

THE THEORY OF HYDROMAGNETIC STABILITY OF TOROIDAL PLASMA CONFIGURATIONS

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We consider criteria for local and convective stability of equilibrium plasma configurations of arbitrary geometry. We study on the basis of these criteria the stability of axially symmetric plasma configurations in the form of a toroid with a longitudinal current, without making the usual restrictions of small toroidality and low plasma pressure. We discuss the problem of the influence of the shape of the cross-section of the plasma torus on the stability.

1. GENERAL CRITERIA FOR LOCAL AND CONVECTIVE STABILITY

CONDITIONS for magnetohydrodynamic stability of an arbitrary equilibrium configuration with respect to local perturbations have been obtained in a number of papers.^[1-5] Local perturbations were taken to mean small displacements of the plasma with respect to its equilibrium position which had an arbitrary shape on the magnetic surface considered and involved a small region of localization in the direction of the normal to the magnetic surface. As another limiting form of perturbation, one can consider perturbations that are constant on each magnetic surface. A derivation of the corresponding criterion for "convective" stability is given in the Appendix. The condition for local stability obtained in^[1-4] has a form that is difficult to visualize. It turns out, however, that we can transform it to a much more convenient form which is close to the condition for convective stability, and this enables us to analyze from a single point of view the well-known criteria of Rosenbluth-Longmire, Kadomtsev, Kruskal-Shafranov, and Suydem. It turns out that all those criteria are limiting cases of a general criterion of local stability although, as far as the physical meaning of their original derivation is concerned, the first two criteria are not connected with any assumption about a local nature.

Using the notation of Greene and Johnson^[4] we can write the equations for the equilibrium of the plasma in the form

$$\nabla p = [jB], \quad j = \text{rot } B, \quad \text{div } B = 0. \tag{1.1}^*$$

The general criteria for local and convective stability derived in the Appendix have in this notation the form

$$\frac{1}{4} S^2 + S \left\langle \frac{jB}{|\nabla V|^2} \right\rangle_c - \Omega \left\langle \frac{B^2}{|\nabla V|^2} \right\rangle_c - \left(\left\langle \frac{j^2}{|\nabla V|^2} \right\rangle_c \left\langle \frac{B^2}{|\nabla V|^2} \right\rangle_c - \left\langle \frac{jB}{|\nabla V|^2} \right\rangle_c^2 \right) > 0, \tag{1.2}$$

$$\frac{S^2}{\tilde{p}^2} (\langle j^2 \rangle \langle B^2 \rangle - \langle jB \rangle^2) + 2S \langle jB \rangle - \Omega \langle B^2 \rangle - \tilde{p}^2 + \frac{\gamma p}{\tilde{p}^2} \langle (Sj - \Omega B)^2 + \tilde{p}^2 \Omega \rangle > 0. \tag{1.3}$$

Here $\gamma = \frac{5}{3}$, V is the moving volume of the system enclosed by closed magnetic surfaces. The quantities p ,

Ω , and S are so-called surface functions, depending solely on V . They are defined by the equations

$$\dot{p} = i\dot{\Phi} - J\dot{\chi}, \quad \Omega = i\ddot{\Phi} - J\ddot{\chi}, \quad S = \dot{\chi}\dot{\Phi} - \Phi\dot{\chi}, \tag{1.4}$$

where the dots indicate differentiation with respect to V . The functions Φ and χ describe the longitudinal and azimuthal magnetic currents, and the functions J and I the longitudinal and azimuthal currents inside the magnetic surfaces bounding the volume V :

$$\Phi = \int B d\sigma_{\perp}, \quad \chi = \int B d\sigma_{\parallel}, \quad J = \int j d\sigma_{\perp}, \quad I = \int j d\sigma_{\parallel}. \tag{1.5}$$

The angle brackets indicate averages over the volume included between two neighboring magnetic surfaces, and angle brackets with an index c averages over a closed magnetic line of force lying on a "rational" magnetic surface:

$$\langle f \rangle = \frac{d}{dV} \int f d\tau, \quad \langle f \rangle_c = \oint f \frac{dl}{B} / \oint \frac{dl}{B}. \tag{1.6}$$

The function

$$S = -\dot{\Phi}^2 \frac{d \dot{\chi}}{dV d\Phi}$$

characterizes the tilting (shear) of the magnetic lines of force. The function Ω , which for $S = 0$ is equal to $\Omega = p \ddot{\Phi} / \dot{\Phi} = p \ddot{\chi} / \dot{\chi}$, is a characteristic of the "minimum \bar{B} ."

Let us analyze the criteria (1.2) and (1.3).

1. In the case of a cylindrical geometry of the plasma, (1.2) goes over into Suydem's^[6] well-known necessary stability condition

$$\rho B_s^2 (\mu' / \mu)^2 + 8p' > 0 \tag{1.7}$$

($\mu(\rho) = B_{\omega} / \rho B_s$, B_{ω} and B_s are the azimuthal and longitudinal magnetic field), which clearly imposes very narrow requirements upon the pressure distribution $p(\rho)$ over the radius ρ of the plasma cylinder. When $\mu'(\rho) = 0$, the criterion (1.3) leads in the case considered to the inequality

$$p' - \gamma p B_s' / B_s > 0, \tag{1.8}$$

so that for stability a "maximum \bar{B} " is necessary.

2. For configurations with closed magnetic lines of force, when $S = 0$, the stability conditions (1.2) and (1.3) take the form

$$-\Omega \left\langle \frac{B^2}{|\nabla V|^2} \right\rangle_c - \left(\left\langle \frac{j^2}{|\nabla V|^2} \right\rangle_c \left\langle \frac{B^2}{|\nabla V|^2} \right\rangle_c - \left\langle \frac{jB}{|\nabla V|^2} \right\rangle_c^2 \right) > 0, \tag{1.9}$$

* $[jB] \equiv j \times B$.

$$-\Omega \langle B^2 \rangle - p^2 + \gamma p (\langle B^2 \rangle \Omega^2 / p^2 + \Omega) > 0. \quad (1.10)$$

As the quantity in the round brackets in inequality (1.9) is positive, it is necessary for stability that $\Omega < 0$. The condition $\Omega < 0$ is a "minimum \bar{B} " requirement, first obtained in the papers of Rosenbluth and Longmire^[7] and Kadomtsev.^[8] Equation (1.9) shows that for small plasma pressures when we can neglect the quantity within the round brackets, the condition $\Omega < 0$ is a sufficient condition for local stability when $S = 0$.

The condition (1.10) for convective stability was obtained by Kadomtsev^[9] when considering configurations with closed lines of \mathbf{B} assuming a constant displacement ξ_{\perp} along the lines of force. He showed in^[8] that the condition close to $\Omega < 0$ and weaker than $\Omega < 0$,

$$\Omega (\bar{p}^2 - \gamma p \Omega) < 0, \quad (1.11)$$

is a necessary and sufficient condition for arbitrary perturbations in a low pressure plasma with closed magnetic lines of force.

5. The remaining terms on the left-hand side of inequality (1.2) can be interpreted as follows. The first term describes the stabilizing action of the shear of the lines of force. The term proportional to Ω characterizes the stabilizing action of the minimum \bar{B} . The term within the round brackets describes the destabilization connected with the finite pressure of the plasma. The second term expresses the simultaneous action of the shear and a longitudinal current flowing in the plasma.

In the case of a plasma configuration with a longitudinal current, the term proportional to Ω becomes of the same order of magnitude as the last term within the round brackets of inequality (1.2) even for vanishingly small pressures, and may prove to be insufficient for stabilization. For instance, in the case of a cylindrical geometry with $S = 0$ we obtain the requirement $\bar{p} > 0$. The stabilizing action of the shear described by the first term in (1.2) is negligibly small for configurations with a weakly inhomogeneous longitudinal current. The main stabilizing effect for such systems is determined by the second and third terms of expression (1.2). The minimum \bar{B} principle determining the stability of Stellarator-type configurations, where the magnetic surfaces are formed by the external magnetic fields, thus turns out to be insufficient for the stabilization of Tokamak-type systems, the magnetic surfaces of which are produced by the current in the plasma.

The insufficiency of only the single condition of minimum \bar{B} for configurations with a longitudinal current agrees with the fact that there exists yet another, additional stability condition—the Shafranov-Kruskal criterion^[10,11] which was first obtained for the case of a cylindrical geometry of the plasma and afterwards extended to the case of a torus of sufficiently large radius R . This condition has the form

$$\rho_{\Sigma} B_s / R B_{\omega} > 1. \quad (1.12)$$

Here ρ_{Σ} is the radius of the plasma filament, B_{ω} and B_s are, as in Eq. (1.7), the azimuthal and longitudinal magnetic fields. It was further shown that the criteria (1.2) and (1.3) applied to a toroidal configuration lead to a restriction on a longitudinal current of the same type as condition (1.12).

The criteria (1.2) and (1.3) are only necessary conditions for the stability of the plasma since in their

derivation we considered perturbations of a special form. However, they lead to sufficiently narrow restrictions and contain as particular cases all hydromagnetic stability criteria obtained under different assumptions which are known at the present time. We note in conclusion that the criterion (1.2), which for a cylindrical plasma gives Suydem's condition (1.7) for a toroidal plasma, leads to an appreciably weaker restriction on the magnitude of the longitudinal current of the kind (1.12) (see Sec. 3).

2. TOROIDAL EQUILIBRIUM CONFIGURATIONS FOR THE CASE OF AXIAL SYMMETRY

1. General Relations

It is convenient to describe axially symmetric configurations in a cylindrical system of coordinates r, ϕ, z by means of the functions $\psi = rA_{\phi}$, $I_A = rB_{\phi}$, and p which depending solely on ψ , i.e., which are surface functions. The components of the vectors \mathbf{B} and \mathbf{j} are then determined by the relations^[12]

$$B_z = \partial\psi/\partial r, \quad B_r = -\partial\psi/\partial z, \quad B_{\phi} = I_A/r, \quad (2.1)$$

$$j_z = I_A' \partial\psi/r \partial r, \quad j_r = -I_A' \partial\psi/r \partial z, \quad j_{\phi} = rp' + I_A I_A'/r. \quad (2.2)$$

The prime denotes here and henceforth differentiation with respect to ψ .

The function $\psi(r, z)$ satisfies the equation

$$\frac{\partial^2 \psi}{\partial z^2} + r \frac{\partial}{\partial r} \frac{1}{r} \frac{\partial \psi}{\partial r} = -r^2 p' - I_A I_A', \quad (2.3)$$

where p and I_A are arbitrary functions of ψ which determine the distribution of the longitudinal field B_{ϕ} and the longitudinal current density j_{ϕ} along the cross section of the plasma.

The main surface functions Φ, χ, J , and I , which occur in the stability theory, can in the case of an axially symmetric problem be written in the form

$$\chi = -2\pi \int B_z r dr = -2\pi\psi, \quad I = -2\pi \int j_z r dr = -2\pi I_A \quad (2.4)$$

$$\Phi = \int I_A dz dr/r, \quad J = \int \left(rp' + \frac{I_A I_A'}{r} \right) dz dr. \quad (2.5)$$

If we introduce, apart from the volume V bounded by the closed magnetic surface $\psi = \text{const}$, a function U such that

$$V = 2\pi \int r dz dr, \quad U = 2\pi \int (1/r) dz dr, \quad (2.6)$$

then the differentials of Φ and J can be expressed in terms of the differentials of V and U :

$$2\pi d\Phi = I_A dU, \quad 2\pi dJ = p' dV + I_A I_A' dU. \quad (2.7)$$

To find the surface quantities determined by Eqs. (1.4) as functions of ψ for given $p(\psi)$ and $I_A(\psi)$ it is thus sufficient to evaluate the integrals (2.6). The expressions for S and Ω can then be written in the form

$$V^3 S = -(I_A U')', \quad V^3 \Omega = -I_A' (I_A U')' - p' V''. \quad (2.8)$$

2. The Solution in the Vicinity of the Magnetic Axis

We expand $-p'$ and $-I_A I_A'$ in a power series in ψ :

$$-p' = a + a'\psi + \dots, \quad -I_A I_A'/R^2 = b + b'\psi + \dots, \quad (2.9)$$

and look for a solution of the equilibrium equation (2.3) near the magnetic axis ($r = R, z = 0$) as a power series in z and $r^2 - R^2$. Imposing the requirement of symmetry with respect to the $z = 0$ plane we get, up to and including cubic terms

$$\psi = \frac{AR^2}{2} \left(1 + C \frac{r^2 - R^2}{R^2} \right) z^2 + \frac{a + b - A}{8} (r^2 - R^2)^2 - \frac{b - (1 - C)A}{24R^2} (r^2 - R^2)^3 + \dots \quad (2.10)$$

We see that in our approximation the terms in the expansion (2.9) which are linear in ψ do not influence the equilibrium configuration. The cross sections of the magnetic surfaces $\psi = \text{const}$ near the magnetic axis are ellipses and the ratio of their semi-axes $l_z/l_r = \epsilon$ determines the constant $A = (a + b)/(1 + \epsilon^2)$. The second arbitrary constant C depends on the position of the separatrix of the magnetic surfaces; the coordinates r_S and z_S of its edges are given by the formulae

$$\frac{r_S^2}{R^2} - 1 = -\frac{1}{C}, \quad \frac{z_S^2}{R^2} = \frac{1}{4C^3} \left[(1 + 2\epsilon^2)C - \frac{\epsilon^2 - a/b}{1 + a/b} \right], \quad (2.11)$$

$$\frac{r_S^2}{R^2} - 1 = \frac{2\epsilon^2}{C + (\epsilon^2 - a/b)/(1 + a/b)}, \quad z_S = 0. \quad (2.12)$$

The form of the magnetic surfaces of the equilibrium configuration (2.10) depends thus on the three constant parameters ϵ, C , and b/a . The ratio b/a is determined by the current distribution in the plasma. In particular, the position of the maximum of the current density j_ϕ as function of r depends according to (2.2) on b/a : $r_0 = R\sqrt{b/a}$.

One verifies easily that if one chooses the constants specially such that the coefficient of $(r^2 - R^2)^3$ vanishes, Eq. (2.10) solves Eq. (2.3) exactly with $p' = -a'$ and $I_A I_A' = -bR^2$. Putting $b = (1 - C)A, c_0 = AC$, we get the exact solution

$$\psi = (bR^2 + c_0 r^2) z^2 / 2 + (a - c_0) (r^2 - R^2)^2 / 8. \quad (2.13)$$

We note two particular cases.

1) When $b = 0$ both edges of the separatrix contract

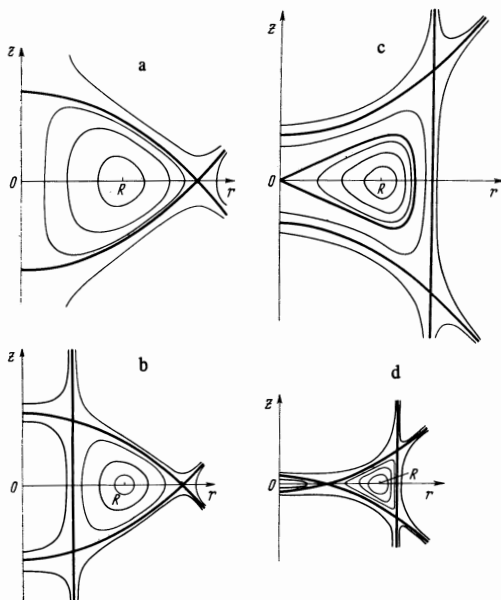


FIG. 1. Cross-sections of the magnetic surfaces of the configuration (2.10): a) $C = 1, b/a = 1$; b) $C = 4/3, b/a = 1$; c) $C = 4/5, b/a = 1$; d) $C = -8/3, b/a = 1$.

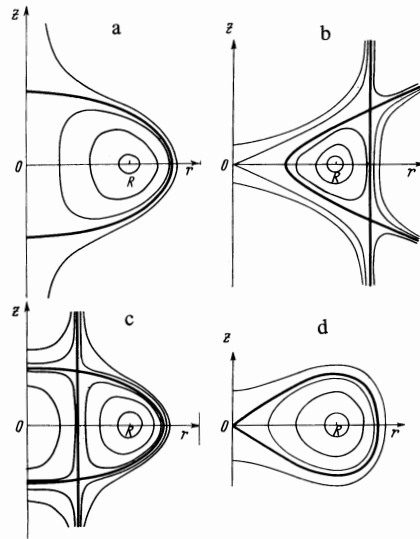


FIG. 2. Cross-sections of the magnetic surfaces of the configuration (2.13): a) $b/a = 0$; b) $b/a = -1/7$; c) $b/a = -7$; d) $b/a = 1$.

to the z axis. The corresponding equilibrium configuration, which was considered in [13], is described by the function

$$\psi = c_0 r^2 z^2 / 2 + (a - c_0) (r^2 - R^2)^2 / 8. \quad (2.14)$$

2) In the case $c_0 = 0$, the separatrix has only one edge situated at the origin $r = 0, z = 0$, and then

$$\psi = bR^2 z^2 / 2 + a(r^2 - R^2)^2 / 8. \quad (2.15)$$

The equilibrium configuration (2.15) is remarkable in that one can evaluate for it relatively simply all surface functions occurring in the stability criteria, (1.2) and (1.3).

The cross sections of the magnetic surfaces of the equilibrium configurations (2.10) and (2.13) are shown in Figs. 1 and 2 for $\epsilon = 1$. The magnetic surfaces of configuration (2.13) differ from the magnetic surfaces of the general case (2.10) in that the edge of the separatrix (2.12) which lies at $z = 0$ is absent.

In "polar" coordinates ($R - r = \rho \cos \omega, z = \rho \sin \omega$) connected with the magnetic axis $r = R$, the solution (2.10) can be written, up to terms of order ρ^3 , in the form

$$\psi = (\lambda_1 \sin^2 \omega + \lambda_2 \cos^2 \omega) \rho^2 + (\lambda_3 \sin^2 \omega + \lambda_4 \cos^2 \omega) \cos \omega \rho^3 + \dots, \quad (2.16)$$

where the parameters λ_i can be expressed in terms of the parameters of the configuration (2.10) by the formulae

$$\lambda_1 = AR^2/2, \quad \lambda_2 = (a + b - A)R^2/2, \quad \lambda_3 = ACR_A, \quad \lambda_4 = (3a + b - A - 2AC)R/6. \quad (2.17)$$

3. Expressions for the Surface Functions in a System of Coordinates Connected with the Magnetic Axis

For the calculation of the surface functions of an axially symmetric equilibrium configuration we can use a polar system of coordinates ρ, ω, φ which is connected with the magnetic axis, $R - r = \rho \cos \omega, z = \rho \sin \omega$, in which the components of the magnetic

field can be expressed in terms of ψ through the formulae $B_\rho = \partial\psi/\rho r \partial\omega$ and $B_\omega = -\partial\psi/r \partial\rho$. Integrating over a layer between two neighboring magnetic surfaces $\delta\psi = \delta\rho \partial\psi/\partial\rho$ we get the following expressions for V' and U' :

$$V' = 2\pi \int_0^{2\pi} \frac{r\rho d\omega}{\partial\psi/\partial\rho}, \quad U' = 2\pi \int_0^{2\pi} \frac{\rho d\omega}{r \partial\psi/\partial\rho}. \quad (2.18)$$

Apart from V' and U' the surface quantities indicated by angle brackets enter into the stability conditions (1.2)(1.3). One checks easily that in the case considered where there is axial symmetry $\langle f \rangle_C = \langle f \rangle$. Indeed,

$$\oint f dl/B = \oint f \rho d\omega/B_\omega = - \oint f r \rho d\omega / (\partial\psi/\partial\rho), \quad (2.19)$$

and we get from (2.18)

$$\langle f \rangle = \frac{2\pi}{V'} \int_0^{2\pi} f \frac{r\rho d\omega}{\partial\psi/\partial\rho}. \quad (2.20)$$

From this it follows in particular that $\langle 1/r^2 \rangle = U'/V'$. For a known function ψ one can thus express the surface quantities $\langle f \rangle$, similarly to V' and U' , in terms of a single integral taken with the limits from 0 to 2π .

4. Expressions for the Surface Functions in Cylindrical Coordinates

It is more convenient to calculate the surface functions for the exact solution (2.13) directly in a cylindrical system of coordinates r, φ, z . If we know the expression for z as function of r and ψ , we can obtain, by taking r as the independent variable, the single integrals

$$V = 4\pi \int_{r_1}^{r_2} z(r, \psi) r dr, \quad U = 4\pi \int_{r_1}^{r_2} z(r, \psi) \frac{dr}{r}, \quad (2.21)$$

where the integration is taken between points where $z(r, \psi)$ vanishes. The derivatives V' and U' can similarly to the preceding be written in the form

$$V' = 4\pi \int_{r_1}^{r_2} \frac{r dr}{\partial\psi/\partial z}, \quad U' = 4\pi \int_{r_1}^{r_2} \frac{dr}{r \partial\psi/\partial z}. \quad (2.22)$$

The surface functions denoted by angle brackets are then expressed in terms of the integrals

$$\langle f \rangle = \frac{4\pi}{V'} \int_{r_1}^{r_2} f \frac{r dr}{\partial\psi/\partial z}. \quad (2.23)$$

5. The Surface Functions V and U for the Equilibrium Configuration (2.13)

Using the substitution $r^2 = R^2(1 + \sqrt{\Psi} \cos \vartheta)$, where $\Psi = 8\psi^2/R^4(a - c_0)$, we can write the expressions for the functions V and U for the configuration (2.13) in the form

$$V = \frac{8\pi\psi}{R\sqrt{(a-c_0)(b+c_0)}} \int_0^\pi \frac{\sin^2 \vartheta d\vartheta}{\sqrt{1+k_0 \cos \vartheta}}, \quad (2.24)$$

$$U = \frac{8\pi\psi}{R^3\sqrt{(a-c_0)(b+c_0)}} \int_0^\pi \frac{\sin^2 \vartheta d\vartheta}{(1+\kappa k_0 \cos \vartheta)\sqrt{1+k_0 \cos \vartheta}}.$$

Here $k_0 = \sqrt{\kappa}$ and $\kappa = 1 + b/c_0$. The integrals obtained reduce to elliptic ones:^[14]

$$V = \frac{64\pi\psi}{R\sqrt{(a-c_0)(b+c_0)(1+k_0)}} \left[\left(2 - \frac{2}{k^2}\right) \frac{K-E}{3k^2} - \frac{E}{3k^2} \right], \quad (2.25)$$

$$U = \frac{64\pi\psi}{R^3\sqrt{(a-c_0)(b+c_0)(1+k_0)}} \frac{1}{1+\kappa k_0} \times \left[\frac{K-E}{k^4 \sin^2 \alpha} - \frac{KE(a, k) - EF(a, k)}{k^4 \sin^3 \alpha \cos \alpha / \sqrt{1-k^2 \sin^2 \alpha}} \right], \quad (2.26)$$

where the modulus k and $\sin \alpha$ are defined by the expressions $k^2 = 2k_0/(1+k_0)$, $\sin^2 \alpha = (\kappa + \kappa k_0)/(1 + \kappa k_0)$.

By differentiating the integrals (2.24) with respect to the parameter, we find easily the expansion of V and U in the vicinity of the magnetic axis:

$$V = \frac{4\pi^2\psi}{R\sqrt{(a-c_0)(b+c_0)}} \left[1 + \frac{3}{32} k_0^2 + \dots \right], \quad (2.27)$$

$$U = \frac{4\pi^2\psi}{R^3\sqrt{(a-c_0)(b+c_0)}} \left[1 + \left(\frac{3}{32} + \frac{\kappa}{8} + \frac{\kappa^2}{4} \right) k_0^2 + \dots \right].$$

When $\epsilon = 1$, it follows from this, when $c_0 = (a-b)/2$ that

$$V = \frac{8\pi^2\psi}{R(a+b)} \left\{ 1 + \frac{3}{4} \frac{(a-b)^2}{(a+b)^3} \frac{2\psi}{R^4} + \dots \right\},$$

$$U = \frac{8\pi^2\psi}{R^3(a+b)} \left\{ 1 + \left[\frac{3}{4} \left(\frac{a-b}{a+b} \right)^2 + \frac{3a+b}{a+b} \right] \frac{2\psi/R^4}{a+b} + \dots \right\}. \quad (2.28)$$

In the case of circular near-axis cross-sections of the magnetic surfaces both functions V and U can for the configuration (2.14) be expressed in terms of complete elliptic integrals:

$$V = \frac{8\pi R^3 \Psi}{(1+\sqrt{\Psi})^{1/2}} \left[\frac{2k^2-2}{3k^4} K + \frac{2-k^2}{3k^4} E \right],$$

$$U = \frac{8\pi R \Psi}{(1+\sqrt{\Psi})^{3/2}} \left[\frac{2-k^2}{k^4} K - \frac{2}{k^4} E \right] \quad (2.29)$$

Here $b = 0$, $a = 2c_0$, $\Psi = 16\psi/aR^4$, $k^2 = 2\sqrt{\Psi}/(1+\sqrt{\Psi})$.

For the case of the equilibrium configuration (2.15), V and U are elementary functions for any eccentricity of the near-axis cross-sections of the magnetic surfaces:

$$V = \frac{4\pi^2}{\sqrt{ab}R} \psi, \quad U = \pi^2 R \sqrt{\frac{a}{b}} \left(1 - \sqrt{1 - \frac{8\psi}{aR^4}} \right). \quad (2.30)$$

3. STABILITY OF AXIALLY SYMMETRIC TOROIDAL CONFIGURATIONS

Taking into account the expressions obtained in the above for the surface functions we can write the criteria for local and convective stability (1.2) and (1.3) for axially symmetric configurations in the form¹⁾

$$1/4 V'^6 S^2 - p' \{ I^2 [U''Q_1 - V''Q_2 + p'(Q_2Q_3 - Q_1^2)] + U'(I'Q_1 + p'Q_3 - V'') \} > 0, \quad (3.1)$$

$$S^2 [I^2 (U'Q_4 - V'^2) + Q_4Q_5] + SI \left(2p' + I' \frac{U'}{V'} + \frac{I'}{I} \frac{Q_5}{V'} \right) + \frac{p'V''}{V'^4} (I^2U' + Q_5) - \frac{p'^2}{V'^2} + \nu p \left\{ S^2 V'Q_4 + SI' \frac{2V''}{V'} + \frac{I'}{I} \right\} + \frac{V''}{V'^3} \left[(I^2U' + Q_5) \frac{V''}{V'^2} - p' \right] > 0. \quad (3.2)$$

In accordance with relations (2.8) we have here $S = -(IU')'/V'^3$, $\Omega = I'S - p'V''/V'^3$ and we have denoted by Q_i the following quantities:

$$\frac{Q_1}{V'} = \left\langle \frac{1}{|\nabla\psi|^2} \right\rangle, \quad \frac{Q_2}{V'} = \left\langle \frac{1}{r^2 |\nabla\psi|^2} \right\rangle, \quad \frac{Q_3}{V'} = \left\langle \frac{r^2}{|\nabla\psi|^2} \right\rangle,$$

$$\frac{Q_4}{V'} = \langle r^2 \rangle, \quad \frac{Q_5}{V'} = \left\langle \frac{|\nabla\psi|^2}{r^2} \right\rangle. \quad (3.3)$$

¹⁾In the following we drop for the sake of simplicity the index A of the function $I_A(\Psi) = rB_\varphi$.

For given functions $p(\psi)$ and $I(\psi)$ the function ψ is determined by the solution of the equilibrium Eqs. (2.3) and the functions $V(\psi)$, $U(\psi)$, and $Q_i(\psi)$ are expressed by the integrals (2.18) to (2.23).

1. Stability of Toroidal Configurations Near the Magnetic Axis

To study the stability near the magnetic axis by expanding in powers of ψ it is expedient to write the functions V and Q_i in the form

$$V = R^2 U - W, \quad Q_2 = \frac{Q_1}{R^2} - \frac{V'}{R^2} \left\langle \frac{r^2 - R^2}{r^2 |\nabla\psi|^2} \right\rangle,$$

$$Q_3 = R^2 Q_1 + V' \left\langle \frac{r^2 - R^2}{|\nabla\psi|^2} \right\rangle, \quad Q_4 = R^2 V' + V' \langle r^2 - R^2 \rangle,$$

separating explicitly their main part and the corrections which can easily be evaluated by the well-known expansion of ψ in powers of the displacement ρ from the magnetic axis.

Retaining in the inequalities (3.1) and (3.2) only the main terms, we obtain the following stability conditions near the magnetic axis $r = R$:

$$p' \left[p' + \frac{II'}{R^2} + \frac{I^2}{R^2} \left(\frac{W''}{V'} + p' \frac{q}{V'} \right) \right] > 0, \quad (3.4)$$

$$- \left(p' + \frac{II'}{R^2} \right)^2 - I^2 \left[\frac{p' W''}{R^2 V'} + \left(p' + \frac{II'}{R^2} \right) \frac{U''}{V'} \right] + \nu p \left[\left(\frac{I^2 W''}{R^2 V'} + \frac{II'}{R^2} \right) \frac{W''}{V'} - \left(p' + \frac{II'}{R^2} \right) \frac{V''}{V'} \right] > 0. \quad (3.5)$$

Here W and q are surface functions defined by the relations

$$W' = R^2 U' - V' = 2\pi \int_0^{2\pi} \frac{(R^2 - r^2) \rho d\omega}{r \partial\psi / \partial\rho},$$

$$q = V' \left\langle \frac{(r^2 - R^2)^2}{r^2 |\nabla\psi|^2} \right\rangle = 2\pi \int_0^{2\pi} \frac{(r^2 - R^2)^2 \rho d\omega}{r |\nabla\psi|^2 \partial\psi / \partial\rho}. \quad (3.6)$$

Expressions (3.4) to (3.6) show that for evaluating all quantities occurring in the condition for local stability (3.4) it is sufficient to know the expansion of the function ψ up to terms of order ρ^3 . However, for the condition of convective stability it is necessary to evaluate apart from W'' also separately either V'' or U'' and this requires taking into account in the function ψ quantities of order ρ^4 .

A. We consider first the condition of local stability (3.4). The current function ψ is for an equilibrium configuration which is symmetric with respect to the $z = 0$ plane given by Eq. (2.16):

$$\psi = f_1(\omega) \rho^2 + f_2(\omega) \cos \omega \rho^3 + \dots$$

In the approximation considered $\rho = \sqrt{(\psi/f_1)}$ $\times (1 - \sqrt{(\psi/f_1)} f_2/2)$, and the functions V' , W'' , and q occurring in (3.4) can be written in the form

$$V' = \pi R \int_0^{2\pi} \frac{d\omega}{f_1}, \quad W'' = \frac{\pi}{R} \int_0^{2\pi} \left(4R \frac{f_2}{f_1} + 1 \right) \cos^2 \omega \frac{d\omega}{f_1^2},$$

$$q = 4\pi R \int_0^{2\pi} \frac{\cos^2 \omega d\omega}{4f_1^2 + f_1'^2 f_1}. \quad (3.7)$$

Evaluation of these integrals gives

$$V' = \frac{2\pi R}{\lambda_1 + \lambda_2} \frac{1 + \varepsilon^2}{\varepsilon}, \quad q = \frac{2\pi^2 R}{(\lambda_1 + \lambda_2)^3} \frac{(1 + \varepsilon^2)^3}{(1 + \varepsilon)\varepsilon^4}, \quad (3.8)$$

$$W'' = \frac{\pi^2 (1 + \varepsilon^2)^2}{R(\lambda_1 + \lambda_2)^2 \varepsilon^3} \left\{ 1 + \frac{R}{\lambda_1 + \lambda_2} \frac{(1 + \varepsilon^2)^3}{4\varepsilon^2} \left[\lambda_3 + 3\lambda_4 + 6\lambda_4 \frac{1 - \varepsilon^2}{1 + \varepsilon^2} - (\lambda_3 - 3\lambda_4) \left(\frac{1 - \varepsilon^2}{1 + \varepsilon^2} \right)^2 \right] \right\}, \quad (3.9)$$

where $\varepsilon = l_z/l_r = \sqrt{(\lambda_2/\lambda_1)}$ is the ratio of the semi-axes of the elliptical cross sections of the magnetic surfaces near the magnetic axis. Expressing the coefficients λ_i in terms of the parameters a , b , and C of the equilibrium configuration (2.10), we get

$$V' = \frac{4\pi^2}{R(a+b)} \frac{1 + \varepsilon^2}{\varepsilon}, \quad q = \frac{16\pi^2}{R^3(a+b)^3} \frac{(1 + \varepsilon^2)^3}{(1 + \varepsilon)\varepsilon^4}, \quad (3.10)$$

$$W'' = \frac{4\pi^2 (1 + \varepsilon^2)^3}{R^3(a+b)^2 \varepsilon^5} \left[\frac{3a+b}{a+b} - \frac{1 - \varepsilon^2}{1 + \varepsilon^2} (1 + 2C) \right]. \quad (3.11)$$

Substituting these expressions into inequality (3.4) we get the following condition for local stability near the magnetic axis:

$$\frac{R^2 j_0^2}{B_0^2} < \left(1 + \frac{1}{\varepsilon} \right)^2 \left[\frac{2}{1 + \varepsilon} + \frac{1 - \varepsilon a - b}{1 + \varepsilon a + b} - \frac{1 - \varepsilon^2}{1 + \varepsilon^2} (1 + 2C) \right], \quad (3.12)$$

where the values of j^2 and B^2 are chosen on the magnetic axis itself, $r = R$. According to (3.12), the limiting current density near the magnetic axis depends on the parameters ε , b/a , and C . The quantity ε is determined by the eccentricity of the near-axis cross sections of the magnetic surfaces, the ratio b/a depends on the current distribution along the cross section of the plasma, and the constant C characterizes the third harmonic of the function ψ , determining the position of the separatrix.

In the case of circular near-axis cross-sections of the magnetic surfaces, $\varepsilon = 1$, the dependence on the parameters b/a and C disappears, and the stability condition (3.12) takes the very simple form $R^2 j_0^2 / B_0^2 < 4$. Assuming small toroidality and uniformity of the longitudinal current and of the longitudinal magnetic field, this condition is exactly the same as the restriction (1.12).

To elucidate the dependence of the limiting current on the eccentricity of the near-axis cross-sections of the magnetic surfaces we choose the constant C such that the expansion of ψ near the magnetic axis (2.10) is the same as the exact solution (2.13), and then $C = (a/b - \varepsilon^2)/(a/b + 1)$.

We consider particular cases for different ratios b/a .

1. If $a = 0$, i.e., the magnetic configuration near the axis is force-free,

$$R^2 j_0^2 / B_0^2 < 2(1 + \varepsilon^2)(2 - \varepsilon^2) / \varepsilon^2,$$

and for stability it is necessary that $\varepsilon < \sqrt{2}$.

2. For $a = b$, when the maximum longitudinal current density as function of r occurs at $r = R$, we have

$$R^2 j_0^2 / B_0^2 < (5 + 3\varepsilon - 2/\varepsilon - \varepsilon^2 - \varepsilon^3)(1 + \varepsilon^2) / \varepsilon^2(1 + \varepsilon).$$

The first factor on the right-hand side has here a maximum when $\varepsilon = 1$ and goes through zero near $\varepsilon = 1/2$ and $\varepsilon = 3/2$ so that stability occurs only for magnetic surface cross sections which are sufficiently close to circular.

3. In the case $b = 0$ (we have (for the equilibrium configuration (2.14))

$$R^2 j_0^2 / B_0^2 < 2(\varepsilon^2 + 3\varepsilon - 2)(1 + \varepsilon^2) / \varepsilon^3(1 + \varepsilon).$$

For stability it is then necessary that $\epsilon > 0.56$.

4. When $a/b = \epsilon^2$ (for the equilibrium configuration (2.15)), we get

$$R^2 j_0^2 / B_0^2 < 4(1 + \epsilon^2) / \epsilon^2(1 + \epsilon).$$

In that case there is on the right-hand side of the inequality a monotonically decreasing function of ϵ which does not tend to zero.

Let us now assume that the plasma pressure vanishes on some magnetic surface ψ_Σ , which has a minimum distance from the z axis equal to R_1 ; then

$$p = p_0(1 - \psi/\psi_\Sigma), \quad I^2 = I_0^2 + 2II'(\psi - \psi_\Sigma), \quad (3.13)$$

$$\psi_\Sigma = \frac{\epsilon^2 R^4}{8} \frac{a+b}{1+\epsilon^2} \left(1 - \frac{R_1^2}{R^2}\right)^2 \Theta, \quad \Theta = 1 + \frac{1 - R_1^2/R^2}{3\epsilon^2} \left(\frac{\epsilon^2 - a/b}{1 + a/b} + C\right). \quad (3.14)$$

These equations enable us to obtain the condition for local stability in terms of the ratio of the plasma to the magnetic pressure $\beta_0 = 2p_0/B_0^2$ on the magnetic axis:

$$\beta_0 < \frac{(1 + \epsilon^2)(1 - R_1^2/R^2)^2}{4\epsilon^2(1 + b/a)} \left[\frac{2}{1 + \epsilon} + \frac{1 - \epsilon}{1 + \epsilon} \frac{1 - b/a}{1 + b/a} - \frac{1 - \epsilon^2}{1 + \epsilon^2}(1 + 2C) \right] \Theta. \quad (3.15)$$

When the stability condition has this form, the dependence on the parameters b/a and C remains even in the case of circular near-axis magnetic surface cross-sections, $\epsilon = 1$, in which case for stability it is necessary that

$$\beta_0 < \frac{1}{2} \frac{(1 - R_1^2/R^2)^2}{1 + b/a} \left[1 - \left(\frac{1 - b/a}{1 + b/a} - C\right) \frac{1 - R_1^2/R^2}{3} \right]. \quad (3.16)$$

The parameter C depends on the distance from the edge of the separatrix. If both edges of the separatrix (2.11) are inside the torus, $C > 0$ and stability is improved. However, if both edges of the separatrix (2.11) are outside the torus this has a destabilizing influence.

It follows from (3.13) that the ratio b/a can be interpreted as a characteristic for the degree of diamagnetism of the plasma

$$\frac{b}{a} = \frac{I^2(0) - I^2(\psi_\Sigma)}{2R^2 p(0)} = \frac{B_0^2 - B_\epsilon^2}{2p_0}. \quad (3.17)$$

Here B_ϵ is the longitudinal field B_φ on the boundary of the plasma $\psi = \psi_\Sigma$ when $r = R$.

For small toroidality when $1 - R_1^2/R^2 \ll 1$ there remains a strong dependence of β_0 on the parameters R_1/R and b/a :

$$\beta_0 < \frac{1}{2} \frac{(1 - R_1^2/R^2)^2}{1 + b/a}. \quad (3.18)$$

For the equilibrium configurations described by the exact solution (2.13) $\Theta = 1$, and in the case $\epsilon = 1$ we get a condition for local stability which is the same as (3.18).

To estimate separately the influence of the shear and the minimum \bar{B} we consider the local stability of the configuration (2.13) for $\epsilon = 1$. In the case of configuration (2.15) when $V'' = 0$ we get $a = b$, and condition (3.18) gives $\beta_0 < (1 - R_1^2/R^2)^2/4$. In the case when $S = 0$ it follows from (2.8) and (2.28) that

$$R^2 j_0^2 / B_0^2 = [7 + 10a/b + 15(a/b)^2] / (1 + a/b). \quad (3.19)$$

The right-hand side of Eq. (3.19) has a minimum exceeding 4 for $a/b \approx 0.1$, and this contradicts the condition for local stability $R^2 j_0^2 / B_0^2 < 4$. In the case of closed

lines of force of \mathbf{B} the configuration (2.13) with $\epsilon = 1$ thus turns out to be unstable.

B. Let us now consider the condition for convective stability (3.5). In those cases when $V'' = 0$ or $S = 0$, the functions V'' and U'' occurring in (3.5) can be expressed in terms of the function W'' and to evaluate that function it is sufficient to know the expansion of ψ up to terms $\sim \rho^3$. Using Eq. (2.8) we get condition (3.5) for the cases $V'' = 0$ and $S = 0$, respectively, in the form

$$-\left(p' + \frac{II'}{R^2}\right)^2 - \left[\frac{I^2}{R^2} \left(2p' + \frac{II'}{R^2}\right) - \gamma p \left(\frac{I^2}{R^2} \frac{W''}{V'} + \frac{II'}{R^2}\right)\right] \frac{W''}{V'} > 0, \quad (3.20)$$

$$-\left(p' + \frac{II'}{R^2}\right)^2 - \frac{I^2}{R^2} \left[p' \frac{W''}{V'} - \left(p' + \frac{II'}{R^2}\right) \frac{I'}{I}\right] + \gamma p \left[\left(\frac{I^2}{R^2} \frac{W''}{V'} + p' + \frac{2II'}{R^2}\right) \frac{W''}{V'} + \left(p' + \frac{II'}{R^2}\right) \frac{I'}{I}\right] > 0. \quad (3.21)$$

We turn further to the equilibrium configuration (2.13) and limit ourselves here to circular near-axis magnetic surface cross sections $\epsilon = 1$. In the case $V'' = 0$, from (2.28) we have $a = b$, and condition (3.20) with $\epsilon = 1$ gives $R^2 j_0^2 / B_0^2 < 12$. The condition for convective stability near the magnetic axis has thus the same form as the condition for local stability and in the case considered turns out to be weaker. In the case $S = 0$ it follows from Eqs. (3.19) and (3.21) that the configuration considered turns out to be also unstable with respect to convective perturbations when $\beta_0 \ll 1$ and $1 + b/a > 0$.

2. Stability of the Equilibrium Configuration (2.15)

In conclusion we shall consider in more detail the configuration (2.15), the stability of which can be evaluated most simply. Since for the given configuration the volume V is proportional to ψ , we can write Eqs. (3.13) and (3.14) in the form

$$p = p_0(1 - V/V_\Sigma), \quad 2V_\Sigma = \pi^2 \epsilon R^3 (1 - R_1^2/R^2)^2. \quad (3.22)$$

Here V_Σ is the volume bounded by the magnetic surface Ψ_Σ on which $p = 0$.

Putting $C = 0$, $a/b = \epsilon^2$ we get from (3.15) the condition for local stability in the vicinity of $r = R$:

$$\beta_0 < \frac{2\epsilon V_\Sigma / \pi^2 R^3}{(1 + \epsilon)(1 + \epsilon^2)}. \quad (3.23)$$

For fixed V_Σ and R the expression on the right-hand side of inequality (3.23) is a maximum for $\epsilon = l_z/l_r = 0.66$. In practice the optimum ϵ can most expediently be looked for taking into account the given V_Σ and R . If we write the limiting value of β_0 determined by the inequality (3.23) as $\beta_0 \max = \epsilon^2(1 - R_1^2/R^2)^2 / (1 + \epsilon)(1 + \epsilon^2)$ and put equal to zero the variation of V_Σ and $\beta_0 \max$ with respect to ϵ and $R(\epsilon)$ for fixed R_1 we find that the optimum value of ϵ is determined by the equation

$$R_1^2 / 3R^2 = (2 + \epsilon - \epsilon^2) / (2 + 3\epsilon + 4\epsilon^2 + 5\epsilon^3)$$

and depends on the ratio R_1/R . In the case of small toroidality ($R_1/R \rightarrow 1$) the optimum is $\epsilon = 0.66 = 1/1.52$. In the case of limitingly large toroidality ($R_1/R \rightarrow 0$) the optimum is $\epsilon = 1.52$. The value $\epsilon = 1$ becomes optimum for $R/(R - R_1) = 2.9$. We must note,

however, that the optimum value $\beta_{0 \max}$ differs little from the corresponding value for $\epsilon = 1$.

The quantities Q_4 and Q_5 occurring in the criterion (3.2) for convective stability can, similar to V and U , be expressed as elementary functions:

$$Q_4 = \frac{4\pi^2 R}{\sqrt{ab}}, \quad Q_5 = \pi^2 R^3 \sqrt{\frac{a}{b}} \left[\frac{4\psi}{R^4} + b \left(1 - \sqrt{1 - \frac{8\psi}{aR^4}} \right) \right]. \quad (3.24)$$

As we noted earlier, the condition for convective stability near the magnetic axis turns out to be weaker than the condition for local stability.

Bearing in mind that for the configuration considered $V'' = 0$, we can write the criteria (3.1) and (3.2) in the form

$$V'^6 S^2 - 4p' \{ I^2 [U'' Q_1 + p' (Q_2 Q_3 - Q_1^2)] + U' (II' Q_1 + p' Q_3) \} > 0, \quad (3.25)$$

$$S^2 [I^2 (U' Q_4 - V'^2) + Q_4 Q_5] + SI \left(2p' + II' \frac{U'}{V'} + \frac{I'}{I} \frac{Q_5}{V'} \right) - \frac{p'^2}{V'^2} + \gamma p (S^2 V' Q_4 + SI') > 0. \quad (3.26)$$

If we restrict ourselves to the case of circular near-axis magnetic surface cross-sections, $\epsilon = 1$, we obtain easily for the quantities Q_1 , Q_2 , and Q_3

$$Q_1 = \frac{2\pi^2}{3a^2 R^3 \psi} \left[\frac{1}{\sqrt{1-\alpha}} + g_1 \right], \quad (3.27)$$

$$Q_3 = \frac{2\pi^2}{3a^2 R^3 \psi} [\sqrt{1-\alpha} + (1+\alpha)g_1 - \alpha g_2],$$

$$Q_2 = \frac{2\pi^2}{3a^2 R^3 \psi} \left[\frac{1}{(1-\alpha)^{3/2}} - \frac{3\alpha^3}{(1-\alpha^2)^{3/2}} + \frac{g_1 + \alpha g_2}{1+\alpha+\alpha^2} \right]. \quad (3.28)$$

Here $\alpha^3 = 8\psi/aR^4$ and g_1 and g_2 denote the following functions of α :

$$g_1 = \left[\frac{2}{\sqrt{1+\alpha+\alpha^2}} + \frac{2+\alpha}{1+\alpha+\alpha^2} \right]^{1/2},$$

$$g_2 = \left[\frac{2}{\sqrt{1+\alpha+\alpha^2}} - \frac{1+2\alpha}{1+\alpha+\alpha^2} \right]^{1/2}. \quad (3.29)$$

We have thus for the configuration (2.15) considered here for $\epsilon = 1$ the exact expressions (2.30), (3.24), and (3.27) to (3.29) in terms of elementary functions for all quantities occurring in the stability conditions (3.25) and (3.26). These expressions show that the ratio ψ/ψ_S , with ψ_S the value of ψ on the separatrix, is the expansion parameter in the vicinity of the magnetic axis. In the given case $\psi/\psi_S = 8\psi/aR^4$. When $\epsilon = \sqrt{(a/b)} = 1$, we have from (2.30) and (3.24)

$$V' = \frac{4\pi^2}{aR}, \quad U' = \frac{4\pi^2}{aR^3} \frac{1}{\sqrt{1-\alpha^3}}, \quad Q_4 = \frac{4\pi^2 R}{a},$$

$$Q_5 = \pi^2 R^3 a \left(\frac{\alpha^3}{2} + 1 - \sqrt{1-\alpha^3} \right). \quad (3.30)$$

Substituting the functions (3.27) to (3.30) into the stability criteria (3.25) and (3.26), we can check that the stability condition near the magnetic axis considered in the above is the most stringent one. In the case of lim- itingly large toroidality the main stabilizing factor for a plasma on magnetic surfaces near the separatrix ($\alpha \approx 1$) is the shear described by the term proportional to S^2 .

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APPENDIX

One can obtain the condition for hydromagnetic stability of a plasma from a variational principle^[15]

$$\delta w = \frac{1}{2} \int \{ (\text{rot} [\xi \mathbf{B}])^2 + \gamma p (\text{div} \xi)^2 + (\xi \nabla p) \text{div} \xi + [\xi \xi] \text{rot} [\xi \mathbf{B}] \} d\tau > 0 \quad (1)$$

for arbitrary displacement ξ . If we introduce a system of coordinates θ , ζ , and V which are connected with the magnetic surfaces such that θ and ζ are cylindrical coordinates on the magnetic surfaces with periods of unity and while we choose as third coordinate the volume V and require that $[\nabla \theta \nabla \zeta] \nabla V = 1$, we can write the expression for δw in the form^[4]

$$\delta w = \frac{1}{2} \int \left\{ \left(\text{rot} [\xi \mathbf{B}] + \xi_v \frac{[\mathbf{j} \nabla V]}{|\nabla V|^2} \right)^2 + \gamma p (\text{div} \xi)^2 - 2 \xi_v \frac{[\mathbf{j} \nabla V]}{|\nabla V|^4} (\mathbf{B} \nabla) \nabla V \right\} d\theta d\zeta dV. \quad (2)$$

The vectors \mathbf{B} , \mathbf{j} , and ξ are then defined by the expressions

$$\mathbf{B} = \Phi [\nabla V \nabla \theta] + \chi [\nabla \zeta \nabla V], \quad \mathbf{j} = j [\nabla V \nabla \theta] + i [\nabla \zeta \nabla V],$$

$$\xi = \xi_\theta [\nabla \zeta \nabla V] + \xi_\zeta [\nabla V \nabla \theta] + \xi_v [\nabla \theta \nabla \zeta], \quad (3)$$

so that the contravariant components of these vectors are respectively equal to

$$B^i = \{\chi, \Phi, 0\}, \quad j^i = \{i, j, 0\}, \quad \xi^i = \{\xi_\theta, \xi_\zeta, \xi_v\}. \quad (4)$$

If we introduce the notation

$$\mu = \Phi \xi_\theta - \chi \xi_\zeta, \quad \eta = j \xi_\theta - i \xi_\zeta, \quad (5)$$

we get

$$[\nabla V \nabla \theta] = \frac{i\mathbf{B} - \chi \mathbf{j}}{p}, \quad [\nabla \zeta \nabla V] = \frac{\Phi \mathbf{j} - j\mathbf{B}}{p},$$

$$\xi = \frac{\mu \mathbf{j} - \eta \mathbf{B}}{p} + \xi_v [\nabla \theta \nabla \zeta]. \quad (6)$$

One checks easily that the expressions for $\text{div} \xi$ and $\text{curl} [\xi \times \mathbf{B}]$ will be

$$\text{div} \xi = \frac{1}{p} (j \nabla \mu - \mathbf{B} \nabla \eta) + \frac{\partial \xi_v}{\partial V}, \quad (7)$$

$$\text{rot} [\xi \mathbf{B}] = \left(\frac{\partial \mu}{\partial \theta} + \frac{\partial \xi_v \Phi}{\partial V} \right) [\nabla \theta \nabla V] + \left(\frac{\partial \mu}{\partial \zeta} - \frac{\partial \xi_v \chi}{\partial V} \right) [\nabla \zeta \nabla V] + \left(\frac{\partial \xi_v \Phi}{\partial \zeta} + \frac{\partial \xi_v \chi}{\partial \theta} \right) [\nabla \theta \nabla \zeta]. \quad (8)$$

1. To obtain the criterion for convective stability we put $\xi = \xi(V)$. The potential energy (2) then becomes

$$\delta w = \frac{1}{2} \int \left\{ \xi_v^2 (\gamma p + \mathbf{B}^2) - 2 \xi_v \xi_v \mathbf{B} \left(\frac{Sj - \Omega \mathbf{B}}{p} + \frac{[\mathbf{j} \nabla V]}{|\nabla V|^2} \right) + \xi_v^2 \left[\left(\frac{Sj - \Omega \mathbf{B}}{p} + \frac{[\mathbf{j} \nabla V]}{|\nabla V|^2} \right)^2 - 2 \frac{[\mathbf{j} \nabla V]}{|\nabla V|^4} (\mathbf{B} \nabla) \nabla V \right] \right\} d\theta d\zeta dV. \quad (9)$$

We transform the last term of that expression. As $\mathbf{B} \nabla V = 0$, we can use the vector identity $(\mathbf{B} \cdot \nabla) \nabla V + (\nabla \nabla V) \mathbf{B} = [\mathbf{j} \times \nabla V]$ to obtain

$$\Delta \equiv \frac{2}{|\nabla V|^4} [\mathbf{j} \nabla V] (\mathbf{B} \nabla) \nabla V = \frac{[\mathbf{j} \nabla V]}{|\nabla V|^4} ([\mathbf{j} \nabla V] + \mathbf{K}), \quad (10)$$

where $\mathbf{K} = (\mathbf{B} \cdot \nabla) \nabla V - (\nabla \nabla V) \mathbf{B}$. From the definition of the coordinate V we have

$$\nabla V_i = \{0, 0, 1\}, \quad \nabla V^i = g^{i3}, \quad |\nabla V|^2 = g^{33}.$$

Hence we have, since $\mathbf{K}^i = (\mathbf{B} \cdot \nabla) \nabla V^i - (\nabla \nabla) \mathbf{B}^i$,

$$K^i = \{\mathbf{B} \nabla g^{i3} - g^{33} \dot{\chi}_i, \mathbf{B} \nabla g^{23} - g^{33} \dot{\Phi}, \mathbf{B} \nabla g^{33}\},$$

and hence

$$[\nabla V] \mathbf{K} = |\nabla V|^4 \left(\Omega - \mathbf{B} \nabla \frac{I g^{23} - J g^{13}}{g^{33}} \right) = |\nabla V|^4 \left(\Omega - \mathbf{B} \nabla \frac{[\mathbf{j} \nabla V]_3}{|\nabla V|^2} \right). \quad (11)$$

Substituting this relation into (10) we find

$$\Lambda = \frac{[\mathbf{j} \nabla V]^2}{|\nabla V|^4} + \Omega - \mathbf{B} \nabla \frac{[\mathbf{j} \nabla V]_3}{|\nabla V|^2} = \frac{j^2}{|\nabla V|^2} + \Omega - \text{div} \frac{[\mathbf{j} \nabla V]_3}{|\nabla V|^2} \mathbf{B}. \quad (12)$$

When $\xi_V = \xi_V(\mathbf{V})$ the last term in (12) does not contribute to the integral (9) and therefore

$$\delta \omega = \frac{1}{2} \int \left\{ \xi_V^2 (\gamma p + \mathbf{B}^2) - \frac{2 \xi_V \dot{\xi}_V}{\dot{p}} [\mathbf{B} (S \mathbf{j} - \Omega \mathbf{B}) - \dot{p}^2] + \frac{\xi_V^2}{\dot{p}^2} [(S \mathbf{j} - \Omega \mathbf{B})^2 + \dot{p}^2 \Omega] \right\} d\theta d\xi dV. \quad (13)$$

The quantity $\delta \omega$ will be positive if the integral over the volume δV included between two neighboring magnetic surfaces is positive. If we indicate the average over the volume δV of the layer by angle brackets, the condition that (13) be positive definite is thus

$$-\langle \mathbf{B} (S \mathbf{j} - \Omega \mathbf{B}) - \dot{p}^2 \rangle + \langle \gamma p + \mathbf{B}^2 \rangle \langle (S \mathbf{j} - \Omega \mathbf{B})^2 + \dot{p}^2 \Omega \rangle > 0. \quad (14)$$

Expanding the brackets we get from this the criterion for convective stability in the form

$$\frac{S^2}{\dot{p}^2} \langle \langle j^2 \rangle \langle \mathbf{B}^2 \rangle - \langle \mathbf{j} \mathbf{B} \rangle^2 \rangle + 2S \langle \langle \mathbf{j} \mathbf{B} \rangle - \Omega \langle \mathbf{B}^2 \rangle - \dot{p}^2 + \gamma p \left\langle \left\langle \frac{\mathbf{j} S - \mathbf{B} \Omega}{\dot{p}} \right\rangle^2 + \Omega \right\rangle > 0. \quad (15)$$

2. The condition for local stability, obtained in ^[1-4] has the form

$$\left(\frac{S}{2} \oint \frac{dl}{B} + \oint \frac{\mathbf{j} \mathbf{B}}{|\nabla V|^2 B} dl \right)^2 - 2 \oint \frac{\mathbf{B}^2}{|\nabla V|^2 B} dl \oint \frac{[\mathbf{j} \nabla V] (\mathbf{B} \nabla) \nabla V}{|\nabla V|^4} \frac{dl}{B} > 0, \quad (16)$$

where the integration is along closed lines of force on rational magnetic surfaces. According to Eqs. (10) and (12) we can, as the last term in (12) vanishes when integrated along the line of force, write condition (16) in the form

$$\left(\frac{S}{2} \oint \frac{dl}{B} + \oint \frac{\mathbf{j} \mathbf{B}}{|\nabla V|^2 B} dl \right)^2 - \oint \frac{\mathbf{B}^2}{|\nabla V|^2 B} dl \left(\oint \frac{j^2}{|\nabla V|^2 B} dl - \Omega \oint \frac{dl}{B} \right) > 0. \quad (17)$$

In the expression obtained the invariant surface function Ω determining stabilization by a minimum \bar{B} is clearly separated off, and it is also clear that the positive square of the second term in the first bracket to a large extent compensates the first term in the second bracket in (17). Introducing the notation

$$\langle f \rangle_c = \oint f \frac{dl}{B} / \oint \frac{dl}{B}$$

we are led to the condition for local stability in the form

$$\frac{1}{4} S^2 + S \left\langle \frac{\mathbf{j} \mathbf{B}}{|\nabla V|^2} \right\rangle_c - \Omega \left\langle \frac{\mathbf{B}^2}{|\nabla V|^2} \right\rangle_c - \left(\left\langle \frac{j^2}{|\nabla V|^2} \right\rangle_c \left\langle \frac{\mathbf{B}^2}{|\nabla V|^2} \right\rangle_c - \left\langle \frac{\mathbf{j} \mathbf{B}}{|\nabla V|^2} \right\rangle_c^2 \right) > 0. \quad (18)$$

The criteria for convective stability (15) and local stability (18) were used earlier in the text of the present paper.

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