over the ensemble of random potentials V(**x**), W(**x**, **x**′; **y**, **y**′) = G(**x**, **x**′)$\overline{G(\mathbf{y},\mathbf{y}')}$ = G $\otimes$ $\overline{G}$—direct product of the operators G and $\overline{G}$, and the bar over G(**y**, **y**′) indicates complex conjugation. A statistical approach of this kind is at present universally accepted.

The usual procedure consists in the following[4,5,7,16]. We construct by iteration a retarded solution of Eq. (1) in the form of a (Born) perturbation-theory series:

$$G = G_0 + G_0VG_0 + G_0VG_0VG_0 + \dots \quad (3)$$

We use here symbolic operator notation, and

$$G_0(\mathbf{x}-\mathbf{y}) = -\frac{1}{4\pi}\frac{e^{ik_0|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|} \quad (4)$$

is the Green's function of equation (1) for V = 0 and $k_0 = \sqrt{E} + i0$. We further average the series (3) or its product by the complex-conjugate series and obtain an expression for $\mathcal{G}$ and $\mathcal{W}$ in terms of the moments $\langle V(\mathbf{x}_1) \dots V(\mathbf{x}_S)\rangle$ of the random field $V(\mathbf{x})$. For what follows, it is convenient to express these moments in terms of the correlation functions $k_S(\mathbf{x}_1, \dots, \mathbf{x}_S)$ using the group-expansion formulas[16,18]

$$\langle V(\mathbf{x}_1)\dots V(\mathbf{x}_s)\rangle = \sum_{\nu}\sum_{\text{div}} k_{s_1}(\Gamma_1)\dots k_{s_\nu}(\Gamma_\nu). \quad (5)$$

The summation in (4) is over all the breakdowns of the points $\mathbf{x}_1, \dots, \mathbf{x}_S$ into groups $\Gamma_1, \dots, \Gamma_\nu$. We assume that the values of the field V at the points $\mathbf{x}$ and $\mathbf{y}$ become independent if $|\mathbf{x}-\mathbf{y}| \gtrsim l$ (l—characteristic correlation scale). In this case $k_S(\mathbf{x}_1, \dots, \mathbf{x}_S) \to 0$ when $|\mathbf{x}_i - \mathbf{x}_0| \gtrsim l$. This property facilitates the analysis of the approximations as a function of G, to the construction of which we shall proceed later.

We have thus $\mathcal{G}$ and $\mathcal{W}$ in the form of series in the correlation functions k_S. The terms of these series can be represented by diagrams[16] of the type of Fig. 1, on which the segments of the lower and upper horizontal lines denote the functions G_0 and $\overline{G}_0$, respectively, and points that enter in the same correlation function k_S are unified into groups.

The series that result can be regarded as series in powers of a certain parameter γ, numerically equal to unity, if we ascribe to each function k_S a factor γ. Then the power of γ will indicate the number of correlation groups in the given term of the expansion.

The sum of the irreducible diagrams[1)] of the series for $\mathcal{G}$ and $\mathcal{W}$, without the extreme lines forms the so called mass operator M and intensity operator K[16], which determines $\mathcal{G}$ and $\mathcal{W}$ in terms of the Dyson equations

$$\mathcal{G} = G_0 + G_0M\mathcal{G}, \quad (6)$$

and of the Bethe-Salpeter equations

$$\mathcal{W} = \mathcal{G}\otimes\overline{\mathcal{G}} + (\mathcal{G}\otimes\overline{\mathcal{G}})K\mathcal{W} \quad (7)$$

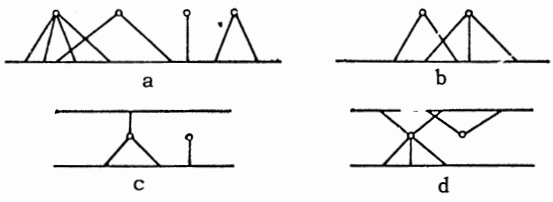

FIG. 1.

(the names are borrowed from quantum electrodynamics). Analogous constructions were made in[5,7,8] for the normal potential $V(\mathbf{x})$, when $k_3 = k_4 = \dots = 0$.

If the potential $V(\mathbf{x})$ is produced by individual scatterers located at random points $\mathbf{x}_i$ and having a self-field $U_i(\mathbf{x}) \equiv U(\mathbf{x}-\mathbf{x}_j)$, then

$$V(\mathbf{x}) = \sum_i U_i(\mathbf{x}). \quad (8)$$

In this case, besides the general description with the aid of the correlation functions k_S, there is also another possible and more adequate description with the aid of the functions $g_S(\mathbf{x}_1, \dots, \mathbf{x}_S)$[17,18], which characterize the correlation of the position of the scatterers.

Let us expand the Green's function G in a series in the scattering multiplicity[19,15]

$$G = G_0 + G_0\Big[\sum_{i_1} T_{i_1} + \sum_{i_1}\sum_{i_2\neq i_1} T_{i_1}G_0T_{i_2} + \sum_{i_1}\sum_{i_2\neq i_1}\sum_{i_3\neq i_2} T_{i_1}G_0T_{i_2}G_0T_{i_3} + \dots\Big]G_0. \quad (9)$$

In this formula T_i is the operator of scattering by the i-th isolated scatterer, which is determined by the equation

$$T = V + VG_0T \quad (10)$$

with $V = U_i$. The averaging necessary for the calculation of $\mathcal{G}$ and $\mathcal{W}$ reduces now to the calculation of the mean values of expressions of the form

$$\sum_{i_1,\dots,i_s}{}' \Phi(\mathbf{x}_1, \dots, \mathbf{x}_s),$$

which are equal to

$$\Big\langle \sum_{i_1,\dots,i_s}{}' \Phi(\mathbf{x}_1, \dots, \mathbf{x}_s)\Big\rangle = \int d^3x_1 \dots \int d^3x_s f_s(\mathbf{x}_1, \dots, \mathbf{x}_s)\Phi_s(\mathbf{x}_1, \dots, \mathbf{x}_s), \quad (11)$$

where none of the points in the sum Σ' coincide, and the function Φ is symmetrical. We denote by $f_S(\mathbf{x}_1, \dots, \mathbf{x}_S)$ the distribution function of the centers of the scatterers, which is connected with the probability dw of observing one scatterer in each of the volume elements $d^3x_1, \dots, d^3x_S$, by means of the relations[18]

[1)] A diagram is called irreducible if, after discarding the extreme segments of the propagation lines, it cannot be cut up into two parts without cutting at the same time the lines representing the functions k_S. In Fig. 1 the diagrams a and c are reducible, while b and d are irreducible.

$$dw = [f(\mathbf{x}_1, \ldots, \mathbf{x}_s) + O(d^3x)]\, d^3x_1 \ldots d^3x_s.$$

The functions f_S are expressed in terms of g_S in the same manner as the moment functions are expressed in terms of the correlation functions. Namely[17,18],

$$f_s(\mathbf{x}_1, \ldots, \mathbf{x}_s) = \sum_{\nu} \sum_{\text{div}} g_{s_1}(\Gamma_1) \ldots g_{s_\nu}(\Gamma_\nu). \qquad (12)$$

Once the operators $\mathcal{G}$, $\mathcal{W}$, M, and K for a medium made up of random scatterers have been expressed as series in the correlation function g_S,[2)] it becomes possible to introduce into this expansion the parameter γ, which is assigned as a factor to each of the functions g_1. The power of any term of the expansion will then be equal to the number of the correlation groups of these scatterers. We have already used such a device above, introducing a parameter (which we also denoted γ) into the expansion in the correlation functions of the potential k_S. Consequently, in the case (8) we have two expansions in powers of γ: one of them has been obtained with the aid of k_S and the other with the aid of g_S. We shall show that these expansions coincide. To this end, we turn to the formulas relating k_S with g_S. These formulas can be found in[18]. It follows from them that the k_S are linear in the g_S. Consequently, in the case of (8) both methods of introducing the parameter γ are equivalent.

Thus, we have formal expansions of $\mathcal{G}$, $\mathcal{W}$, M, and K in terms of the number of correlation groups, that is, in powers of γ and in case (8) we have two variants of the expansion, one of which uses the correlation functions of the potential and the other the correlation functions of the scatterer distribution. We shall obtain further approximations of the Green's function $\mathcal{W}$, based on the first term of the group expansion of the mass operator M_1. We shall not consider here the average double Green's function. We note also that the approximation of $\mathcal{W}$ in which the first term of the group expansion of the intensity operator K_1 is used turns out to be useful in the derivation of a radiation-transport equation in which account is taken of the scatterer correlation.

To conclude this section, we present expressions for M_1 and K_1. In the general case these will be series in k_S, represented by the diagrams of Figs. 2 and 3. The mean value $\langle V \rangle = k_1$ is best excluded by lumping it in the energy and thus replacing E by $E - \langle V \rangle$ (in particular, in the function G_0 [Eq. (3)] k_0^2 is now equal to $E - \langle V \rangle$). We shall therefore not assume henceforth that $k_1 = 0$. In the case (8), in addition, we can express M_1 and K_1 in the form of series in the functions g_S, which recall the group integrals introduced by Mayer in statistical physics[20].

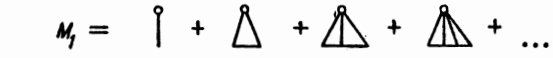

FIG. 2.

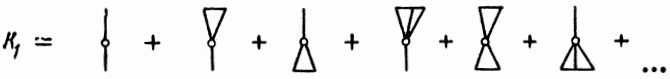

FIG. 3.

These expressions can be obtained by carrying out the group expansion not only in the distribution function f_S but also during an earlier stage, directly in the operators $G(\{\mathbf{x}_i\})$ and $G(\{\mathbf{x}_i\}) \otimes \overline{G(\{\mathbf{x}_i\})}$ which are to be averaged and which depend on the positions of the scatterers. We use to this purpose a procedure proposed in[11] (see also[21]), and represent $G(\{\mathbf{x}_i\})$ (the procedure is similar for $G \otimes \overline{G}$) in the form

$$G(\{\mathbf{x}_i\}) = G_0 + \sum_{\nu=1}^{\infty} \sum_{\text{subgr } \nu} G_\nu^{\text{gr}}, \qquad (13)$$

where $\sum_{\text{subgr } \nu}$ denotes summation over all possible subgroups of ν points from among the points $\{\mathbf{x}_i\}$. After writing down formulas (13) for the cases when the system of scatterers contains 1, 2, 3, ..., centers, we can obtain their inversion

$$G_\nu^{\text{gr}}(\mathbf{x}_1, \ldots, \mathbf{x}_\nu) = G(\mathbf{x}_1, \ldots, \mathbf{x}_\nu) - \sum_{\text{subgr}(\nu-1)} G(\mathbf{x}_{i_1}, \ldots, \mathbf{x}_{i_{\nu-1}}) + \sum_{\text{subgr}(\nu-2)} G(\mathbf{x}_{i_1}, \ldots, \mathbf{x}_{i_{\nu-2}}) - \ldots + (-1)^\nu G_0. \qquad (14)$$

The averaging of (13) and of the corresponding representation for $\mathcal{W}$ with the aid of (11) leads to the formulas

$$\mathcal{G} = \sum_{s=0}^{\infty} \frac{1}{s!} \int d^3x_1 \ldots \int d^3x_s\, G_s^{\text{gr}}(\mathbf{x}_1, \ldots, \mathbf{x}_s) f_s(\mathbf{x}_1, \ldots, \mathbf{x}_s),$$

$$\mathcal{W} = \sum_{s=0}^{\infty} \frac{1}{s!} \int d^3x_1 \ldots \int d^3x_s\, W_s^{\text{gr}}(\mathbf{x}_1, \ldots, \mathbf{x}_s) f_s(\mathbf{x}_1, \ldots, \mathbf{x}_s).$$

It remains to substitute here expression (12) for the distribution functions f_S in terms of the correlation g_S, and to calculate the operators M and K in the approximation linear in g_S. To this end it is convenient to use expressions (6) and (7) in the form

$$M = G_0^{-1} - \mathcal{G}^{-1},$$
$$K = (\mathcal{G} \otimes \overline{\mathcal{G}})^{-1} - \mathcal{W}^{-1}.$$

[2)] For independent scatterers this was done in [15].

The calculations yield finally

$$M_1 = \sum_{s=1}^{\infty} \frac{1}{s!} \int d^3x_1 \ldots \int d^3x_s\, g_s(\mathbf{x}_1, \ldots, \mathbf{x}_s)\, T_s^{gr}(\mathbf{x}_1, \ldots, \mathbf{x}_s),$$

$$K_1 = \sum_{s=1}^{\infty} \frac{1}{s!} \int d^3x_1 \ldots \int d^3x_s\, g_s(\mathbf{x}_1, \ldots, \mathbf{x}_s) \times [T_s(\mathbf{x}_1, \ldots, \mathbf{x}_s) \otimes \overline{T_s(\mathbf{x}_1, \ldots \mathbf{x}_s)}]^{gr}. \quad (15)$$

Here $T_S(\mathbf{x}_1, \ldots, \mathbf{x}_S)$ is the operator for scattering by the complex of s scatterers situated at the points $\mathbf{x}_1, \ldots, \mathbf{x}_S$, and is defined by (10) with $V = \sum_{i=1}^{S} U_i$. The symbol gr denotes the results of using the subtraction procedure provided for by formula (14), which results in exclusion of the effects of scattering by all subgroups containing only a part of the scatterers. If different scatterers are independent, then $g_2 = g_3 = \ldots = 0$, $g_1 = n$ (scatterer concentration), and formulas (15) go over into the approximation used in[15] and earlier in[22] for the model of pointlike scatterers.

3. AVERAGE GREEN'S FUNCTION IN THE CASE OF SMALL-SCALE INHOMOGENEITIES

We denote by $\mathcal{G}(M)$ the average Green's function defined by the operator M through expression (6). We wish to ascertain here under what conditions $\mathcal{G}(M_1)$ will differ little from $\mathcal{G}(M)$. To this end we go over to a Fourier representation in which the operator $\mathcal{G}$ is diagonal by virtue of the assumed statistical homogeneity. Solving Dyson's equation (6), we get

$$\mathcal{G}(\mathbf{p}|M_1) = \frac{1}{k_0^2 - p^2 - M_1(\mathbf{p})} = \int e^{-i\mathbf{p}(\mathbf{x}-\mathbf{x}')} \mathcal{G}(\mathbf{x}-\mathbf{x}'|M_1)\, d^3x_1$$

$$\mathcal{G}(\mathbf{p}|M) = \frac{1}{k_0^2 - p^2 - M(\mathbf{p})} = \mathcal{G}(\mathbf{p}|M_1) \times \left[1 - \frac{M(\mathbf{p}) - M_1(\mathbf{p})}{k_0^2 - p^2 - M_1(\mathbf{p})}\right]^{-1}. \quad (16)$$

It is therefore clear that replacement of M(p) by $M_1(\mathbf{p})$ leads to an approximation of the Green's function $\mathcal{G}$ which is uniformly exact for all $\mathbf{p}$, if[3)]

$$\max_{\mathbf{p}} \left| \frac{M(p) - M_1(p)}{k_0^2 - p^2 - M_1(p)} \right| \ll 1. \quad (17)$$

We now assume that $M_1(\mathbf{p})$ is a sufficiently small and smooth function of p, namely

$$\left| \frac{M_1(p)}{p^2} \right|_{p=k_0} \ll 1, \quad (18)$$

$$\left| \frac{dM_1(p)}{dp^2} \right|_{p=k_0} \ll 1. \quad (19)$$

Conditions (18) and (19), together with the assumption that the numerator of (17) varies sufficiently slowly with p, an assumption to which we shall return later, enables us to estimate the maximum in (17) and to transform this condition into

$$\left| \frac{M(p) - M_1(p)}{\operatorname{Im} M_1(p)} \right|_{p=k_0} \ll 1. \quad (20)$$

Of course, we cannot calculate $M(p) - M_1(p)$, since this difference is the sum of an infinite number of diagrams M(p), which are not accounted for in $M_1(p)$. It remains for us to follow the procedure of selecting those terms of the series for $M - M_1$, which are asymptotically principal with respect to some parameter. This can be done in the limiting case of long waves, when

$$k_0 l \ll 1. \quad (21)$$

The difference $M - M_1$ is represented by the sum of all possible diagrams containing not less than two correlation functions k_S. Let us compare all such diagrams with a given set of function $k_{S_1}, \ldots, k_S$ with one another. An analysis, which is conveniently carried out in the coordinate representation because of the localization of the functions $k_S(x_1, \ldots, x_S)$, shows that the principal among these topologically different diagrams are those having the form of a loop. These are obtained if one or several single-group insertions are made in one of the internal lines of the single-group diagrams for M_1. An example of such a diagram, containing four groups, is shown in Fig. 4a. In the integrals corresponding to these diagrams, the integrands increase most slowly if the different groups of points unified by the correlation functions k_S are made mutually remote. Other diagrams (for example, Fig. 4b) lead to integrals that converge more rapidly, as a result of which they acquire, compared with the single-loop diagrams, additional small factors $(k_0 l)^m$, $m > 0$ or $(k_0 l)^m \ln(1/k_0 l)$.

Retaining only those diagrams of the operator $M - M_1$ which are principal in the limit $k_0 l \ll 1$, we now sum these diagrams. As a result of the summation of the diagrams over all possible single-group insertions into a given internal line, this line is replaced by

$$G_0 M_1 G_0 + G_0 M_1 G_0 M_1 G_0 + \ldots = \mathcal{G}(M_1) - G_0. \quad (22)$$

Since the function (22) enters into the integrals for $M - M_1$ together with $k_S(x_1, \ldots, x_S)$, it is sufficient

[3)] This condition is obtained also if one stipulates that the operators $\mathcal{G}(M)$ and $\mathcal{G}(M_1)$ differ little in their norm.

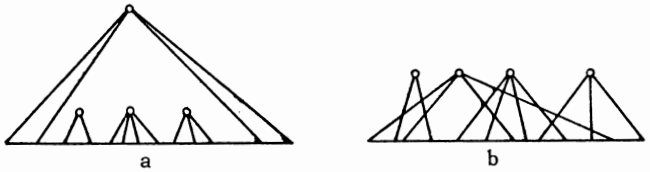

FIG. 4.

to know this function at values of the argument $\lesssim l$. Calculating for $x \to 0$ the integral

$$\mathcal{G}(x|M_1) - G_0(x) = \frac{1}{(2\pi)^3} \times \int d^3p\, e^{ipx} \frac{M_1(p)}{[k_0^2 - p^2 - M_1(p)][k_0^2 - p^2 + i0]}$$

in first order in the small parameter $(M_1/p^2)_{p=k_0}$, we get:

$$[\mathcal{G}(x|M_1) - G_0(x)]_{x\to 0} \cong \frac{iM_1(p)}{8\pi k_0}\Big|_{p=k_0}.$$

We consequently arrive at the conclusion that the difference $M - M_1$, which is asymptotically exact when $k_0 l \to 0$, is represented by a sum of single-group diagrams of the mass operator, in each of which one of the propagation functions (4) has been replaced by $iM_1(p)/8\pi k_0|_{p=k_0}$ in all possible manners.

We return to the denominator of the condition (20), that is, to $\mathrm{Im}M_1(p)$. Let us consider the diagram M_1 containing the function k_S. In the corresponding integral we can write, as a result of the condition $k_0|x_j - x_{j+1}| \ll 1$,

$$\mathrm{Im}[G_0(x_1 - x_2)\ldots G_0(x_{s-1} - x_s)\exp\{ip(x_1 - x_s)\}] \cong \left(-\frac{1}{4\pi}\right)^{s-1} \times \frac{k_0|x_1 - x_2| + \ldots + k_0|x_{s-1} - x_s| + p(x_1 - x_s)}{|x_1 - x_2|\ldots|x_{s-1} - x_s|}$$

The term $p(x_1 - x_S)$ vanishes after integration over the angles, and we arrive at the conclusion that the quantity $\mathrm{Im}M_1(p)$, which is asymptotically exact when $k_0 l \to 0$, is represented by a sum of single-group diagrams of the mass operator, in each of which one of the propagation functions (4) has been replaced by $-k_0/4\pi$ in all possible manners.

Comparison with the analogous representation of $M - M_1$ leads to the conclusion that when $pl \ll 1$ we have

$$\frac{M(p) - M_1(p)}{\mathrm{Im}\,M_1(p)} \cong -\frac{iM_1(p)}{2k_0^2}. \qquad (23)$$

Thus, (20) reduces to the condition (8), which was already assumed by us. The inequality (19) also follows from (18) and (21), inasmuch as when $pl \ll 1$, we have $dM_1/dp^2 \sim l^2 M_1$, owing to the localization of $M_1(x - x')$ in a region with dimensions $\sim l$. Finally, in the approximation (23), the smooth dependence of the numerator (17) on p adds nothing to the conditions (18) and (19).

The final conclusion is consequently that for small-scale inhomogeneities (that is, in the limit when $k_0 l \ll 1$), the condition for the applicability of the group approximation of M_1 of the mass operator is the inequality (18).

4. AVERAGE GREEN'S FUNCTION IN THE CASE OF LARGE-SCALE INHOMOGENEITIES

In the analysis of the short-wave asymptotic value of the average Green's function it is expedient to use the method of approximate summation of the perturbation-theory series which was employed in the case of the normal potential $V(\mathbf{x})$[4,9]. To this end we go over in the Green's function to the t-representation, performing a Fourier transformation with respect to the energy:

$$G(\mathbf{x}, \mathbf{x}', E) = \int_{-\infty}^{\infty} dt\, e^{i(E+i0)(t-t')} G(\mathbf{x}, \mathbf{x}', t - t'). \qquad (24)$$

Actually, for the retarded Green's function we have $G(\mathbf{x}, \mathbf{x}', t - t') = 0$ when $t < t'$. We shall need in what follows the Green's function $G(\mathbf{x}, \mathbf{p})$ in the mixed coordinate-momentum representation, when the Fourier transformation is carried out in terms of the coordinate difference

$$G(\mathbf{x}, \mathbf{p}) = \int d^3x'\, e^{-ip(x-x')} G(\mathbf{x}, \mathbf{x}').$$

The potential $V(\mathbf{x})$ and the function G_0 will also be taken in the p-representation:

$$V(\mathbf{x}) = \int d^3q\, e^{iqx} V(\mathbf{x}),$$
$$G_0(\mathbf{p}, t) = -ie^{-itp^2}, \quad t > 0. \qquad (25)$$

The momenta q_j from the Fourier transformations of the potential $V(x_j)$ will henceforth be called virtual.

Having agreed on the notation, we proceed to calculate the average Green's function. We use the expression obtained in[4] [4)]

$$G(\mathbf{x}, \mathbf{p}, t) = -i\exp\left[-itp^2 - i\int_0^t dt'\, V(\mathbf{x} - 2\mathbf{p}t')\right] \times \left\{1 + \int_0^t dt' \int_0^{t'} dt''\, \nabla^2 V(\mathbf{x} - 2\mathbf{p}t + 2\mathbf{p}t'') - i\int_0^t dt' \left[\int_0^{t'} dt''\, \nabla V(\mathbf{x} - 2\mathbf{p}t + 2\mathbf{p}t'')\right]^2 + \ldots\right\}, \qquad (26)$$

[4)] The signs differ from those in [4] because of Eqs. (1) and (24).

which leads to the approximation of the equivalent screen method (e.s.m.) for the wave function and which presupposes satisfaction of the inequality $pl \gg 1$. We note with respect to this requirement that, in general, the averaging of expression (26) can lead to a change in the conditions of its applicability. In particular, it is necessary to require a smooth dependence on the coordinates not for the inhomogeneous potential $V(\mathbf{x})$, but for its correlation functions k_s, which is not at all the same. For example, smoothly varying k_s can describe a potential produced by sharply bounded scatterers, say spheres.

To average the principal term of formula (26), we use the expression for the characteristic functional of a random field $V(\mathbf{x})$ in terms of the correlation functions k_s [18]:

$$\mathcal{F}[\alpha(\mathbf{y})] = \left\langle \exp\left[i \int d^3y\, \alpha(\mathbf{y}) V(\mathbf{y}) \right] \right\rangle$$

$$= \exp\left[\sum_{s=1}^{\infty} \frac{i^s}{s!} \int d^3y_1 \ldots \int d^3y_s\, k_s(\mathbf{y}_1, \ldots, \mathbf{y}_s) \alpha(\mathbf{y}_1) \ldots \alpha(\mathbf{y}_s) \right].$$

The correction terms (26) are averaged with the aid of the relation

$$\left\langle V(\mathbf{x}_1) \ldots V(\mathbf{x}_\nu) \exp\left[i \int d^3y\, \alpha(\mathbf{y}) V(\mathbf{y}) \right] \right\rangle = (-i)^\nu \frac{\delta^\nu \mathcal{F}[\alpha(\mathbf{y})]}{\delta\alpha(\mathbf{x}_1) \ldots \delta\alpha(\mathbf{x}_\nu)}.$$

As a result of the calculations we arrive at the following expression (the dependence on x vanishes as a result of the statistical homogeneity):

$$\langle G(\mathbf{x}, \mathbf{p}, t) \rangle = \mathcal{G}_{\text{pr}}(\mathbf{p}, t)[1 + P_1(\mathbf{p}, t) + P_2(\mathbf{p}, t)], \quad (27)$$

where

$$\mathcal{G}_{\text{pr}}(\mathbf{p}, t) = -i \exp\left[-itp^2 + \sum_{s=1}^{\infty} \frac{(-i)^s}{s!} \int_0^t dt_s \ldots \int_0^t dt_1\, k_s(2\mathbf{p}t_s, \ldots, 2\mathbf{p}t_1) \right], \quad (28)$$

$$P_1(\mathbf{p}, t) = i \sum_{s=1}^{\infty} \frac{(-i)^s}{s!} \int_0^t dt_s \ldots \int_0^t dt_1 \sum_{m=1}^{s} \sum_{n=1}^{s} [t - \max(t_m, t_n)] \times (\nabla_m, \nabla_n) k_s(2\mathbf{p}t_s, \ldots, 2\mathbf{p}t_1), \quad (29)$$

and

$$P_2(\mathbf{p}, t) = i \sum_{s=2}^{\infty} \sum_{\sigma=2}^{\infty} \frac{(-i)^{s+\sigma}}{s!\sigma!} \int_0^t dt_s \ldots \int_0^t dt_1 \int_0^t d\tau_\sigma \ldots \int_0^t d\tau_1 \times \sum_{n=1}^{s} \sum_{\nu=1}^{\sigma} [t - \max(t_n, \tau_\nu)] (\nabla_n k_s(2\mathbf{p}t_s, \ldots, 2\mathbf{p}t_1) \times \nabla_\nu k_\sigma(2\mathbf{p}\tau_\sigma, \ldots, 2\mathbf{p}\tau_1)). \quad (30)$$

For a normally distributed potential $V(x)$ we have $k_3 = k_4 = \ldots 0$, and formulas (27)—(30) go over into the corresponding formulas of [4,9].

Let us proceed to ascertain the conditions under which it is possible to confine oneself in (27) to the principal term $\mathcal{G}_{\text{pr}}(\mathbf{p}, t)$. This can be done, as in [4] by stipulating that the correction be small, $|P_1(\mathbf{p}, t) + P_2(\mathbf{p}, t)| \ll 1$, in the entire time domain in which the function $\mathcal{G}_{\text{pr}}(\mathbf{p}, t)$ is not yet negligibly small compared with unity. If $\mathcal{G}_{\text{pr}}(\mathbf{p}, t)$ attenuates within times that are short compared with the time required for the wave to cover the correlation scale ($\sim l/p$), then, as shown by analysis, without assuming the potential normal, it is no longer possible to expect that the attenuation of $\mathcal{G}$ with time is described by the Gaussian law $\exp[-\langle V^2 \rangle t^2/2]$. From this point of view, the formulas obtained for the spectrum of the electron in the presence of random impurities [2-4] from the function $\mathcal{G} \sim \exp[-\langle V^2 \rangle t^2/2]$, can be applicable only for sufficiently large impurity concentrations n, satisfying the condition $nl^3 \gg 1$ (l is the effective radius of the impurity-center potential). In this case the potential $V(\mathbf{x})$ turns out to be Gaussian because it is made up of a set of independent impurity potentials. (Compare with the condition $na_0^3 \gg 1$ [2], and also with analogous reasoning in the case of the problem of multiple Coulomb scattering [24]).

We shall henceforth be interested in a different case, when the attenuation time of $\mathcal{G}_{\text{pr}}(\mathbf{p}, t)$ is large compared with l/p. We wish to obtain in this case the conditions for the applicability from a somewhat different point of view, which is analogous to that assumed in the preceding section. Namely, we stipulate that the function $\mathcal{G}_{\text{pr}}(\mathbf{p}, E)$ give a uniform approximation of the exact function $\mathcal{G}(\mathbf{p}, E)$ for all energies E:

$$\max_E \left| \frac{\mathcal{G}(\mathbf{p}, E) - \mathcal{G}_{\text{pr}}(\mathbf{p}, E)}{\mathcal{G}(\mathbf{p}, E)} \right| \leqslant 1\ ^{5)}. \quad (31)$$

The functions $\mathcal{G}(\mathbf{p}, E)$ and $\mathcal{G}_{\text{pr}}(\mathbf{p}, E)$ are expressed by the Fourier integral (24). In the case of weak attenuation of $\mathcal{G}(p, t)$ the main contribution to the integral is made by the region $t > l/p$. In this region $\mathcal{G}_{\text{pr}}(p, t)$, $P_1(p, t)$, and $P_2(\mathbf{p}, t)$ have the following asymptotic form:

$$\mathcal{G}_{\text{pr}} = -i \exp[-itp^2 - i\langle V \rangle t - i(A_0 t + B_0)], \quad P_1 = -i(A_1 t + B_1), \quad P_2 = -i(A_2 t + B_2), \quad (32)$$

---

[5] If we regard $\mathcal{G}(p, t - t')$ as a linear operator in the space of functions of the time, then this means that the norms of $\mathcal{G}_{\text{pr}}$ and $\mathcal{G}$ differ little [23].

where

$$A_0=\sum_{s=2}^{\infty}\frac{(-i)^s}{(s-1)!}\int_0^{\infty}dt_s\ldots\int_0^{\infty}dt_2\,k_s(2\mathbf{p}t_s,\ldots,2\mathbf{p}t_2,0),$$

$$A_1=\sum_{s=2}^{\infty}\frac{(-i)^s}{(s-1)!}\int_0^{\infty}dt_s\ldots\int_0^{\infty}dt_2\sum_{m=2}^{s}\sum_{n=2}^{s}\max(t_m,t_n)\,(\nabla_m\cdot\nabla_n)\,k_s,$$

$$A_2=-\Big[\sum_{s=2}^{\infty}\frac{(-i)^s}{(s-1)!}\int_0^{\infty}dt_s\ldots\int_0^{\infty}dt_2\sum_{m=2}^{s}t_m\nabla_m k_s\Big]^2$$

$$=\frac{1}{4}\Big(\frac{dA_0}{dp}\Big)^2. \tag{33}$$

We shall not present here any formulas for B_i, for they can be neglected under the conditions formulated below.

As shown by simple analysis, the inequality (31) can be satisfied by stipulating fulfillment of the conditions:

$$pl\gg 1, \tag{34}$$

$$|\operatorname{Im}A_0|\sim|A_0|\ll p/l, \tag{35}$$

$$|dA_0/dp|^2\ll|A_0|. \tag{36}$$

An examination of the examples allows us to assume that conditions (34)—(36), can apparently be made more precise by a more thorough analysis. Thus, in the case of independent scattering centers, they take the form

$$|A_0|\ll p/l,\quad |A_0/\operatorname{Im}A_0|\ll pl.$$

The same conditions remain in force also in the case of a normal potential $V(\mathbf{x})$.

Under the assumptions (34)—(36) we can, without violating (31), extend the asymptotic first formula (32) to include all values of t, assuming by the same token

$$\mathcal{G}(\mathbf{p},E)=\frac{1}{E-\langle V\rangle-p^2-A_0(\mathbf{p})}. \tag{37}$$

In concluding this section we present the expression into which the first formula of (33) goes over in the case of scattering by correlated scattering centers:

$$A_0(\mathbf{p})=\sum_{s=1}^{\infty}\frac{1}{(s-1)!}\int d^3x_1\ldots\int d^3x_s\,g_s(\mathbf{x}_1\ldots\mathbf{x}_s)\,U(\mathbf{x}_1)$$

$$\times(1+\beta(\mathbf{x}_1))\,\beta(\mathbf{x}_2)\ldots\beta(\mathbf{x}_s),$$

$$\beta(\mathbf{x})=\exp\Big[-i\int_0^{\infty}dtU(2\mathbf{p}t+\mathbf{x})\Big]-1.$$

To derive this expression it is necessary to average the Green's function $G(\mathbf{x},\mathbf{p},t)$ with the e.s.m. (26), using a generating functional[18] for the system of random scatterers, and then calculate the asymptotic form of the argument of the exponential in $\mathcal{G}_{\mathrm{pr}}(\mathbf{p},t)$.

5. CONNECTION BETWEEN THE e.s.m. AND THE ONE-GROUP APPROXIMATION IN THE MASS OPERATOR

The Green's function (37) recalls the expression (16) for $\mathcal{G}$ in terms of the mass operator. We shall now show that these are not only outwardly similar. To this end, we calculate the operator M_1 in the e.s.m. approximation. We shall use the equality $M_1=\langle T\rangle_1$, which can be verified, for example, by expanding in a power series in the potential. Here $\langle T\rangle_1$ is the one-group term of the expansion of $\langle T\rangle$ in the correlation functions. The operator T is defined by (10) and is connected with G by the relation

$$T=G_0^{-1}(G-G_0)G_0^{-1}.$$

Substituting here the initial approximation of the e.s.m. (26) for the Green's function G, and going over to the momentum representation, we get

$$\langle T(\mathbf{q},\mathbf{p},E)\rangle=-i\delta(\mathbf{p}-\mathbf{q})$$

$$\times\int_0^{\infty}dt\,(k_0^2+i0-p^2)^2\Big\langle\exp\Big[-i\int_0^t dt'V(2\mathbf{p}t')\Big]-1\Big\rangle_1.$$

Hence

$$M_1(\mathbf{p},E)\,|_{k_0=p}$$

$$=i\lim_{\varepsilon\to+0}\varepsilon^2\int_0^{\infty}dt\,e^{-\varepsilon t}\Big\langle\exp\Big\{-i\int_0^t dt'\,V(2\mathbf{p}t')\Big\}-1\Big\rangle_1.$$

By integrating by parts and using the well known Abel formula[23]

$$\lim_{\varepsilon\to+0}\varepsilon\int_0^{\infty}e^{-\varepsilon t}f(t)\,dt=\lim_{t\to\infty}f(t),$$

we get

$$M_1(\mathbf{p},E)\,|_{k_0=p}=i\lim_{t\to\infty}\frac{df(t)}{dt},$$

where

$$f(t)=\Big\langle\exp\Big[-i\int_0^t dt'\,V(2\mathbf{p}t')\Big]-1\Big\rangle_1$$

$$=\sum_{s=2}^{\infty}\frac{(-i)^s}{s!}\int_0^t dt_s\ldots\int_0^t dt_1\,k_s(2\mathbf{p}t_s,\ldots,2\mathbf{p}t_1).$$

Comparing this expression with (28) and the first formula of (33), we get $f(t)=-i(A_0t+B_0)$ as $t\to\infty$, hence

$$M_1(\mathbf{p},E)\,|_{k_0=p}=A_0(\mathbf{p}). \tag{38a}$$

We analogously calculate

$$\frac{\partial M_1(\mathbf{p},E)}{\partial E}\Big|_{k_0=p}=\lim_{t\to\infty}\Big(f-t\frac{df}{dt}\Big)=-iB_0(\mathbf{p}). \tag{38b}$$

6. NEGLECT OF SPATIAL DISPERSION IN THE AVERAGE GREEN'S FUNCTION

The average Green's function $\mathcal{G}(M_1)$ corresponds to a certain effective homogeneous absorbing medium, for which $M_1(\mathbf{x}-\mathbf{x}')$ plays the role of an effective potential. In the acoustic or electromagnetic case, this place is occupied by the effective refractive index or the effective dielectric constant of the medium. $M_1(\mathbf{x}-\mathbf{x}')$ has a temporal dispersion (dependence on k_0) and a spatial dispersion (nonlocality, or, in the p-representation, a dependence on p). On the other hand, "absorption" (in fact, scattering) of the average field is described by $\mathrm{Im}M_1$.

We wish to ascertain here under what conditions we can make the substitution

$$M_1(p,E)\to M_1(k_0,E), \tag{39}$$

which denotes neglect of the spatial dispersion. As in Secs. 3—4, we start from the condition that the substitution (39) introduces in $\mathcal{G}(M_1)$ an error that is uniformly small for all p, that is,

$$\max_p \left| \frac{M_1(p)-M_1(k_0)}{k_0^2-p^2-M_1(k_0)} \right| \ll 1. \tag{40}$$

Estimating the left side of (40), we linearize the numerator

$$M_1(p)-M_1(k_0)\cong \frac{dM_1}{dp^2}\Big|_{p=k_0}(p^2-k_0^2).$$

In this approximation, the maximum can be calculated exactly. It is attained when

$$p^2=p_0^2\equiv k_0^2-\frac{|M_1(k_0)|^2}{\mathrm{Im}\,M_1(k_0)}$$

and is equal to

$$\left|\frac{dM_1}{dp^2}\frac{M_1(p)}{\mathrm{Im}\,M_1(p)}\right|_{p=k_0}=\left|\frac{M_1(p_0)-M_1(k_0)}{M_1(k_0)}\right|. \tag{41}$$

It is clear therefore that if expression (41) is much smaller than unity, then the linearization of $M_1(p) - M_1(k_0)$ is valid, and by the same token the condition (40) is satisfied. If, to the contrary, this expression is $\gtrsim 1$, then the maximum of (41) lies in a region where the linearization is not valid. In this case it is sufficient to estimate the numerator of (40) for $|p^2-k_0^2| \sim |\mathrm{Im}M_1(k_0)|$, where $|\mathrm{Im}M_1(k_0)|$ is the width of the sharp function $|k_0^2-p^2-M_1(k_0)|^{-1}$. With this, we get from the condition (40)

$$|dM_1/dp^2|_{p=k_0}\ll 1. \tag{42}$$

Satisfaction of the inequality (42) can be readily verified in both cases considered above, $k_0 l \ll 1$ and $k_0 l \gg 1$. When $k_0 l \ll 1$, it coincides with the already verified condition (19), and when $k_0 l \gg 1$, it follows directly from (34)—(36).

After the foregoing simplifications, it is clear that in the coordinate representation

$$\mathcal{G}(\mathbf{x}) = -e^{ik_1|\mathbf{x}|}/4\pi|\mathbf{x}|, \tag{43}$$

where

$$k_1=\sqrt{k_0^2-M_1(k_0)}\cong k_0-M_1(k_0)/2k_0.$$

We shall not stop here to discuss the limitations imposed by formula (43) on the permissible distances $|\mathbf{x}|$.

7. METHOD OF SMOOTH PERTURBATIONS FOR THE AVERAGE GREEN'S FUNCTION

As is well known, the averaging of the Green's function written out in first approximation of the method of smooth perturbations (s.p.m.) leads, even in the case of a normal potential V(x), to an incorrect asymptotic form at large times ($\mathcal{G}(\mathbf{p}, t)$) or large distances ($\mathcal{G}(\mathbf{x}, E)$)[9]. The use of certain definite methods of summing the perturbation-theory series directly for the averaged functions $\mathcal{G}$ makes it possible to obtain an expression that is free of the foregoing difficulty[4,9]. It can be shown that this expression is equivalent to calculating $\ln \mathcal{G}(\mathbf{p}, t)$ in first order in the correlation function $k_2(\mathbf{x}_1, \mathbf{x}_2)$ of the potential V(x) (we refer here to a normally distributed potential), and it is therefore natural to connect it with the s.p.m. for the average Green's function.

Using the expansion in terms of correlation groups, we are able to generalize the foregoing expression to the case of an arbitrarily distributed potential V(x). To this end, we represent $\mathcal{G}(\mathbf{p}, t)$ in the form of a power series in the parameter γ, the degree of which is equal to the number of the correlation groups in the given term:

$$\mathcal{G}(\mathbf{p},t)=-ie^{-itp^2}\left(1+\sum_{s=1}^{\infty}\gamma^s ie^{itp^2}\mathcal{G}_s(\mathbf{p},t)\right),$$

and take logarithms

$$\ln\left[i\exp(itp^2)\,\mathcal{G}(\mathbf{p},t)\right]=\sum_{s=1}^{\infty}\gamma^s\left[ie^{itp^2}\mathcal{G}_s-\frac{1}{2}\times\sum_{s_1\geqslant 1}\sum_{s_2\geqslant 1;\ s_1+s_2=s} ie^{itp^2}\mathcal{G}_{s_1}\cdot ie^{itp^2}\mathcal{G}_{s_2}+\ldots\right]. \tag{44}$$

Starting from (44), and assuming the correlation radius of the potential V(x) to be finite, we can show that each term of the group expansion of $\ln \mathcal{G}(p, t)$ takes on an asymptotic value that is linear in t as $t \to \infty$. In turn, the linear dependence of $\ln \mathcal{G}(p, t)$ on the time as $t \to \infty$ is equivalent to definite analytic properties of the function $\mathcal{G}(\mathbf{p}, E)$ in the upper half-plane of the complex variable E. Namely, it implies that the singularity of the function $\mathcal{G}(\mathbf{p}, E)$

$= (E - \langle V \rangle - p^2 - M(\mathbf{p}, E))^{-1}$ which is closest to the real axis is the pole $E = E(\mathbf{p})$, and expansion (44) of $\ln \mathcal{G}(\mathbf{p}, t)$ reflects the expansion of the solution $E = E(p)$ of the dispersion equation

$$E - \langle V \rangle - p^2 - M(\mathbf{p}, E) = 0$$

in a series in a number of correlation groups. (We note that from this point of view the proper place of the corrections $P_1(\mathbf{p}, t)$ and $P_2(\mathbf{p}, t)$ in formula (27) is precisely the argument of the exponential.)

Let us discuss in somewhat greater detail the one-group approximation in the expansion (44), which can be written in the form

$$\mathcal{G}(\mathbf{p}, t) = -i \exp\left[-itp^2 - i\int_0^t d\tau (t-\tau) e^{i\tau p^2} M_1(\mathbf{p}, \tau)\right]. \quad (45)$$

From this we get as $t \to \infty$,

$$\mathcal{G}(\mathbf{p}, t) = -i \exp[-itp^2 - itM_1(\mathbf{p}, E)|_{k_0=p} + (\partial M_1(\mathbf{p}, E)/\partial E)|_{k_0=p}]$$

(cf. the formulas (38)). Formula (45), as follows from a comparison with the approximations considered above for the average Green's function in the case of long and short waves, goes over, if conditions (21) and (18) [or (34)—(36)] into these approximations. On the other hand, formula (45) (as well as (27)) gives the exact value of $\mathcal{G}(\mathbf{p}, t)$ for a random potential V that does not depend on $\mathbf{x}$. Thus, it possesses a sufficiently broad region of applicability.

It is curious to trace the connection between (45) and the summation of the perturbation series (3). Omitting the details of the rather cumbersome proof, which is conveniently carried in the t-representation and we present the result.

Let us imagine an arbitrary diagram for the average Green's function $\mathcal{G}(\mathbf{p}, E)$, for example that shown in Fig. 1a. To each segment of the propagation line in the $\mathbf{p}$-representation there corresponds a factor $1/(k_0^2 - \mathbf{q}^2)$, in which the momentum $\mathbf{q}$ consists of the external momentum $\mathbf{p}$ which enters into the diagram, and of the virtual momenta $\mathbf{q}_j$ (arguments of the functions k_s) in accordance with the rule for the momentum conservation at the nodes. The expression for q^2 contains squares of the virtual momenta q_j^2 and their pairwise products $\mathbf{q}_i\mathbf{q}_j$. We retain from among the latter only those products in which both factors pertain to one group (that is, are arguments of one correlation function k_s), and leave out all the pairwise products of virtual momenta with different correlation groups. The sum of all the diagrams transformed in this manner gives the function (45).

8. CONCLUSION

We wanted to show in the present paper that the approximations usually employed in the calculation of the average Green's functions are not connected with any of the customary assumptions, either that the potential $V(\mathbf{x})$ has a normal distribution or that the scattering centers are independent. We were able to generalize these approximations to include an arbitrarily distributed random potential. It turned out here that the different approximations are based on the single-group term of the expansion of the mass operator M in the number of correlation groups, denoted by M_1. The obtained applicability conditions do not contain limitations on the individual terms of the series representing M_1, and therefore admit of cases in which the effects of higher correlations are not small. We hope in the future to show, by means of very simple models, that such a situation is indeed possible.

Our estimates as applied to the normal potential V lead to the applicability conditions obtained by Andreev[4,9], but differ, and always on the more stringent side, from the results of other authors[6,7,15-17]. We note in this connection that the required uniform approximation of the Green's function $\mathcal{G}$, from which we have always started, seems to us to be sufficiently justified.

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