

WAVE PROCESSES IN A PLANE-PARALLEL CAPILLARY FILLED WITH

He II

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Wave processes which take place in a plane-parallel capillary filled with He II are considered. The velocities and damping coefficients of fourth sound and the thermal wave are calculated for capillaries possessing different ratios of viscous wavelength to width. The relation between the amplitudes of first and second sound and the viscous wave in fourth sound and in the thermal wave is obtained. The theoretical results are compared with the experimental data.

IN the consideration of wave processes in He II, one must distinguish between the following cases:

The dimensions of the vessel filled with He II are sufficiently large (so that the effects of the boundaries can be neglected).

The characteristic dimensions of the vessel are such that the normal liquid is clamped. For this, it is necessary that either the length of the viscous wave $\lambda_v = \sqrt{2\eta/\omega\rho_n}$ or the length of the free path of the elementary excitations l be much greater than the characteristic dimensions of the vessel d .

Finally, an intermediate case is possible, where the dimensions of the vessel are such that the normal liquid is only partially clamped.

In the first case, three types of wave motion are possible: first sound, where the superfluid and the normal components move as a whole—the pressure and the density oscillate while the temperature remains constant; second sound, where both components move against each other, so that the liquid as a whole is at rest, while the temperature oscillates.¹⁾ Finally, viscous waves, in which only the normal component is oscillating. The viscous waves are strongly damped (transverse) waves which appear on the walls of the vessel and which are important for the determination of the damping coefficient of first sound.

The velocity and damping coefficient of these three types of waves are computed from the complete linearized set of hydrodynamic equations:^[1, 2]

$$\begin{aligned} \dot{\rho} + \text{div } \mathbf{j} &= 0, \\ \frac{\partial j_i}{\partial t} + \nabla_i p &= \eta \frac{\partial}{\partial x_k} \left(\frac{\partial v_{ni}}{\partial x_k} + \frac{\partial v_{nk}}{\partial x_i} - \frac{2}{3} \delta_{ik} \frac{\partial v_{nl}}{\partial x_l} \right), \\ \dot{\mathbf{v}}_s + \nabla \mu &= 0, \quad (\dot{\sigma}) + \sigma \rho \text{ div } \mathbf{v}_n = 0. \end{aligned} \tag{1}$$

Here ρ is the density, σ the specific entropy, p the pressure, T the temperature, η the coefficient of viscosity, \mathbf{v}_n and \mathbf{v}_s the velocities of the normal and superfluid components, $\mathbf{j} = \rho_n \mathbf{v}_n + \rho_s \mathbf{v}_s$, $\rho_{n(s)}$ the density of the normal (superfluid) component, and $\nabla \mu = -\sigma \nabla T + \nabla p / \rho$.

From among the dissipative mechanisms, we keep only the first viscosity, since the clamping of the nor-

mal component, which leads to the modification of first sound into fourth, takes place as the result of it, as well as modification of second sound into a thermal wave. Account of other dissipative mechanisms does not lead to additional difficulties. The role of volume mechanisms of dissipation (second viscosity) and thermal resistance of the boundary for the case of propagation of fourth sound were studied in ^[3].

If the normal component is clamped (second case), then the character of the sound propagation in helium is materially altered. As Pellam^[4] and Atkins^[5] have shown, in this case, wave motion is possible which is propagated only over the superfluid liquid. This wave motion is obtained from the set (1). Here the complete set of equations consists of the first, third, and fourth equations, in which \mathbf{v}_n must be set equal to zero. The velocity of fourth sound, obtained in this manner, is, according to Atkins,^[5] equal to

$$u_{40}^2 = \frac{\rho_s}{\rho} u_1^2 + \frac{\rho_n}{\rho} u_2^2, \tag{2}$$

where $u_1 = \sqrt{\partial p / \partial \rho}$ is the velocity of first sound, and $u_2 = [\sigma^2 \rho_s / \rho_n (\partial \sigma / \partial T)]^{1/2}$ is the velocity of second sound.

Fourth sound is essentially a modification of first (ordinary) sound. With decrease in the thickness of the capillary or decrease in the frequency, i.e., with increase in length of the viscous wave, the velocity of first sound falls off from its ordinary value to the velocity of fourth sound u_{40} . In addition, a modification of second sound also takes place. Its velocity falls off with decrease in the dimensionless parameter d/λ_v , while the damping increases. When the wavelength and the damping distance are comparable, second sound ceases to exist. The damped vibrations of the temperature have gained the name of thermal waves, the velocity of propagation of which was measured in the experiments of Pollack and Pellam.^[6] The transition from first sound to fourth sound and from second sound to the thermal waves can be followed by using the set (1), if we write $-\omega \rho r \mathbf{v}_n$ in place of the dissipative term in the right-hand side of the second equation.^[6, 7] Here ω is the frequency of the sound vibrations and r is a dimensionless parameter which describes the clamping of the normal component; $r = 0$ corresponds to the

¹⁾In this research, we shall neglect the thermal expansion, the smallness of which justifies the separation described above.

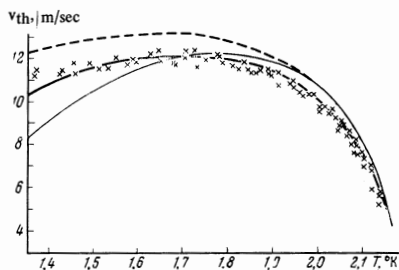


FIG. 1. Temperature dependence of the velocity of thermal waves $v_{th}(T)$. Powder with transverse dimension $d_r = 3.5 \times 10^{-2}$ cm, $d\sqrt{\omega} = 6.2 \times 10^{-2}$ cm-sec $^{-1/2}$. Points – experimental data of Pollack and Pellam; [6] thin line – theoretical values of Pollack and Pellam; [6] heavy line – theoretical values from Eq. (18); dashed curve – thermal waves for $d\sqrt{\omega} \rightarrow \infty$ (second sound).

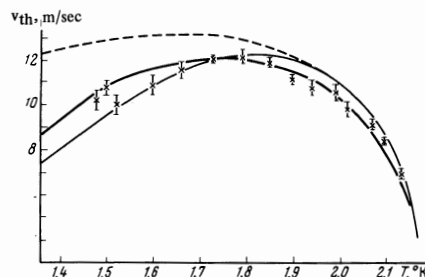


FIG. 2. The same as in Fig. 1, for $d_r = 1.5 \times 10^{-2}$, $d\sqrt{\omega} = 4.3 \times 10^{-2}$ cm-sec $^{-1/2}$.

first case and $r = \infty$ to the second. Naturally the parameter r must depend on the dimensions of the capillary and the quantities which characterize the helium.

Partial clamping of the normal component in the experiments of Pollack and Pellam was brought about by placing the helium in a volume filled with fine powder-rouge. A different degree of clamping of the normal component was achieved by a change in the transverse dimensions of the powder (from $d_r = 2 \times 10^{-3}$ cm to $d_r = 3.5 \times 10^{-2}$ cm).²⁾ Eliminating (each time) the dependence of the velocity of thermal waves as a function of temperature for powders of different transverse dimension, Pollack and Pellam obtained a series of graphs $v_{th}(T)$ (for constant frequency ω and d_r). Under the assumption that the parameter r does not depend on the temperature, the theoretical curves of $v_{th}(T)$ were constructed (for $r = \text{const}$). Each time the constant was so chosen that the theoretical curve best corresponded to the experimental results. It was shown that the lower the temperature and the more strongly the normal component is clamped (i.e., the finer the powder), the better is the agreement between theory and experiment. In the case of sufficiently coarse powder and high temperature, the agreement of theory with experiment has a more qualitative character (Figs. 1, 2, 3, 4).

Such a theory is inadequate in that the parameter r remains undetermined. For the determination of the explicit form of r , a more detailed consideration is necessary. For large r , this has been done in our work (with Sanikidze).^[3] The propagation of sound in an isolated capillary and in a system of identical capillaries parallel to one another was considered. The velocity and the absorption coefficient of fourth sound were considered under the assumption that $\lambda_v \gg d \gg l$. By comparing the results obtained in ^[3] with the results of ^[6], one could determine the parameter r , which was shown to be equal to $3\eta/\omega\rho d^2$. As to the thermal waves, it was shown that they are very rapidly damped in the case considered ($\lambda_v \gg d$) and their velocity is very small.

In the present work, wave processes are considered for a plane-parallel capillary of width $2d$, filled with He II. It is assumed that the hydrodynamic case is realized ($d \gg l$). The set of equations (1) is solved with the corresponding boundary conditions. The absorption coefficients and the velocities of fourth sound and the thermal wave are obtained for capillaries of arbitrary width. Consideration is given to the dispersion of the sound and thermal waves brought about by a change in the ratio of the length of the viscous wave λ_v to the thickness of the capillary d . Results are obtained which, strictly speaking, are applicable only for plane-parallel capillaries. However, one can make some generalization also to the case of a system of curved capillaries (which is the volume filled with rouge), introducing an effective d and taking it into account that the sound travels along the curved channels. The resultant data are in excellent agreement with the experiment of Pollack and Pellam.^[6]

The capillary is unbounded in two directions; we choose the z axis along the third direction, while the x axis is along the direction of propagation of the wave. The boundary conditions on the walls of the capillary (for $z = \pm d$) in the given case are the vanishing of the velocities:

$$v_n = 0, \quad v_{sz} = 0. \tag{3}$$

The deviation of the quantities from their equilibrium values will be sought in the form

$$\begin{aligned} v_n &= L_1 \nabla Q_1 + L_2 \nabla Q_2 + U, \\ v_s &= N_1 \nabla Q_1 + N_2 \nabla Q_2, \quad \sigma' = M_1 Q_1 + M_2 Q_2, \end{aligned} \tag{4}$$

where Q_i and U are functions of the coordinates and the time; the time dependence is chosen in the form $e^{-i\omega t}$; L_i and N_i ($i = 1, 2$) are amplitudes.

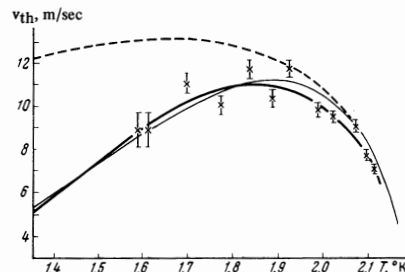


FIG. 3. The same as in Fig. 1, for $d_r = 4.4 \times 10^{-3}$ cm, $d\sqrt{\omega} = 2.1 \times 10^{-2}$ cm-sec $^{-1/2}$.

²⁾We note that the clamping of the normal component can also take place in a thin film of He II (sound in a thin film of He II is known as third sound).^[5] The approach to the solution of this problem is similar to that described above.^[8,9]

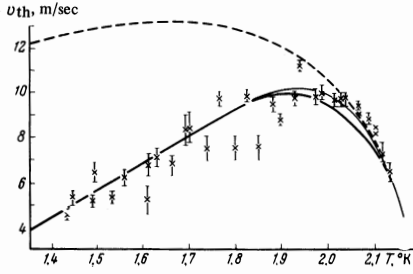


FIG. 4. The same as in Fig. 1, for $d_r \approx 2.0 \times 10^{-3}$ cm, $d\sqrt{\omega} = 1.5 \times 10^{-2}$ cm-sec $^{-1/2}$.

Substitution of (4) in the set (1) leads to the system of equations for Q_i and U and to a coupling between the amplitudes:

$$\begin{aligned} \Delta Q_1 + k_1^2 Q_1 = 0, \quad \Delta Q_2 + k_2^2 Q_2 = 0, \\ \Delta U + k_3^2 U = 0, \quad \text{div } U = 0; \end{aligned} \quad (5)$$

$$N_i = P_i L_i, \quad M_i = D_i L_i \quad (i = 1, 2). \quad (6)$$

Thanks to the smallness of the dissipation term, P_i and D_i can be written down in the following way:

$$\begin{aligned} P_1 = 1 + \frac{4i\eta\omega}{3\rho_n(u_1^2 - u_2^2)}, \quad P_2 = -\frac{\rho_n}{\rho_s} + \frac{4i\eta\omega}{3\rho_s(u_1^2 - u_2^2)}, \\ D_1 = \frac{4i\eta\omega^2\rho_s\sigma}{3\rho\rho_n u_1^2(u_1^2 - u_2^2)}, \\ D_2 = i\frac{\omega\sigma}{u_2^2} \left[1 + \frac{4i\eta\omega(\rho_s u_1^2 - \rho u_2^2)}{3\rho\rho_n u_2^2(u_1^2 - u_2^2)} \right]. \end{aligned} \quad (7)$$

Here k_1^2 is the square of the wave vector corresponding to first sound, k_2^2 to second, and k_3^2 to the viscous wave:^[2, 10]

$$\begin{aligned} k_1^2 &\equiv k_{\perp 1}^2 + k_{\parallel}^2 = -\frac{\omega^2}{u_1^2} \left(1 + \frac{4i\eta\omega}{3\rho u_1^2} \right), \\ k_2^2 &\equiv k_{\perp 2}^2 + k_{\parallel}^2 = \frac{\omega^2}{u_2^2} \left(1 + \frac{4i\eta\omega\rho_s}{3\rho\rho_n u_2^2} \right), \\ k_3^2 &\equiv k_{\perp 3}^2 + k_{\parallel}^2 = i\omega\rho_n/\eta. \end{aligned} \quad (8)$$

According to (4)–(8), the wave motions in the capillary can be regarded as the superposition of three types of oscillations: first sound (the first components in (4)), second sound (the second components in (4)), and the viscous wave (the third component in (4)). The contribution of each of the three oscillations is determined by the relations between the amplitudes, which are in turn determined by the boundary conditions (3).

We now turn our attention to the fact that the viscosity η enters twice: in the expression for k_3^2 —in the denominator, and in the remaining quantities—in the numerator, in the form of small additions. In the latter case, it describes the volume mechanism of absorption and, strictly speaking, the corresponding components should be omitted, since they are of the same order as the terms containing the second viscosity, which have already been omitted. In the derivation of the dispersion equations (see below), we shall take this into account.

We shall seek solutions for Q_i and U satisfying the set of equations (1) and the boundary conditions (3) in the form

$$\begin{aligned} Q_i = e^{i(k_{\parallel}x - \omega t)} \cos k_{\perp i} z, \\ U_x = A i e^{i(k_{\parallel}x - \omega t)} \cos k_{\perp 3} z, \quad U_z = A \frac{k_{\parallel}}{k_{\perp 3}} e^{i(k_{\parallel}x - \omega t)} \sin k_{\perp 3} z. \end{aligned} \quad (9)$$

It should be noted that generally the solutions for Q_i and U in the form

$$Q_i = e^{i(k_{\parallel}x - \omega t)} \sin k_{\perp i} z, \quad U_x = A i e^{i(k_{\parallel}x - \omega t)} \sin k_{\perp 3} z$$

already satisfy (1) and (3). However, such solutions are of no physical interest. The fact is that in capillaries whose transverse dimensions are much smaller than the sound wavelength (we shall be interested in just this case in what follows), such solutions actually reveal an almost complete absence of wave motion, since the amplitudes of the oscillations of all quantities are equal to zero on the axis of the capillary.

Substituting (9) in (4), taking (6) into account, and using (3), we get a set of homogeneous equations in the amplitudes L_i and A . The condition of compatibility of this set of equations is the vanishing of the determinant of the coefficients. Expanding this determinant, and replacing $k_{\perp i}$ and $k_{\perp 3}$ in these expressions by k_{\parallel} through (8), we get an equation in k_{\parallel} . We shall now consider the solution of the resultant equation for different relations between d and λ_V .

We shall be interested in those cases in which the ratio d/λ_V changes from zero to values of the order of and greater than unity, while the sound wavelength is much greater than the transverse dimensions of the capillaries. Such assumptions are valid since

$$|\lambda_3/\lambda_i|^2 = \omega\eta/u_i^2\rho_n \ll 1.$$

Furthermore, as we shall see below, the solutions obtained for k_{\parallel}^2 are much smaller in magnitude than k_3^2 . In evaluating the determinant, we can consider the quantities $|k_{\perp i}d|$ and $|k_{\perp i}/k_3|$ to be small ($i = 1, 2$):

$$\begin{aligned} |k_i d| \ll 1, \quad |k_i/k_3| \ll 1, \\ |k_{\parallel} d| \ll 1, \quad |k_{\parallel}/k_3| \ll 1; \end{aligned} \quad (10)$$

Here $d/\lambda_V = |k_3 d|/\sqrt{2}$ can change from zero to values of the order of and greater than unity. With account of (10), the equation for the desired k_{\parallel} can be written in the form

$$\begin{aligned} k_{\parallel}^4 \left(1 - \frac{\text{tg } k_3 d}{k_3 d} \right) - k_{\parallel}^2 \left[\frac{\omega^2}{u_1^2} + \frac{\omega^2}{u_2^2} - \frac{\text{tg } k_3 d}{k_3 d} \left(\frac{\rho_n \omega^2}{\rho u_2^2} + \frac{\rho_s \omega^2}{\rho u_1^2} \right) \right] \\ + \frac{\omega^4}{u_1^2 u_2^2} = 0. \end{aligned} \quad (11)$$

When the dimensions of the capillaries are much smaller than the length of the viscous wave ($|k_3 d| \ll 1$), then the solution of Eq. (11), which has a small imaginary part, corresponds to fourth sound and can be written in the form

$$k_{\parallel}^2 = \frac{\omega^2}{u_0^2} \left[1 + i d^2 \frac{\omega\rho_n^2\rho_s}{3\eta\rho^2} \frac{(u_1^2 - u_2^2)^2}{u_0^4} \right] \quad (12)$$

and is naturally identical (in the corresponding approximation) with the results of [3].

Introducing (for convenience) the dimensionless parameter

$$r = -i \frac{\rho_n}{\rho} \frac{\text{tg } k_3 d}{k_3 d - \text{tg } k_3 d}, \quad (13)$$

we can write Eq. (11) in the form

$$\frac{\omega^4}{k_{\parallel}^4} \left(1 + i \frac{\rho r}{\rho_n} \right) - \frac{\omega^2}{k_{\parallel}^2} \left[u_1^2 + u_2^2 + i r \left(\frac{\rho_s}{\rho_n} u_1^2 + u_2^2 \right) \right] + u_1^2 u_2^2 = 0. \quad (14)$$

The latter equation is the same as the equation obtained by Pollack and Pellam.^[6] Thus the dimensionless pa-

parameter r , introduced in ^[6], depends on the temperature and is (mainly) a complex quantity (see (13); it was assumed in ^[6] that r is real).

The solution of Eq. (14) with accuracy to terms proportional to u_2^4/u_1^4 ($u_1 \gg u_2$) can be written in the form

$$k_{||}^2 = \omega^2 \frac{1 + ir\rho_n/(\rho_n + ir\rho_s)}{u_1^2 - u_2^2 r^2 \rho_s \rho_n / (\rho_n + ir\rho_s)^2}, \quad (15)$$

$$k_{\perp}^2 = \frac{\omega^2}{u_2^2} \left(1 + ir \frac{\rho_s}{\rho_n} \right). \quad (16)$$

The first solution corresponds to fourth sound for small d/λ_V ($|r| \gg 1$) and to first sound for large d/λ_V ($|r| \ll 1$). The second corresponds to thermal waves for small and intermediate values of d/λ_V , and to second sound for large d/λ_V .

We now investigate the second solution. For this purpose, we separate the real and imaginary parts of the parameter r . In what follows, we shall need two forms of writing them:

$$r = \frac{\rho_n}{\rho} (m_1 + im_2) = \frac{\rho_n}{\rho} \frac{a + i[a^2 - b(1-b)]}{a^2 + (1-b)^2}, \quad (17)$$

where

$$b = \frac{\text{sh } 2\delta + \sin 2\delta}{4\delta(\cos^2 \delta + \text{sh}^2 \delta)}, \quad a = \frac{\text{sh } 2\delta - \sin 2\delta}{4\delta(\cos^2 \delta + \text{sh}^2 \delta)},$$

$$\delta = \frac{d}{\lambda_n} = \frac{|k_3|d}{\gamma\sqrt{2}}.$$

Thus a and b are functions of the dimensionless parameter δ .

The dependence of a and b on δ is shown in Fig. 5. For small δ , the value of a tends to zero as $2\delta^2/3$ and b to unity as $1 - 8/15\delta^4$. For large δ , both quantities a and b approach zero as $1/2\delta$. So far as the function $m_1(\delta)$ is concerned, when $\delta \lesssim 2$ it is well approximated by the function $3/2\delta^2$. For large δ , it approaches zero as $1/2\delta$. Finally, the function $m_2(\delta)$ for $\delta \lesssim 2$ oscillates about the value -0.2 with an amplitude of the order of 0.01 (for $\delta = 0$, $m_2 = -0.2$), while for further increase of δ , it approaches zero as $-1/2\delta$.

It therefore follows that for fixed values of d and ω , the real part of r depends on the temperature as the coefficient of the first viscosity η , if δ remains smaller than 2 in the temperature range studied; the temperature dependence of the imaginary part of r is determined by the factor ρ_n .

Using (16) and (17), and taking it into account that $m_1^2 \gg m_2^2$ for $\delta \lesssim 2$, and that the inequality $2|m_2| \gg \rho_S m_1^2/\rho$ for $\delta > 2$, we get for the velocity and the absorption coefficient of thermal waves

$$v_{th} = \sqrt{2} u_2 \left[\left(\left(1 - 2 \frac{\rho_s}{\rho} m_2 + \frac{\rho_s^2}{\rho^2} m_1^2 \right)^{1/2} + 1 - \frac{\rho_s}{\rho} m_2 \right)^{1/2} \right], \quad (18)$$

$$\text{Im } k_{||} = \frac{\omega}{u_2 \sqrt{2}} \left[\left(\left(1 - 2 \frac{\rho_s}{\rho} m_2 + \frac{\rho_s^2}{\rho^2} m_1^2 \right)^{1/2} - 1 + \frac{\rho_s}{\rho} m_2 \right)^{1/2} \right]. \quad (19)$$

If we set $m_2 = 0$, then Eqs. (18) and (19) are identical to the corresponding expressions given in ^[6]. Such an approximation is correct for not too high temperatures and not very large δ , when one can assume that $\rho_S m_1^2/\rho \gg 2|m_2|$.

It is not difficult to estimate the order of magnitude of δ in the experiments of Pollack and Pellam.^[6] For fixed ω and d , the parameter δ depends on the tem-

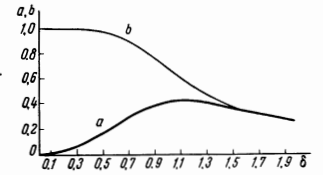


FIG. 5. Dependence of the values of a and b on the parameter $\delta = d/\lambda_V$.

perature. We carry out the estimate for $T = 1.7^\circ \text{K}$. (Measurements of the velocity of the thermal wave $v_{th}(T)$ in ^[6] were carried out from 1.4 to 2.15°K .) In the estimate, it must be taken into account that, according to Allen and Misener,^[11] the irregularity of the geometry of the channels leads to an effective decrease in the diameter of the capillary, which is equal, in order of magnitude, to the transverse dimensions of the rough powder, d_r . In correspondence with ^[11], we shall assume that $d \approx d_r \times 10^{-1}$. In ^[6] the frequencies used were ~ 10 kHz. For powders of the order of $d_r = 10^{-2}$ cm, we obtain

$$\delta = d\sqrt{\omega\rho_n/2\eta} \approx 3. \quad (20)$$

The bending of the capillaries leads not only to a difference in the effective width of the capillary from the real width, but also decreases somewhat (in comparison with calculation by Eq. (18)) the value of the velocity of the wave. Therefore, in comparison with the experimental data, the calculated value of the velocity must be multiplied by the transmission coefficient. For spherical powder, this coefficient is equal to $2/\pi$.^[6] This circumstance is taken into account in the construction of the graphs in Figs. 1–4, on which are plotted the curves of the temperature dependence of the velocity of the thermal waves for different values of $d\sqrt{\omega}$; the theoretical curves of Pollack and Pellam and their experimental results are also shown. The comparison shows that the curves obtained from Eq. (18) are virtually identical with the theoretical curves obtained from the formula of Pollack and Pellam for sufficiently small $d\sqrt{\omega}$ and temperatures below 1.9°K (Figs. 3–4). However, even at $d\sqrt{\omega} = 6.2 \times 10^{-2} \text{ cm-sec}^{-1/2}$ (Fig. 1) and $d\sqrt{\omega} = 4.3 \times 10^{-2} \text{ cm-sec}^{-1/2}$ (Fig. 2) the curves differ appreciably, while the curve obtained from Eq. (18) (Fig. 1) is in excellent agreement with the experimental data.³⁾

So far as the absorption of thermal waves is concerned,^[6] it increases with decrease in temperature and decrease in the diameter of the powder, which agrees with Eq. (19) (see Fig. 6). We now consider the solution (15) of Eq. (14). Taking it into account that $u_1^2 \gg u_2^2$, we can rewrite Eq. (15) in the form

$$k_{||}^2 = \frac{\omega^2}{u_1^2} \left(1 + \frac{ir\rho_n}{\rho_n + ir\rho_s} \right). \quad (21)$$

Substituting in (21) the expressions for the real and imaginary parts of r (17), we get

$$k_{||}^2 = \frac{\omega^2}{u_1^2} \frac{\rho - b\rho_n}{(\rho - b\rho_n)^2 + a^2\rho_n^2} \left(1 + i \frac{a\rho_n}{\rho - b\rho_n} \right). \quad (22)$$

From the last expression, it is seen that the damping

³⁾We note that the condition $d \gg l$ is satisfied throughout the temperature range in the experiments under discussion.

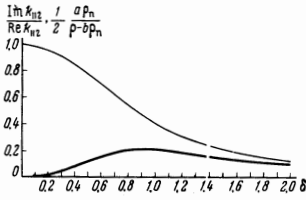


FIG. 6. Dependence of the ratio $\text{Im } k_{||2} / \text{Re } k_{||2}$ on the parameter $\delta = d/\lambda_v$ (thin line), temperature 1.7°K ; dependence of the value of $a\rho_n/2(\rho - b\rho_n)$ on the parameter δ (heavy line), temperature 2°K .

of fourth sound, which is due to the slipping past of the normal component, increases with increase in δ , reaches a maximum for $\delta \sim 1$, and thereafter falls off upon further increase of δ (Fig. 6). For fixed d and ω , the damping coefficient of fourth sound increases with increase in temperature (Fig. 7). These results are in agreement with [7] and [12].

For the case of not very high temperatures (below 2.1°K) and for δ not too close to unity, the inequality $a\rho_n < \rho - b\rho_n$ is satisfied and Eq. (22) can be rewritten in the form

$$k_{||2}^2 = \frac{\omega^2}{u_1^2} \frac{\rho}{\rho - b\rho_n} \left(1 + i \frac{a\rho_n}{\rho - b\rho_n} \right). \quad (23)$$

We then obtain the following for the velocity and absorption coefficient of fourth sound:

$$u_4^2 = u_1^2 \left(1 - b \frac{\rho_n}{\rho} \right), \quad (24)$$

$$\text{Im } k_{||2} = \frac{1}{2} \frac{\omega}{u_4} \frac{a\rho_n}{\rho - b\rho_n}. \quad (25)$$

Figure 8 shows graphs of u_4/u_1 as a function of δ ($T = 2^\circ\text{K}$), while Fig. 9 gives the dependence of u_4 on the temperature for fixed d and ω ($d\sqrt{\omega} = 2.8 \times 10^{-2} \text{ cm-sec}^{-1/2}$). For comparison, the dependences of v_{th}/u_4 on δ (Fig. 8) and also the curves for u_{40} and u_S (Fig. 9) are given in these drawings.

In conclusion, let us consider the relation between the amplitudes L_1 and L_2 in fourth sound and in thermal wave.

As has already been said, according to (4)–(8) fourth sound and the thermal waves can be regarded as the superposition of three types of oscillations: first sound (first components in (4)), second sound (the second components in (4)) and the viscous wave (third component in (4)). The contribution of each of the waves (i.e., the relation between the amplitudes L_1 , L_2 , and A) is determined by the boundary conditions (3) and changes as a function of d/λ_v . Setting the expression for v_{SZ} equal to zero, we get

$$\frac{N_1}{N_2} = - \frac{k_{12} \sin k_{12}d}{k_{11} \sin k_{11}d}$$

or, since $|k_{11}d| \ll 1$,

$$\frac{N_1}{N_2} = - \frac{k_{12}^2}{k_{11}^2} = - \frac{k_2^2 - k_{||}^2}{k_1^2 - k_{||}^2}. \quad (26)$$

In the case of propagation of fourth sound, i.e., for $k_{||}^2 = k_{||1}^2$, taking it into account that $u_2^2 \ll u_1^2$ and dropping the imaginary part, which describes the dissipation, we get from Eq. (23)

$$\frac{N_1}{N_2} = \frac{u_1^2 \rho - b\rho_n}{u_2^2 b\rho_n} \quad (27)$$

With increase in d/λ_v , which corresponds to a decrease in b (we recall that $b < 1$), this ratio increases, how-

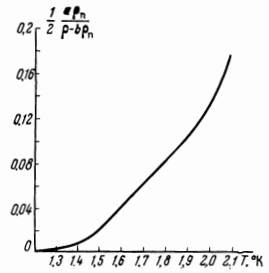


FIG. 7. Temperature dependence of the quantity $a\rho_n/2(\rho - b\rho_n)$ for $d\sqrt{\omega} = 2.8 \times 10^{-2} \text{ cm-sec}^{-1/2}$.

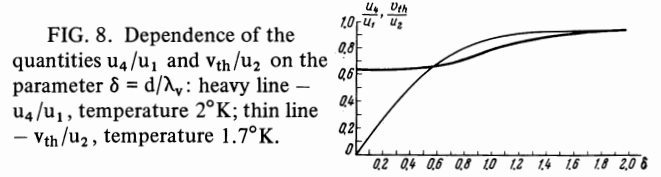


FIG. 8. Dependence of the quantities u_4/u_1 and v_{th}/u_2 on the parameter $\delta = d/\lambda_v$: heavy line – u_4/u_1 , temperature 2°K ; thin line – v_{th}/u_2 , temperature 1.7°K .

ever, $N_2 \ll N_1$ even for $b = 1$. In the limiting case of large $\delta = d/\lambda_v$, the quantity $N_2 \rightarrow 0$. Thus the contribution of second sound to fourth sound is always much less than the contribution of first sound, and naturally decreases with increase in δ .

We also note that the spatial distribution (over the cross section of the capillary) of all three waves is different. The oscillations of first and second sound are concentrated near the axis of the capillary, while the viscous wave differs appreciably from zero near the walls: its superposition on the first sound guarantees the clamping of the normal component at the wall.

It is not difficult to find the amplitude of oscillations of the entropy for propagation of fourth sound. Since $M_1 = 0$ (we recall that we are not considering small effects associated with volume dissipation) we get, according to (4), (6) and (27):

$$\frac{\sigma'}{v_{sx}} = \frac{M_2}{ik_{11}(N_1 + N_2)} = - \frac{\rho_s \sigma}{\rho u_4} b. \quad (28)$$

It should be noted that a change in the boundary conditions leads to a material change in the relation between the amplitudes of first and second sound. Thus, in the limiting case of an absolutely thermally conducting wall ($\kappa_c \rightarrow \infty$) the oscillations of the entropy in the capillary are absent in first approximation ($\sigma' = 0$). Hence, since $M_1 = 0$, we have $M_2 = 0$ (4). In this case, second sound is generally lacking in the capillary. The velocity of propagation of fourth sound also changes somewhat here. This problem has been discussed in detail in the reference cited.^[3] A similar situation arises in the intermediate case. For infinite thermal conductivity of the walls of the capillary, thermal waves cannot be propagated and Eq. (22) is exact (compare with (15)), since the components which contain u_2 and the velocity of fourth sound are due to the presence of second sound in the capillary.^[4]

Finally, we consider the relation between the amplitudes in the thermal wave, i.e., for $k_{||}^2 = k_{||2}^2$. Substitution of (16) in (26) gives

$$\frac{N_1}{N_2} = -i \frac{r\rho_s}{\rho_n + ir\rho_s}. \quad (29)$$

⁴⁾We note that the ratio of the first component to the second in Eq. (2) is again identical with (27) for $b = 1$.

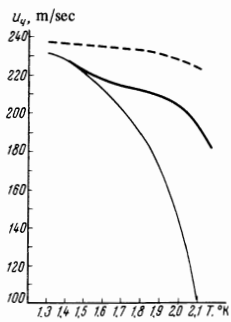


FIG. 9. Temperature dependence of the velocity of fourth sound for various values of $d\sqrt{\omega}$: thin line - u_{40} , fourth sound as $d\sqrt{\omega} \rightarrow 0$; heavy line - u_4 , fourth sound for $d\sqrt{\omega} = 2.8 \times 10^{-2}$ cm-sec $^{-1/2}$; dashed line - u_1 , fourth sound as $d\sqrt{\omega} \rightarrow \infty$ (first sound).

With increase in $\delta = d/\lambda_V$ (decrease of r) this ratio approaches zero, i.e., the contribution of first sound to the thermal wave decreases. If now $\delta \rightarrow 0$ ($r \rightarrow \infty$), then Eq. (29) gives $N_1/N_2 = -1$. Thus, in the limiting case of small δ in the thermal wave, in first approximation, the normal and superfluid components are motionless while the oscillations of temperature and pressure propagate with small velocity and high attenuation.

Note added in proof (July 4, 1967). For the case of a cylindrical capillary, the dispersion equation is formally the same as the dispersion equation (16) for a plan-parallel capillary if the parameter r is assumed to be equal to

$$r_{\text{cyl}} = -i \frac{\rho_n}{\rho} \frac{2J_1(k_3 d)}{k_3 d J_0(k_3 d) - 2J_1(k_3 d)}.$$

This substitution changes the path of the dispersion curves slightly; However, the difference does not exceed several percent. In the comparison of

the theoretical curves with the experimental, it can be seen that the formulas which refer to the propagation of the waves in cylindrical capillaries give a better description of the wave in the system of curved capillaries, which is the case of the rouge, than the similar formulas for plane capillaries.

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