AN EXACT STATIC AXIAL SOLUTION OF THE EINSTEIN EQUATIONS

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An exact solution of the equations of GTR is presented for a special case of an axially symmetric gravitational field. Its connection with the well known solutions of Schwarzschild, Einstein, and de Sitter is shown; expressions are given for the proper pressure p_0 and density ρ_{00} and for the dependence of the Gaussian curvature on p_0, ρ_{00}.

WE have found an exact solution of the equations of gravitation for a special case of static axial symmetry of the field, which is an extension of the well known internal solution of Schwarzschild.[1]

The explicit form of this solution in the "polar" coordinate system[2,3] r, ϑ, φ, τ is

$$ds^2 = -\frac{1}{[\alpha+\beta(1-r^2/R^2)^{1/2}]^2}\left\{\frac{dr^2}{1-r^2/R^2} + r^2(d\vartheta^2+\sin^2\vartheta d\varphi^2) - [\eta+\gamma(1-r^2/R^2)^{1/2}+\delta r\cos\vartheta]^2 d\tau^2\right\}, \quad (1)$$

where α, β, γ, δ, η, $c_0 = R^{-2}$ are arbitrary constants of integration, whose physical meaning can be found by going to the limit and connecting the internal and external solutions for a concrete axially symmetric problem.

As for the coordinate system r, ϑ, φ, τ, it is assumed that $r = 0$ is the axis of symmetry of the distribution of matter, τ is the timelike coordinate, and φ is an angle coordinate, $0 \le \varphi \le 2\pi$. The axial character of the solution is connected with the presence of the constant of integration δ in Eq. (1).

For certain values of the constants of integration the solution (1) goes over into previously known solutions for a gravitational field with spherical symmetry. For $\alpha = 1$, $\beta = \delta = 0$ we get the well known internal solution of Schwarzschild:

$$ds^2 = -\frac{dr^2}{1-r^2/R^2} - r^2(d\vartheta^2+\sin^2\vartheta d\varphi^2) + [\eta+\gamma(1-r^2/R^2)^{1/2}]^2 d\tau^2. \quad (2)$$

For $\alpha = \eta = 1$, $\beta = \delta = \gamma = 0$ we get the expression for the Einstein line element:

$$ds^2 = -\frac{dr^2}{1-r^2/R^2} - r^2(d\vartheta^2+\sin^2\vartheta d\varphi^2) + d\tau^2. \quad (3)$$

Finally, for $\alpha = \gamma = 1$, $\beta = \eta = \delta = 0$ we get the de Sitter line element:

$$ds^2 = -\frac{dr^2}{1-r^2/R^2} - r^2(d\vartheta^2+\sin^2\vartheta d\varphi^2) + (1-r^2/R^2)d\tau^2. \quad (4)$$

Comparing the solutions (2)–(4) with (1), we can conclude that with the least restrictions on the constants of integration the solution (1) generalizes the internal Schwarzschild solution (2) to the case of static axial symmetry of the field.

We remark that the solution (1) was found by us with the so-called g-method approach to the equations of GTR.[4] This method for solving the solutions of GTR agrees in general principle with the method of Tolman,[5] and also with the method of formulation of the solution of internal problems as given by Fock.[6]

After lengthy calculations the corresponding expressions for the proper pressure p_0 and density ρ_{00} in the model with the metric (1) take the forms

$$8\pi p_0 = -\frac{1}{R^2}(\alpha^2-\beta^2) - \frac{2}{R^2}\left\{\frac{(\alpha\gamma-\beta\eta)[\beta+\alpha(1-r^2/R^2)^{1/2}]+\delta r\cos\vartheta}{[\eta+\gamma(1-r^2/R^2)^{1/2}+\delta r\cos\vartheta]}\right\}+\Lambda,$$

$$8\pi\rho_{00} = \frac{3}{R^2}(\alpha^2-\beta^2)-\Lambda. \quad (5)$$

Without giving a detailed analysis of these formulas, we point out only one interesting case of a distribution of density and pressure: if $\alpha^2 - \beta^2 = 0$, then $\rho_{00} = 0$ (for $\Lambda = 0$), $p_0 \neq 0$.

We shall give the expressions for some geometric invariants for the metric (1).

Following Eddington,[2] we write the invariant

$$R_0 = R^{\varepsilon}_{\mu\nu\sigma}R_{\varepsilon}^{\mu\nu\sigma} = \frac{3}{R^4}(\alpha^2-\beta^2)^2 + \{(\alpha\gamma-\beta\eta)[\beta+\alpha(1-r^2/R^2)^{1/2}] + (\alpha^2-\beta^2)\delta r\cos\vartheta\}^2 + \{[\alpha+\beta(1-r^2/R^2)^{1/2}]+2\}^2 \times R^{-4}[\eta+\gamma(1-r^2/R^2)^{1/2}]+\delta r\cos\vartheta]^{-2}. \quad (6)$$

The invariant $G = G_{\mu\nu}g^{\mu\nu}$ (the Gaussian curvature) for the metric (1) is of the form

$$G = \frac{6}{R^2}(\alpha^2-\beta^2) + \frac{6}{R^2}\left\{\frac{(\alpha\gamma-\beta\eta)[\beta+\alpha(1-r^2/R^2)^{1/2}]+(\alpha^2-\beta^2)\delta r\cos\vartheta}{[\eta+\gamma(1-r^2/R^2)^{1/2}+\delta r\cos\vartheta]}\right\} \quad (7)$$

or when we use the expressions (5) for p_0 and ρ_{00},

$$G = -8\pi(3p_0-\rho_{00})+4\Lambda. \quad (8)$$

We point out that in the model filled only with isotropic radiation $p_0 = \rho_{00}/3$ and therefore $G = 4\Lambda$, i.e., for $\Lambda \to 0$ we have $G \to 0$, although the curvature components $R^{\epsilon}_{\mu\nu\sigma}$ do not go to zero.

Finally we note that the solution (1) belongs to the type T_I according to Petrov's classification.[7]

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[4]J. L. Synge, Relativity, The General Theory, North-Holland Publishing Company, Amsterdam, 1960, pp. 189–193.
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