STABILITY OF SIMPLE STATIONARY WAVES RELATED TO THE NONLINEAR DIFFUSION EQUATION

V. M. ELEONSKIĬ

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The stability of solutions of the simple-wave type for a nonlinear diffusion equation is investigated by the quasiclassical approximation method.

A number of problems, such as the propagation of waves in semiconductors with negative differential resistance, or the propagation of a flame front or a thermal-ionization front, make it necessary to find and investigate solutions of the type of a simple steady-state wave for a nonlinear diffusion equation of the type

$$\left(\frac{\partial}{\partial t}-D\frac{\partial^2}{\partial x^2}-D\Delta_\perp\right)c=Q(c). \tag{1}$$

Here $c(t, x, r_\perp)$ is the field of concentrations at the point $(x, r_\perp)$, D is the diffusion coefficient, and Q(c) is the nonlinear source due to the presence in the system of slow creation and annihilation processes. The Laplace operator is represented in the form of a sum of the longitudinal and transverse operators. We shall consider below simple steady-state waves with plane fronts.

For the case of a constant-sign source that vanishes at two points corresponding to the state of equilibrium of the system, solutions of the type of a simple steady-state wave connected with the asymptotic transition system from one equilibrium state to the other were first investigated by Kolmogorov, Petrovskiĭ, and Piskunov [1]. Subsequently, simple waves were investigated for linear sources of this kind in connection with different models of flame-front propagation [2,3]. Recently simple steady-state waves were considered for the case of an alternating-sign source Q(c) in connection with a study of slow waves in distributed systems with N-shaped current-voltage characteristic. [4,5]

We investigate in this paper the stability of simple steady-state waves connected with the nonlinear diffusion equation (1).

We use the method of quasiclassical approximation for the analysis of the stability of simple steady-state waves. We determine the region of decrements or increments for the perturbed problem, for which in the quasiclassical approximation the eigenfunctions are bounded in the entire space. For the instability region we determine the conditions under which the instability is of the drift type. The main difference between the simple steady-state waves connected with the nonlinear equation of diffusion and other types of simple waves, such as those considered by Berezin and Karpman [6], is that in the present case both the unperturbed and perturbed problems are not self-conjugate and do not admit the existence of first integrals of the conservation-law type.

Let the nonlinear source be a positive and convex function of the concentration vanishing at c = 0 and c = 1. Obviously, the derivative of the nonlinear source with respect to the concentration is positive at c = 0 and negative at c = 1. Putting

$$c(t,x,r_\perp)=c(\eta)\equiv c(x+ut)$$

where u is the constant rate of propagation of the simple steady-state wave, and using (1), we find that the distribution of the concentrations for the wave under consideration is determined by a solution of the equation

$$\left(-D\frac{d^2}{d\eta^2}+u\frac{d}{d\eta}\right)c=Q(c) \tag{2}$$

under the following boundary conditions:

$$\lim_{\eta\to+\infty}c=1,\qquad \lim_{\eta\to-\infty}c=0. \tag{3}$$

This boundary-value problem determines the simple steady-state wave connected with the asymptotic transition of the system from one state of equilibrium to another. It is known [1] that in phase plane $(dc/d\eta, c)$ there corresponds to this wave a trajectory that emerges from a saddle-like singular point and goes to a node-type singular point. It is shown in the cited paper that solutions of the type of a simple steady-state wave exists if the proper parameter of the problem, the wave propagation velocity, is given by

$$u\geqslant u_{min}=2\left[D\frac{dQ}{dc}\Big|_{c=0}\right]^{1/2}.$$

This raises the question of the stability of simple steady-state waves against small perturbations. Let us consider the perturbed state of the system

$$c(t,x,r_\perp)=c(x+ut)+\Psi(t,x,r_\perp)\equiv c(\eta)+\Psi(\eta,t,r_\perp). \tag{4}$$

Assuming the perturbation to be small, we obtain a linear equation which determines the development of the small perturbations

$$\left(\frac{\partial}{\partial t}+\mathscr{L}\right)\Psi=0,\quad \mathscr{L}\equiv-D\frac{\partial^2}{\partial\eta^2}+u\frac{\partial}{\partial\eta}+U(\eta)-D\Delta_\perp. \tag{5}$$

We have introduced here the notation $U(\eta) = dQ/dc|_{c(\eta)}$. The obtained equation admits of a solution of the type

$$\Psi(\eta,t,r_\perp)=\Psi_\gamma(\eta,k_\perp)\exp\{ik_\perp r_\perp-\gamma t\} \tag{6}$$

and leads to the following eigenvalue problem for the parameter γ:

$$(L-\gamma)\Psi_\gamma=0,\ L\equiv-D\frac{d^2}{d\eta^2}+u\frac{d}{d\eta}+U(\eta)+Dk_\perp^2. \tag{7}$$

The boundary conditions should correspond to boundedness of the eigenfunction at $-\infty<\eta<+\infty$. Let us ascertain the form of the function $U(\eta)$. Simple graphical constructions show that in the (U, η) plane the curve has the form of a "potential wall" with horizontal asymptotes

$$U(-\infty)=-\frac{dQ}{dc}\Big|_{c=0}<0,\quad U(+\infty)=-\frac{dQ}{dc}\Big|_{c=1}>0,$$

located on opposite sides of the η axis.

Let us assume that $U(\eta)$ is a slowly varying function; then in the quasiclassical approximation the principal term of the asymptotic expansion of the solutions of (7) takes the form[7]

$$\Psi_{\gamma^\pm}(\eta)\sim\exp\left[\int d\eta k_\pm(\eta)\right]\Big/\left[\frac{u^2}{4D}+U(\eta)+Dk_\perp{}^2-\gamma\right]^{1/4}. \quad (8)$$

The local wave numbers are equal to

$$k_\pm(\eta)=\frac{u}{2D}\pm\left[\frac{u^2/4D+U(\eta)+Dk_\perp{}^2-\gamma}{D}\right]^{1/2}. \quad (9)$$

The qualitative behavior of the quasiclassical solutions is determined in many respects by the number and arrangement of the turning points, at which the values of the local wave numbers coincide, $k_+(\eta) = k_-(\eta)$, and the points at which the real parts of the local wave numbers reverse signs. In such a case, the equation determining the turning point is of the form

$$u^2/4D+U(\eta)+Dk_\perp{}^2=\gamma, \quad (10)$$

and the points at which the wave number $k_-(\eta)$ reverses sign are determined by the solutions of the equation

$$U(\eta)+Dk_\perp{}^2=\gamma. \quad (11)$$

Let us assume for the time being that $k_\perp = 0$ (the results of the analysis of the stability at $k_\perp \neq 0$ will be given later). We consider first the region of increments $\gamma < 0$. It is easy to verify that $u^2/4D + U(\eta) > 0$ in the region of existence of simple steady-state waves and there are no turning points. The local wave numbers are real, $k_+(\eta)$ is positive throughout, and $k_-(\eta)$ is negative throughout if $\gamma > U(-\infty)$. It is obvious that when $\gamma > U(-\infty)$ there exist no solutions that are bounded in all of space. However, in the region $0 > \gamma > U(-\infty)$ the equation (11) has a solution and there exists a point at which the local wave number $k_-(\eta)$ reverses sign. Since $k_-(+\infty) < 0$ and $k_-(-\infty) > 0$, the solution $\Psi_\gamma^-(\eta)$ corresponding to the given local wave number is bounded in all of space. Thus, Ψ_γ^- is the quasiclassical approximation of the eigenfunction of the boundary problem under consideration, and corresponds to a continuous strip of increments

$$0>\gamma>-dQ/dc|_{c=0}.$$

Consequently, simple steady-state waves propagating with velocity $u > u_{min}$ are unstable if $k_\perp = 0$.

Let us consider the region of decrements $\gamma > 0$. It is clear that when $u^2/4D + U(+\infty)$ there are no bounded solutions. There are likewise no bounded solutions in the region $U(+\infty) < \gamma < u^2/4D + U(+\infty)$, where a turning point exists but $k_-(\eta)$ does not reverse sign on the right of the turning point and is a positive quantity. However, in the region of decrements $0 < \gamma \lesssim -dQ/dc|_{c=1}$ there exists both a turning point and a point situated to the right of the turning point, at which the local wave number $k_-(\eta)$ reverses sign, and $k_-(+\infty) < 0$. Since the real part of the local wave number is positive, on the left of the turning point, we arrive at the conclusion that for the given strip of decrements there exists solutions that are bounded in all of space. The quasiclassical eigenfunctions should be further determined in the usual manner in the immediate vicinity of the turning point.

We note, finally, that the eigenfunction of the neutral solution ($\gamma = 0$) is in our problem the distribution function of the diffusion flux in the simple wave. The latter circumstance is the consequence of the coincidence of the operator of the perturbed problem and the operator acting on $dc/d\eta$. Indeed, differentiating the equation of the unperturbed problem (2) we can verify the correctness of this statement[1)].

When $k_\perp \neq 0$, the region of instability becomes narrower, the largest increment decreases to a value $dQ/dc|_{c=0} - Dk_\perp^2$, and the instability disappears at sufficiently large values of the transverse wave number. Thus, perturbations that are localized in a plane orthogonal to the direction of propagation of the simple wave lead to a partial stabilization of the instability in question.

Let us clarify the character of the instability. It is obvious that the latter is of the drift type if

$$\lim_{t\to\infty}\int_{\gamma<0} d\gamma\, c(\gamma)\Psi_\gamma(\eta)e^{-\gamma t}=0 \quad (12)$$

at x = const, otherwise the instability will be absolute. Here $c(\gamma)$ is the amplitude of the elementary solution, and the integration is carried out over all the admissible values of the increments. Since the local wave number $k_-(\eta) \to k_-(+\infty)$ as $\eta \to +\infty$, we have asymptotically $\int d\eta k_-(\eta) \sim k_-(+\infty)\eta$ and the behavior of the integral (12) is determined essentially by the sign of the difference $k_-(+\infty)u - \gamma$. Consequently, the instability is of the drift type if the following inequality is satisfied for all the permissible values of the increments:

$$D\gamma^2<u^2\left[-\frac{dQ}{dc}\Big|_{c=1}+Dk_\perp{}^2\right]. \quad (13)$$

For the case $k_\perp = 0$, the last inequality leads to the following requirement, which must be satisfied by the propagation velocity of a simple steady-state wave:

$$u>\frac{1}{2}u_{min}\left[-\frac{dQ}{dc}\Big|_{c=0}\Big/\frac{dQ}{dc}\Big|_{c=1}\right]^{1/2}. \quad (14)$$

It is obvious that for a symmetrical nonlinear source the instability is always of the drift type. It should be noted that a complete investigation of the type of instability should be based on an analysis of the asymptotic behavior of the overall solution of the unperturbed problem as an initial-value problem.

We have investigated above the case of a source of constant sign. In many cases it is necessary to investigate the stability of simple steady-state waves, for the diffusion equation with a source of alternating sign. Let the function $-Q(c)$ correspond to an N-shape curve, which vanishes at three points $c = 0$, $c = c_s$, and $c = 1$, and reverses sign on going through the unstable equilibrium position $c_s < 1$. The equation determining the distribution of the concentrations in the simple steady-state wave $c(x - ut) \equiv c(\xi)$ is of the form

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[1)]The author is grateful to A. F. Volkov for a communication concerning the aforementioned property of the neutral solution.

$$-\left(D\frac{d^2}{d\xi^2}+u\frac{d}{d\xi}\right)c=Q(c)$$

and should be solved under the following boundary conditions:

$$\lim_{\xi\to+\infty} c=0,\quad \lim_{\xi\to-\infty} c=1,\quad \lim_{\xi\to\pm\infty}(dc/d\xi)=0.$$

In the phase plane $(dc/d\xi, c)$ a simple wave of this type corresponds to a trajectory representing a common separatrix of two singular saddle points corresponding to the initial and final states of equilibrium of the system. In our case the proper parameter of the unperturbed problem (the propagation velocity of the simple steady-state wave) can assume only one value, for which, as a rule, it is impossible to obtain a closed analytic expression. Obviously, the expressions given above for the principal term of the asymptotic expansion, the local wave numbers, the turning points, and the point at which one of the local wave numbers reverses sign, remain valid also in the given case (with the obvious substitutions $\eta \to \xi$ and $u \to -u$). The statement concerning the property of the neutral solution likewise remains in force. It is easy to establish graphically that the curve $U(\xi)$ corresponds to a curve of the potential-well type, the bottom of which is dropped below the ξ axis by an amount $\min U(\xi) = -\max_c (dQ/dc)$, and both horizontal asymptotes (at $\xi \to \pm\infty$) are located above the ξ axis.

Let $k_\perp = 0$. We can then show that the bottom of the potential well $u^2/4D + U(\xi)$, which determines the turning points, is located below the ξ axis, and consequently the solution of the unperturbed problem should be such that

$$\frac{u^2}{4D}-\max\frac{dQ}{dc}<0.$$

Let us assume the opposite; then for all the values of γ, such that $\gamma < u^2/4D + \min U(\xi)$, there are no turning points. Here, however, one or two points at which the local wave number $k_+(\xi)$ reverses sign, can exist. If the sign of the local wave number reverses twice, then no bounded solutions can exist. If the local wave number reverses sign once, then a solution corresponding to the given local wave number is bounded. However, a situation in which the sign reversal occurs once is possible only for the region bounded from above and below by the quantities $U(+\infty)$ and $U(-\infty)$. Since both horizontal asymptotes are located above the ξ axis, it follows that for values of γ smaller than the lower of the two asymptotes there exist again two points at which $k_+(\xi)$ reverses sign, and consequently there are no solutions that are bounded everywhere.

The absence of eigenfunctions for the indicated region of values of the parameter γ, in which the value $\gamma = 0$ is located, contradicts the fact that when $\gamma = 0$ the problem has a neutral solution, which is bounded everywhere and has an eigenfunction that coincides with the distribution of the diffusion flux in the simple wave. The contradiction is eliminated if it is assumed that the bottom of the potential well $u^2/4D + U(\xi)$ is located below the ξ axis. Thus, the propagation velocity of the simple steady-state wave is bounded from above by the quantity $u_{max} = 2(D\cdot\max_c(dQ/dc))^{1/2}$, and if the instability does exist, then the largest increment does not exceed $\max(dQ/dc) - u^2/4D$.

In conclusion we note that in spite of the fact that we used the quasiclassical approximation method for the investigation of the stability of simple waves, a considerable part of the conclusions remains in force also for those cases when $dQ/dc|_{c(\eta)}$ is not everywhere a slowly varying function. The latter remark is due to the fact that in most cases all that is essential is the asymptotic behavior of the solutions at $\eta \to \pm\infty$.

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