A REGGEON DIAGRAM TECHNIQUE

V. N. GRIBOV

A. F. Ioffe Physico-technical Institute, Academy of Sciences, U.S.S.R.

Submitted February 24, 1967

Zh. Eksp. Teor. Fiz. 53, 654–672 (August, 1967)

A diagram technique for the analysis of branch points in the complex angular momentum plane is developed from an investigation of the asymptotic behavior of Feynman graphs at high energies and fixed momentum transfer. In this method the reggeon is regarded as a nonrelativistic particle characterized by a two-dimensional momentum k. The pole trajectory $\alpha(k^2)$ plays the role of energy, and the angular momentum j plays the role of frequency. The situation arising in the description of the diffraction cone under the assumption of the existence of a Pomeranchuk pole is discussed.

1. INTRODUCTION

It has been shown by Mandelstam[1] and Polkinghorne[2] that under relativistic conditions, poles in the complex angular momentum plane j generate branch points. The branch points have been investigated in [3] assuming a definite structure of the many-particle unitary conditions for complex j. In formulating this hypothesis, the authors of [3] started from the idea that such a branch point is an analytic continuation to noninteger spins of a singularity at $j = \sigma_1 + \sigma_2 - 1$[4,5] for ordinary particles, and from a study of the unitarity condition for complex j on the basis of the simplest graphs of perturbation theory.[6] In [3], the character of the branch points was determined and a unitarity condition for reggeons was obtained which determines the discontinuities on the reggeon singularities corresponding to the formation of several reggeons in the intermediate state, in terms of reggeon production amplitudes. The reggeon unitarity condition was found to be very similar to the unitarity condition for ordinary particles. This gave an indication that it might be possible to work with reggeons as with ordinary particles.

In [7] an attempt was made to investigate the structure of the diffraction cone by taking account of poles and branch points. This study was based on the idea that the branch singularities are enhanced by a pole for values of j close to unity and small t, when the positions of the pole and the branch points almost coincide. Reggeon diagrams were introduced on the basis of the reggeon unitarity condition. However, an incorrect threshold behavior was used in the reggeon production amplitudes, and the result derived in [7] is wrong.

In the present paper we investigate the asymptotic behavior of a large class of Feynman graphs containing exact two-particle amplitudes (cf. below, Figs. 1, 4, 8, and 9). We shall find a connection between the asymptotic form of these amplitudes and the asymptotic form of the graph as a whole. In contrast to the work of Polkinghorne,[2] we do not make use of a representation of the amplitudes within the graph in the form of a ladder or any other explicit form.

It will be shown that if the asymptotic form of the internal amplitudes is determined by a Regge pole, then the asymptotic form of the whole graph contains a contribution from branch points having a form identical with that obtained in [3]. This result is of interest in that it is based on entirely different assumptions than the result of [3] and thus confirms the basic premises of [3]. Moreover, the derivation with the help of Feynman graphs allows one to establish the undetermined factors entering in the definition of the production amplitudes for reggeons, which heretofore prevented the determination of the sign of the discontinuities on the reggeon singularities. Another advantage of this approach is that the result can be written down immediately in the physical region (for negative momentum transfer t), whereas in [3] it was written down for positive t and had to be continued analytically in t.

In Sec. 4 we consider the case of most interest, where the pole and the branch points are close to one another (diffraction cone in arbitrary reactions). It is shown that in this case the pole enhances the contribution of the branch points (in agreement with [7]).

In Secs. 4 and 5 we develop a graph technique for the simultaneous treatment of poles and branch points. In these graphs the reggeon is described as a nonrelativistic particle characterized by a two-dimensional momentum k and an angular momentum j which plays the role of a frequency. The trajectory of the pole $\alpha(t)$ is the analog of the momentum dependence of the energy of the particle. This graph technique is used in the last section in a discussion of the situation arising in the description of the diffraction cone in elastic scattering under the assumption of the existence of a Pomeranchuk pole.

2. TWO-REGGEON BRANCH POINTS

Let us consider the graph shown in Fig. 1. For simplicity we assume that all particles are scalar and have equal masses. We calculate the asymptotic form of this graph for large $s = (p_1 + p_2)^2$ and fixed $t = q^2 = (p_1 - p_3)^2$ as a function of the asymptotic form of the amplitudes $f(k_1, k, k_2)$ and $f'(p_1 - k_1, q - k, p_2 - k_2)$, represented by the bubbles in Fig. 1. The four-momenta k_1, k_2, k_3, and k_4 are defined in the figure, and $k = k_1 - k_3 = k_4 - k_2$.

For the calculation of the asymptotic we use the method of Sudakov.[8] This method is extremely effective, as we shall see below. It consists in the following. The internal momenta k_1, k, and k_2 are decomposed

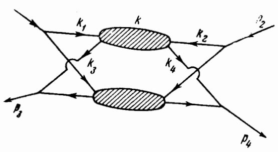


FIG. 1.

into vectors lying in the plane of the large vectors p_1 and p_2 and vectors perpendicular to that plane:

$$k = \alpha p_2' + \beta p_1' + k_\perp, \quad k_1 = \alpha_1 p_2' + \beta_1 p_1' + k_{1\perp}, \quad k_2 = \alpha_2 p_2' + \beta_2 p_1' + k_{2\perp};$$
$$p_1' = p_1 - \frac{m^2}{s} p_2, \quad p_2' = p_2 - \frac{m^2}{s} p_1; \tag{1}$$

$k_\perp$, $k_{1\perp}$, and $k_{2\perp}$ are space-like two-dimensional vectors. It turns out that the integration over the variables α_i and β_i in the plane of the vectors p_1 and p_2 can be carried out asymptotically, and the result has the form of integrals over the two-dimensional vectors in the plane perpendicular to the (p_1, p_2) plane. The fact that in a number of cases the asymptotic is determined by two-dimensional integrals was discovered by Polkinghorne.[2] This is explained by the circumstance that the momentum transfer $q = p_1 - p_3$, on which the asymptotic form of the graphs at high energies depends, is perpendicular to p_1 and p_2. It is easy to show that

$$q = \frac{q^2}{s}(p_2' - p_1') + q_\perp, \quad q^2 \approx q_\perp^2. \tag{2}$$

Following Sudakov, we have introduced in (1), not the vectors p_1 and p_2, but linear combinations of them, p_1' and p_2', which have the convenient property $p_1'^2 \approx p_2'^2 \approx 0$. In the new variables

$$d^4k = {}^1/_2 |s| \, d\alpha d\beta \, d^2k_\perp. \tag{3}$$

We now turn to the integration. We consider the left-hand part of the graph of Fig. 1, which contains an integration over k_1 involving the denominators

$$k_1^2 - m^2 + i\varepsilon = \alpha_1\beta_1 s + k_{1\perp}^2 - m^2 + i\varepsilon, \tag{4a}$$
$$(p_1 - k_1)^2 - m^2 + i\varepsilon = -\alpha_1 s - \beta_1 m^2 + \alpha_1\beta_1 s + k_{1\perp}^2 + i\varepsilon, \tag{4b}$$
$$(k - k_1)^2 - m^2 + i\varepsilon = (\alpha_1 - \alpha)(\beta_1 - \beta)s + (k_{1\perp} - k_\perp)^2 - m^2 + i\varepsilon, \tag{4c}$$
$$(p_1 - k_1 + k - q)^2 - m^2 + i\varepsilon = -(\alpha_1 - \alpha)s - (\beta_1 - \beta)m^2 + (\alpha_1 - \alpha)(\beta_1 - \beta)s + (k_{1\perp} - k_\perp + q_\perp)^2 - q^2(1 - \beta_1 + \beta). \tag{4d}$$

We assume that the amplitudes $f(k_1, k, k_2)$ and $f'(p_1 - k_1, q - k, p_2 - k_2)$ are large when their energy variables $s_1 = (k_1 + k_2)^2 \approx 2k_1k_2$ and $s_2 = (p_1 + p_2 - k_1 - k_2)^2 \approx 2(p_1 - k_1)\cdot(p_2 - k_2)$ are large, i.e., of order s, and the momentum transfers k^2, $(q - k)^2$ and the masses k_1^2, k_2^2, $(k_1 - k)^2$, $(k_2 + k)^2$, ... are of order unity. If any of the last variables becomes of order s the amplitude becomes small and the corresponding region is unimportant in the integral. Under these assumptions it follows from (4) that the region of interest for us must be

$$k_{1\perp}^2 \sim m^2, \quad k_\perp^2 \sim m^2, \quad \alpha_1 \sim m^2/s, \quad \alpha \sim m^2/s, \quad \beta_1 \lesssim 1, \quad \beta \lesssim 1. \tag{5}$$

If we write down the formulas analogous to (4) for the right-hand side of the graph containing the integral over k_2, we find

$$k_{2\perp}^2 \sim m^2, \quad k_\perp^2 \sim m^2, \quad \beta_2 \sim m^2/s, \quad \beta \sim m^2/s, \quad \alpha_2 \lesssim 1, \quad \alpha \lesssim 1. \tag{6}$$

We have used the condition that the variables corresponding to masses and momentum transfers are of order unity (of the order of the mass). The condition that the energy variables be large gives

$$2k_1k_2 \approx \beta_1\alpha_2 s, \quad 2(p_1 - k_1, \ p_2 - k_2) \approx (1 - \beta_1)(1 - \alpha_2)s, \quad \beta_1 \sim 1, \quad \alpha_2 \sim 1. \tag{7}$$

Thus

$$k_\perp^2 \sim k_{1\perp}^2 \sim k_{2\perp}^2 \sim m^2, \quad \alpha \sim m^2/s, \quad \beta \sim m^2/s, \quad \alpha_1 \sim m^2/s, \quad \beta_2 \sim m^2/s, \quad \beta_1 \sim 1, \quad \alpha_2 \sim 1.$$

Using $\beta \ll \beta_1$, we may neglect β in (4) and the quantity α in the analogous equations for the right-hand part of the graph.

We now assume that the amplitudes $f(k_1, k, k_2)$ and $f'(p_1 - k_1, q - k, p_2 - k_2)$ factorize under asymptotic conditions, i.e., have the form

$$f(k_1, k, k_2) = g_1(k_1^2, (k - k_1)^2, k^2) \cdot g_2(k_2^2, (k + k_2)^2, k^2) G(k^2, 2k_1k_2), \tag{8a}$$

$$f'(p_1 - k_1, q - k, p_2 - k_2) = g_1'((p_1 - k_1)^2, (p_1 - k_1 - q + k)^2, (q - k)^2) \times g_2'((p_2 - k_2)^2, (p_2 - k_2 + q - k)^2, (q - k)^2) \times G'((q - k)^2, 2(p_1 - k_1, p_2 - k_2)). \tag{8b}$$

The relations (8) are in any case true if the asymptotic form of the amplitudes f and f' are determined by a Regge pole. We shall see below that they are also effectively true when the asymptotic is determined by the branch points under consideration.

We write the functions $G(k^2, 2k_1k_2)$ and

$$G'((q - k)^2, 2(p_1 - k_1, p_2 - k_2))$$

in the form of a Sommerfeld-Watson integral:

$$G(k^2, 2k_1k_2) = -\int \frac{dl_1}{4i} \xi_{l_1} G_{l_1}(k^2)(2k_1k_2)^{l_1} = -\int \frac{dl_1}{4i} \xi_{l_1} G_{l_1}(k^2)(\alpha_2\beta_1 s)^{l_1}, \tag{9a}$$

$$G'((q - k)^2, 2(p_1 - k_1, p_2 - k_2)) = -\int \frac{dl_2}{4i} \xi_{l_2} G_{l_2}'((q - k)^2)[(1 - \alpha_2)(1 - \beta_1)s]^{l_2}; \tag{9b}$$

ξ_{l_1} and ξ_{l_2} are factors defining the signature. Substituting (8) and (9) in the Feynman integrals corresponding to the graph of Fig. 1 and using

$$k^2 = \alpha\beta s + k_\perp^2 \approx k_\perp^2, \quad (q - k)^2 \approx (q_\perp - k_\perp)^2,$$

we see that the integrations over α_1, β_1, α, $k_{1\perp}$ and $k_{2\perp}$, α_2, β_2, β can be carried out separately. The result of the integrations can be written in the form

$$F(s, q^2) = \frac{i\pi}{2|s|} \int \frac{dl_1}{2i\pi} \int \frac{dl_2}{2i\pi} \xi_{l_1}\xi_{l_2} \times \int \frac{d^2k_\perp}{(2\pi)^2} N_{l_1l_2}^2(q, k_\perp) G_{l_1}(k_\perp^2) G_{l_2}'((q - k_\perp)^2) s^{l_1 + l_2}, \tag{10}$$

$$N_{l_1l_2}(q, k_\perp) = \frac{1}{4(4\pi)^{1/2}} \int \frac{d^2k_{1\perp} d\beta_1 d\alpha_1 \, d\alpha}{(2\pi)^4} s^2\lambda^2 g_1 g_1' \beta_1^{l_1}(1 - \beta_1)^{l_2} \times \{(k_1^2 - m^2)[(p_1 - k_1)^2 - m^2][(k_1 - k)^2 - m^2] \times [(p_1 - k_1 + k - q)^2 - m^2]\}^{-1}; \tag{11}$$

λ is a coupling constant corresponding to the vertices in Fig. 1.

Before investigating (10), we consider the quantity $N_{l_1 l_2}$ and show that it is correctly determined by (11) and is independent of s for $s \to \infty$.

We note first of all that in the integration over α_1, the singularities of all four denominators in (11) lie on different sides of the real axis only if $0 < \beta_1 < 1$. [This is easily seen from (4).] Thus, if we neglect the singularities of g_1 and g_1', the integral over α_1 differs from zero only if $0 < \beta_1 < 1$. The singularities of g_1 and g_1' do not alter this result, since g_1 and g_1' depend on the same quantities

$$k_1^2,\ (k_1-k)^2,\ (p_1-k_1)^2,\ (p_1-k_1+k-q)^2,$$

as the denominators in (11), and the rules for passing the singularities are the same as for these denominators. Hence the quantities $\beta_1^{l_1}$ and $(1-\beta_1)^{l_2}$ are uniquely defined.

We show now that the contribution of the considered region of small α_1 also remains nonvanishing in the integration over α. If it is to remain different from zero, one must require that after integration over α_1 the integrand has singularities in α on both sides of the real axis. It is easily seen from (4) that the singularities of the integral over α stemming from the "pinching" of the integration contour between two singularities in the α_1 plane [(4a) and (4d) or (4b) and (4c)] are indeed located on different sides of the real axis.

Thus the contribution of the region of small α is nonzero even without account of the singularities of g_1 and g_1'. We note that in this connection it was essential that the parts of the graph of Fig. 1 to the right and left of the bubbles, which determine $N_{l_1 l_2}$, contain intersecting lines. If they did not do so, the singularities in α would lie on one and the same side of the real axis and the contribution of the region of small α would be zero. This corresponds to the circumstance that branch points occur only in graphs having a third spectral function.[1-3] We note that $N_{l_1 l_2}$ is real for $t = q^2$ below the thresholds.

We now turn to the consideration of expression (10) for the scattering amplitude. It follows from (10) that the signature of the amplitude $F(s, q^2)$ is determined by the product of the signatures of the amplitudes f and f′, since the quantity $|s|$ arising from the Jacobian transformation does not change sign when s is replaced by $-s$. If $F(s, q^2)$ is written the form of a Sommerfeld-Watson integral,

$$F(s,q^2) = -\int \frac{dj}{4i}\,\xi_j f_j(q^2)\, s^j, \tag{12}$$

then $f_j(q^2)$ is determined by an integral over the absorptive part,[9] which has the following form near the singularity:

$$f_j(q^2) = \frac{2}{\pi}\int\limits_{s_0}^{\infty} ds'\,(s')^{-(j+1)} F_1(s', q^2). \tag{13}$$

To obtain $F_1(s, q^2)$ we use the explicit form of ξ_{l_1} and ξ_{l_2}:

$$\xi_l = \frac{1}{\zeta_l}\exp\left\{-i\frac{\pi}{2}\left(l+\frac{1-P}{2}\right)\right\},\qquad \zeta_l = \begin{cases}\sin(l\pi/2), & P=1\\ \cos(l\pi/2), & P=-1\end{cases} \tag{14}$$

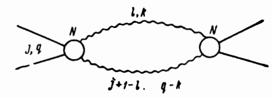


FIG. 2.

(P is the signature; P = 1 for an amplitude with positive signature, and P = −1 for an amplitude with negative signature). From this we find for F_1, the imaginary part of the amplitude for positive s,

$$F_1(s,q^2) = \frac{\pi}{2}\int\frac{dl_1}{2i\pi}\int\frac{dl_2}{2i\pi}\gamma_{l_1l_2}\int\frac{d^2k}{(2\pi)^2}N^2_{l_1l_2}(k,q)\,G_{l_1}(k_1^2)\,G_{l_2}((q-k)^2)\,s^{l_1+l_2-1}, \tag{15}$$

$$\gamma_{l_1l_2} = \operatorname{Re}\xi_{l_1}\xi_{l_2} = \frac{1}{\zeta_{l_1}\zeta_{l_2}}\cos\left[\frac{\pi}{2}\left(l_1+l_2+1-\frac{P_1+P_2}{2}\right)\right]. \tag{16}$$

Substituting (15) in (13), we obtain

$$f_j(q^2) = \int\frac{dl_1}{2i\pi}\int\frac{dl_2}{2i\pi}\gamma_{l_1l_2}\int N^2_{l_1l_2}\frac{G_{l_1}(k^2)\,G_{l_2}'((q-k)^2)}{j+1-l_1-l_2}\frac{d^2k}{(2\pi)^2}. \tag{17}$$

If

$$G_{l_1}(k^2) = \frac{1}{l_1-\alpha(k^2)},\qquad G_{l_2}'((q-k)^2) = \frac{1}{l_2-\beta((k-q)^2)}, \tag{18}$$

then

$$f_j(q^2) = \int\frac{d^2k}{(2\pi)^2}\frac{\gamma_{\alpha\beta}N_{\alpha\beta}^2}{j+1-\alpha(k^2)-\beta((k-q)^2)}. \tag{19}$$

Expression (19) agrees with the result found in [3]. In [3] the factor $\gamma_{\alpha\beta}$ was included in the reggeon production amplitude.

Formula (17) can be rewritten in a somewhat different form if the integration is done, e.g., over l_2. If we recall that the integration contour of integration in the Sommerfeld-Watson integral (9) passes to the right of the singularities of G_{l_1} or G_{l_2}' and (13) holds for Re j > Re $(l_1 + l_2 - 1)$, then it is clear that the pole $j + 1 - l_1 - l_2 = 0$ in (17) lies to the right of the contour of integration. Calculating the residue of this pole, we find

$$f_j(q^2) = \int\frac{dl_1}{2i\pi}\int\frac{d^2k}{(2\pi)^2}G_{l_1}(k^2)\,G'_{j+1-l_1}((q-k)^2)\,N^2_{l_1,j+1-l_1}\gamma_{l_1,j+1-l_1}. \tag{20}$$

The contour of integration in (20) lies to the right of the singularities of G and to the left of the singularities of G′.

The expression (20) corresponds to the Feynman graph shown in Fig. 2. The wavy lines represent the reggeon Green's functions $G_{l_1}(k_1^2)$ and $G_{l_2}'(k_2^2)$; the momentum $q = k_1 + k_2$ is conserved in the vertices; the quantity j + 1 corresponds to the energy and is conserved in the vertices: $j - 1 = l_1 - 1 + l_2 - 1$. It follows from the expression (16) for $\gamma_{l_1 l_2}$ that the denominators ζ_{l_1} and ζ_{l_2} can be included in the Green's functions and the numerators $\sin(\pi j/2)$ or $\cos(\pi j/2)$ can be included in N or can be regarded as common factors reflecting the fact that the branch points do not make a contribution for physical values of the angular momentum. We can therefore write

$$f_j(q^2) = -\sin\left[\frac{\pi}{2}\left(j+1-\frac{P_1+P_2}{2}\right)\right]\int\frac{dl_1}{2i\pi}\int\frac{d^2k_1}{(2\pi)^2}\times g_{l_1}(k_1^2)\,g'_{j+1-l_1}((q-k_1)^2)\,N^2_{l_1,j+1-l_1},\qquad g_l(k^2) = G_l(k^2)/\zeta_l. \tag{21}$$

All the results above were obtained from a study of the simplest graph, Fig. 1. It seems reasonable to us to assume, taking into account the analysis of [3, 6], that the inclusion of other graphs leads only to a change in the expression for N_{l_1, l_2}.

3. MANY-REGGEON BRANCH POINTS

The results obtained in the previous section allow us to find the contribution of many-reggeon branch points. Let us assume that the amplitude

$$f'(p_1 - k_1, q - k, p_2 - k_2)$$

has the previous form (8b), (9b), whereas the amplitude $f(k_1, k, k_2)$ is determined by a two-reggeon branch point and has the form

$$f = \frac{i\pi}{2|s|} \int \frac{dl_1}{2i\pi} \int \frac{dl_2}{2i\pi} \int \frac{d^2k_\perp'}{(2\pi)^2} N_{l_1 l_2}(k_1^2, (k_1 - k)^2, k_\perp, k_\perp') \tag{22}$$
$$\cdot G_{l_1}(k_\perp'^2) G_{l_2}'((k_\perp - k_\perp')^2) N_{l_1 l_2}(k_2^2, (k_2 + k)^2, k_\perp, k_\perp')(\alpha_2 \beta_1 s)^{l_1 + l_2} \xi_{l_1} \xi_{l_2}$$

(here G' and G refer to different reggeons). Expression (22) differs from (10) in that a product of production amplitudes appears instead of a square, since the squares of the masses of the virtual particles are different in the integration, and q is replaced by k, k by k′, and s by $2k_1k_2 = \alpha_2\beta_1 s$. The generalization of (10) to the case of unequal masses, used in writing down (22), is trivial.

Substituting (22) and (8b), (9b) in the integral corresponding to the graph of Fig. 1 and interchanging the order of integration we obtain

$$F(s, q^2) = \frac{\pi}{2} \int \frac{dl_1}{2i\pi} \int \frac{dl_2}{2i\pi} \int \frac{dl_3}{2i\pi} \xi_{l_1} \xi_{l_2} \xi_{l_3} \int \frac{d^2k_\perp}{(2\pi)^2} \int \frac{d^2k_\perp'}{(2\pi)^2} \tag{23}$$
$$\times N^2_{l_1 l_2 l_3}(q, k_\perp, k_\perp') G_{l_1}(k_\perp'^2) G_{l_2}'((k_\perp - k_\perp')^2) G_{l_3}''((q - k_\perp)^2) s^{l_1 + l_2 + l_3 - 2}.$$

In substituting (9b) in the integral we have replaced l_2 by l_3. The expression for $N_{l_1 l_2 l_3}$ is obtained from (11) by replacing g_1 by $N_{l_1 l_2}$, l_1 by $l_1 + l_2 - 1$, and l_2 by l_3. It is clear that the signature of $F(s, q^2)$ is given by the product of the signatures of G_{l_1}, G'_{l_2}, and G''_{l_3}.

Calculating the partial wave amplitude $f_j(q^2)$ in analogy to our previous procedure, we obtain

$$f_j(q^2) = \int dl_1\, dl_2\, dl_3\, d^2k_1\, d^2k_2 \gamma_{l_1 l_2 l_3} N^2_{l_1 l_2 l_3} \cdot$$
$$\times \frac{G_{l_1}(k_1^2) G_{l_2}'(k_2^2) G_{l_3}''((q - k_1 - k_2)^2)}{j + 2 - l_1 - l_2 - l_3},$$
$$\gamma_{l_1 l_2 l_3} = \operatorname{Im} \xi_{l_1} \xi_{l_2} \xi_{l_3} = -\frac{1}{\zeta_{l_1} \zeta_{l_2} \zeta_{l_3}} \tag{24}$$
$$\times \sin\left[\frac{\pi}{2}\left(l_1 + l_2 + l_3 + \frac{3}{2} - \frac{P_1 + P_2 + P_3}{2}\right)\right].$$

Here and in the following the following substitutions are implied:

$$dl \to \frac{dl}{2i\pi}, \quad d^2k \to \frac{d^2k}{(2\pi)^2}.$$

Performing the integration over l_3, we have

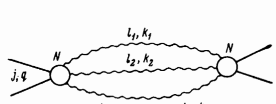


FIG. 3

$$f_j(q^2) = \sin\left[\frac{\pi}{2}\left(j - \sum_{i=1}^{3} \frac{P_i - 1}{2}\right)\right] \int dl_1\, dl_2\, d^2k_1\, d^2k_2 N^2_{l_1, l_2, j+2-l_1-l_2} \tag{25}$$
$$\times g_{l_1}(k_1^2) g_{l_2}(k_2^2) g_{j+2-l_1-l_2}((q - k_1 - k_2)^2).$$

This expression corresponds to the graph of Fig. 3 with the same conservation laws in the vertices as before.

Analogously, one can obtain the contribution of the branch point corresponding to n reggeons:

$$f_j(q^2) = (-1)^{n-1} i (2\pi)^3 \sin\left[\frac{\pi}{2}\left(j - \sum_{i=1}^{n} \frac{P_i - 1}{2}\right)\right] \int dl_1 \ldots dl_n\, d^2k_1 \ldots d^2k_n$$
$$\times \delta\left(q - \sum k_i\right) \delta\left(j + n - 1 - \sum l_i\right) N^2_{l_1 \ldots l_n}(k_1 \ldots k_n) \tag{26}$$
$$\times g_{l_1}(k_1^2) g_{l_2}(k_2^2) \ldots g_{l_n}(k_n^2).$$

If $g_{l_i}(k_i^2) = \zeta_{l_i}^{-1}[l_i - \alpha_i(k_i^2)]^{-1}$, then (26) leads to

$$f_j(q^2) = (-1)^{n-1} \int d^2k_1 \ldots d^2k_n N^2_{\alpha_1 \ldots \alpha_n} (2\pi)^2 \delta\left(q - \sum k_i\right)$$
$$\times \left[j + n - 1 - \sum_{i=1}^{n} \alpha_i(k_i^2)\right]^{-1} \sin\left[\frac{\pi}{2}\left(j - \sum_{i=1}^{n} \frac{P_i - 1}{2}\right)\right] \prod_i \frac{1}{\zeta_i}. \tag{27}$$

We verify that (27) gives a result which coincides with the one obtained in [3]. In the physical region of the s channel ($t < 0$) we have, according to Froissart,[10] $\operatorname{Re} \alpha(t) \ll 1$. Therefore $\operatorname{Re} \sum \alpha_i(k_i^2) \le n$, and the real part of the denominator in (27) is smaller than or equal to $1 - j$. This means that the singularity of $f_j(q^2)$ lies in the region $\operatorname{Re} j \le 1$. We assume that all α are real in the physical region of the s channel. Then the singularity of $f_j(q^2)$ will occur for real j, and the discontinuity on the singularity is equal to

$$\Delta f_j = (-1)^n \pi \sin\left[\frac{\pi}{2}\left(j - \sum_i \frac{P_i - 1}{2}\right)\right] \int d^2k_1 \ldots d^2k_n (2\pi)^2 \delta\left(q - \sum k_i\right)$$
$$\times \delta\left(j + n - 1 - \sum \alpha_i(k_i^2)\right) N^2_{\alpha_1 \ldots \alpha_n} \Big/ \prod_i \zeta_{\alpha_i}.$$

If the poles are equal, $\alpha_i(k_i^2) = \alpha(k_i^2)$, this expression is easily calculated near the singularity. The position of the singularity is determined by the minimal value of $\sum \alpha(k_i^2)$, which is attained for $\mathbf{k}_i = \mathbf{q}/n$ and equals $j_n = n\alpha(q^2/n^2) - n + 1$. For j close to j_n the region of integration is concentrated around the point $\mathbf{k}_i = \mathbf{q}/n$. Expanding $\alpha(k_i^2)$ in powers of $\mathbf{x}_i = \mathbf{k} - \mathbf{q}/n$ and choosing the 1 axis in the two-dimensional space along the direction of **q**, we obtain

$$\Delta f_j(q^2) = (-1)^n \pi \sin\left[\frac{\pi}{2}\left(j - \frac{P - 1}{2} n\right)\right] \int d^2x_1 \ldots d^2x_n (2\pi)^2 \delta\left(\sum x_i\right)$$
$$\times \delta\left(j - j_n + \left(\alpha' + \frac{2q^2}{n^2}\alpha''\right) \sum x_{1i}^2 + \alpha' \sum x_{2i}^2\right) N^2 (\zeta_\alpha)^{-n}$$

$[\alpha' = d\alpha(x)/dx|_{x = q^2/n^2}]$. Introducing the variables

$$z_{1i} = x_{1i}\left(\alpha' + \frac{2q^2}{n^2}\alpha''\right)^{1/2} / (j_n - j), \quad z_{2i} = x_{2i} \frac{\sqrt{\alpha'}}{j_n - j}$$

and taking $N^2(\zeta_\alpha)^{-n}$ outside the integral sign, we have

$$\Delta f_j = (j - j_n)^{n-2} \pi \sin\left[\frac{\pi}{2}\left(j - \frac{P - 1}{2} n\right)\right] B_n N^2_{\alpha \ldots \alpha} (\zeta_\alpha)^{-n}$$
$$\times \left[\alpha'\left(\alpha' + \frac{2q^2}{n^2}\alpha''\right)\right]^{-(n-1)/2},$$
$$B_n = \int d^2z_1 \ldots d^2z_n (2\pi)^2 \delta\left(\sum z_i\right) \delta\left(1 + \sum z_i^2\right) = \frac{n - 1}{(4\pi)^{n-1} n!}, \tag{28}$$
$$\alpha' \equiv \alpha'(q^2/n^2), \quad \alpha'' \equiv \alpha''(q^2/n^2).$$

4. ENHANCEMENT

Let us now investigate how, for example, a pole and a branch point influence each other when they are close to one another. To this end we consider the graph of Fig. 4b, which can contain poles as well as branch points. It is not essential to consider the graph of Fig. 4b; one may at once consider the graph of Fig. 4a, of which the graph of Fig. 4b is a special case (the amplitude f_2 is determined by the branch points).

Setting, in analogy to the foregoing,

$$k_3 = \alpha_3 p_2' + \beta_3 p_1' + k_{3\perp},$$

we analyze the character of the integration over $k_{3\perp}$, α_3, and β_3 in the graph of Fig. 4a. We have

$$k_3^2 = \alpha_3\beta_3 s + k_{3\perp}^2, \quad (q-k_3)^2 = \alpha_3\beta_3 s + (q-k_{3\perp})^2; \tag{29a}$$

$$(p_1 - k_3)^2 = -\alpha_3 s - \beta_3 m^2 + k_{3\perp}^2 + \alpha_3\beta_3 s + m^2,$$
$$(p_2 + k_3)^2 = \beta_3 s + \alpha_3 m^2 + \alpha_3\beta_3 s + k_{3\perp}^2 + m^2. \tag{29b}$$

From the condition that the masses of the virtual particles k_3^2, $(q-k_3)^2$ in the amplitudes $f_1(p_1, k_3, q)$ and $f_2(p_2, k_3, q)$ are of order m^2, we obtain

$$k_{3\perp}^2 \sim m^2, \quad \alpha_3\beta_3 s \sim m^2. \tag{30}$$

It is seen from (29) and (30) that the energy variables in these amplitudes, $s_1 = (p_1 - k_3)^2$ and $s_2 = (p_2 + k_3)^2$, cannot, in contrast to the previous situation, both be of order s. If $\alpha_3 \sim 1$, then $s_1 \sim s$, but then $\beta_3 \sim m^2/s$ and $s_2 \sim m^2$; if $\beta_3 \sim 1$, then $s_2 \sim s$, but $s_1 \sim m^2$. Each of these regions gives a contribution, which is determined by the magnitude of the amplitudes f_1 and f_2. If $f_1 \sim s_1^{\alpha(q^2)}$ and $f_2 \sim s_2^{\beta(q^2)}$, the region $s_1 \sim s$, $s_2 \sim m^2$ will be most important in the integral for $\alpha(q^2) > \beta(q^2)$, and the region $s_1 \sim m^2$, $s_2 \sim s$ will be most important for $\alpha(q^2) < \beta(q^2)$. The asymptotic form of the amplitude F corresponding to the graph of Fig. 4a will be $s^{\alpha(q^2)}$ for $\alpha(q^2) > \beta(q^2)$ and $s^{\beta(q^2)}$ for $\alpha(q^2) < \beta(q^2)$. The integral over $k_{3\perp}$ will in these cases determine either the contribution to the residue of the amplitude F, if the asymptotic of f_1 (f_2) is determined by a pole, or the reggeon production amplitudes, if the asymptotic of f_1((f_2) is determined by a branch point.

The case of most interest is that where $\alpha(q^2)$ and $\beta(q^2)$ are close to one another (pole and branch point are close). In this case the maximal contribution will come from the region

$$m^2/s \ll |\alpha_3| \ll 1, \quad m^2/s \ll |\beta_3| \ll 1. \tag{31}$$

Calculating the integral over this region, we determine the effect of the branch singularity and the pole on one another.

We write the amplitudes f_1 and f_2 in factorized form:

$$f_1(p_1, k_3, q) = -g_1(q^2) g_1'(k_3^2, (q-k_3)^2, q^2) \int \frac{dl}{4i} \xi_l G_l^{(1)}(q^2) s_1^l, \tag{32a}$$

$$f_2(p_2, k_3, q) = -g_2(q^2) g_1''(k_3^2, (q-k_3)^2, q^2) \int \frac{dl'}{4i} \xi_{l'} G_{l'}^{(2)}(q^2) s_2^{l'}. \tag{32b}$$

Using (31), we write instead of (29b)

$$s_1 = -\alpha_3 s, \quad s_2 = \beta_3 s. \tag{33}$$

In the integration over α_3 the singularities $k_3^2 - m^2 + i\varepsilon = 0$, $(q-k_3)^2 - m^2 + i\varepsilon = 0$ and the singularities of g_1' and g_1'' have the location $\alpha_3 = (c - i\varepsilon)/\beta_3 s$ (c is real). The singularities of f_1 corresponding to its right-hand cut are located at $\alpha_3 = (c' + i\varepsilon)/s$, and the ones corresponding to the left-hand cut at $\alpha_3 = (c'' - i\varepsilon)/s$. It follows from this that for $\beta_3 > 0$, only the right-hand cut of f_1 gives a contribution, and for $\beta_3 < 0$, only the left-hand cut does. In both cases the contour may be closed around the singularity of f_1 and an integral over Im f_1 is obtained. If the amplitudes f_1 and f_2 have different signature, the contributions of the two cuts cancel. For amplitudes with equal signature they add. Taking this into account, we obtain the contribution corresponding to the graph of Fig. 4a:

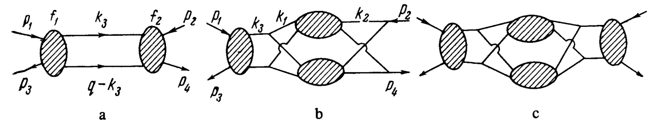


FIG. 4.

$$F(s, q^2) = -g_1(q^2) g_2(q^2) \int \frac{dl}{4i} \int \frac{dl'}{4i} \xi_{l'} \int_{-1}^{-m^2/s} d\alpha_3 \int_{m^2/s}^{1} d\beta_3$$
$$\times (-\alpha_3 s)^l (\beta_3 s)^{l'} G_l^{(1)}(q^2) G_{l'}^{(2)}(q^2) \tag{34}$$
$$\times 2|s| \int \frac{d^2k_3}{(2\pi)^4} \frac{g_1'(k_3^2, (q-k_3)^2, q^2) g_1''(k_3^2, (q-k_3)^2, q^2)}{(k_3^2 - m^2)[(q-k_3)^2 - m^2]}.$$

Since the last integral in (34) depends only on $\alpha_3\beta_3 s$ and $k_{3\perp}^2$, we can introduce the variable $x = -\alpha_3\beta_3 s$ and can perform the integration, e.g., over β_3. For $s \to \infty$, $l \to l'$, we have

$$s^{l'} \int_{m^2/s}^{1} \frac{d\beta_3}{\beta_3} \beta_3^{l'} \left(\frac{x}{\beta_3}\right)^l = \frac{x^l}{l'-l}(s^{l'} - s^l), \tag{35}$$

and we find

$$F(s, q^2) = -\frac{\pi}{2} \int \frac{dl}{2i\pi} \int \frac{dl'}{2i\pi} \xi_{l'} \frac{s^{l'} - s^l}{l'-l} G_l^{(1)}(q^2) G_{l'}^{(2)}(q^2) g_1(q^2) r_l(q^2) g_2(q^2), \tag{36}$$

$$r_l(q^2) = \frac{1}{2} \int \frac{d^2k_3}{(2\pi)^3} dx \frac{g_1'(k_3^2, (q-k_3)^2, q^2) g_1''(k_3^2, (q-k_3)^2, q^2)}{(k_3^2 - m^2)[(q-k_3)^2 - m^2]} x^l. \tag{37}$$

Calculating the partial wave $f_j(q^2)$ with the help of (36), we obtain

$$f_j(q^2) = \int \frac{dl}{2i\pi} \int \frac{dl'}{2i\pi} \frac{G_l^{(1)}(q^2) G_{l'}^{(2)}(q^2) r_j}{(j-l)(j-l')} g_1 g_2$$
$$= g_1(q^2) G_j^{(1)}(q^2) r_j(q^2) G_j^{(2)}(q^2) g_2(q^2). \tag{38}$$

The expression (38) can be represented by the reggeon graph of Fig. 5a which has the character of a correction to the poles $\alpha(q^2)$ and $\beta(q^2)$ owing to the transition from the state α to the state β.

The formulas thus obtained are immediately carried over to the case where a pole and a branch point are present. If we substitute (10) [instead of (32b)] in (34), replace there $N_{l_1 l_2}^2(q, k_\perp)$ by $N_{l_1 l_2}(k_3^2, (k_3 - q)^2, q, k_\perp) \times N_{l_1 l_2}(k_\perp, q)$, and interchange the order of integration, then we obtain instead of (36) and (38)

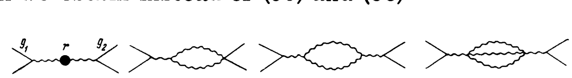


FIG. 5.

$$F(s, q^2) = i\frac{\pi}{2} g_1 \int \frac{dl}{2i\pi} \int \frac{dl_1}{2i\pi} \int \frac{dl_2}{2i\pi} \xi_{l_1}\xi_{l_2} \int \frac{d^2k}{(2\pi)^2} G_l(q^2) r_{l,\, l_1 l_2} \cdot G_{l_1}'(k^2) G_{l_2}''((q-k)^2) N_{l_1 l_2}(q, k) \frac{1}{l_1 + l_2 - l - 1} (s^{l_1+l_2-1} - s^l), \quad (39)$$

$$r_{l,\, l_1 l_2} = \frac{1}{2} \int \frac{d^2k_3}{(2\pi)^3} dx \frac{g_1'(k_3, q) N_{l_1 l_2}(k_3, k, q)}{(k_3^2 - m^2)[(q-k_3)^2 - m^2]} x^l, \quad (40)$$

or, going over to the partial wave,

$$f_j(q^2) = g_1 G_j(q^2) \int \frac{dl_1}{2i\pi} \int \frac{d^2k}{(2\pi)^2} \gamma_{l_1 l_2} r_{j, l_1 l_2} G_{l_1}'(k^2) G_{l_2}''((q-k)^2) N_{l_1 l_2}(q, k), \quad (41)$$

where $l_2 = j + 1 - l$. The expression (41) evidently corresponds to the graph of Fig. 5b.

We may now go further and determine the contribution of the graph of Fig. 4c. If, instead of (32a), we substitute in the expression for the graph of Fig. 4a an amplitude of the form (39) and (40), we obtain

$$f_j(q^2) = g_1 G_j(q^2) \int \frac{dl_1}{2i\pi} \int \frac{d^2k}{(2\pi)^2} r_{j, l_1 l_2} G_{l_1}'(k^2) G_{l_2}''((q-k)^2) r_{j, l_1 l_2} G_j(q^2) g_2, \quad (42)$$

where $l_2 = j + 1 - l_1$, this corresponds to the graph of Fig. 5c.

It is easy to obtain also the contribution of an enhanced three-reggeon branch point corresponding to the graph of Fig. 5d.

Of great interest for the investigation of the diffraction cone for small t, where the pole and the branch point are close together, are the graphs of the type of Fig. 6. These graphs depict the enhancement of branch points by other branch points. It is easy to draw Feynman graphs corresponding to these reggeon graphs and to calculate their contributions. Repeating almost literally the derivation above we obtain, e.g., for the graph of Fig. 6a:

$$f_j(q^2) = g_1 G_j(q^2) \int dl_1 d^2k_1 \int dl_3 d^2k_3 r_{l_1 l_2} \gamma_{l_1 l_2} G_{l_2}((q-k_1)^2) G_{l_1}(k_1^2) \times r_{l_3 l_4} \gamma_{l_3 l_4} G_{l_3}(k_3^2) G_{l_4}((k_1 - k_3)^2) r_{l_3 l_4} G_{l_1}(k_1^2) r_{l_1 l_2} G_j(q^2) g_2,$$

$$dl \to \frac{dl}{2i\pi}, \quad d^2k \to \frac{d^2k}{(2\pi)^2} \quad l_2 = j + 1 - l_1, \quad l_4 = l_1 + 1 - l_3. \quad (43)$$

Graphs of the type shown in Fig. 6 may be called completely enhanced, since they do not contain bubbles which can be enhanced by introducing a pole. By their structure, they represent the contribution to the Green's function of the reggeon caused by the possibility of decay of the reggeon into several others. However, these graphs do not contain the contribution to the Green's function from the change in the vertex part due to the contribution of the branch points.

The problem of the vertex part will be discussed in detail below. For the present we formulate the rules of calculating graphs without vertex parts (of the type of Fig. 6), which follow from the preceding considerations.

Each reggeon line corresponds to a Green's function $G_l(k^2)/(2\pi)^3 i$ which depends on the angular momentum l and the square of the two-dimensional space-like momentum k. All lines have the same direction, cf. Fig. 7. A conservation law of the form $(2\pi)^3 i\delta(l_3 + 1 - l_1 - l_2) \times \delta(k_3 - k_1 - k_2)$ holds in each vertex. The contour of integration over l passes around the singularities of $G_l(k)$ counterclockwise. The singularities of the other quantities depending on l lie outside the contour after the conservation laws are taken into account. Each vertex represents an amplitude for the transformation of a single reggeon into two others, $r_{l_1 l_2}$; vertices of the type of Fig. 7a introduce, moreover, a factor $\gamma_{l_1 l_2}$. The incompletely enhanced graphs considered above satisfy the same rules with $r_{l_1 l_2}$ replaced by $N_{l_1 l_2}$, $N_{l_1 l_2 l_3}$, etc., and $\gamma_{l_1 l_2}$ replaced by $\gamma_{l_1 l_2}$, $\gamma_{l_1 l_2 l_3}$, ….

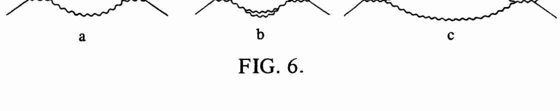


FIG. 6.

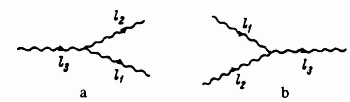


FIG. 7.

5. MORE COMPLICATED GRAPHS

Let us consider more complicated Feynman graphs containing the contribution of several close-lying points. An example for such a graph is that shown in Fig. 8a. In the Appendix we calculate the asymptotic form of the graph of Fig. 8a under the condition that the amplitudes contained in it have the form (8) and (9) with the same assumptions as before, and under the condition that all three branch points of this graph are close to one another.

The answer for the partial wave $f_j(q^2)$ can be written in the form of a sum of two terms, $f_j(q^2) = f_j'(q^2) + f_j''(q^2)$, corresponding to the graphs of Fig. 8b and c, where

$$f_j'(q^2) = \int dl_1 \ldots dl_5 \int d^2k_1 d^2k_2 N_{l_1 l_3} \gamma_{l_1 l_2} r_{l_2 l_5} \gamma_{l_2 l_5} r_{l_1 l_5} N_{l_2 l_4} \times \frac{G_{l_1}^{(1)}(k_1^2) G_{l_3}^{(3)}((q-k_1)^2) G_{l_2}^{(2)}(k_2^2) G_{l_4}^{(4)}((q-k_2)^2) G_{l_5}^{(5)}((q-k_1-k_2)^2)}{(j+1-l_1-l_3)(j+2-l_1-l_2-l_5)(j+1-l_2-l_4)}, \quad (44a)$$

$$f_j''(q^2) = \int dl_1 \ldots dl_5 \int d^2k_3 d^2k_4 N_{l_1 l_3} \gamma_{l_1 l_3} r_{l_4 l_5} \gamma_{l_4 l_5} r_{l_3 l_5} N_{l_2 l_4} \times \frac{G_{l_1}^{(1)}((q-k_3)^2) G_{l_3}^{(3)}(k_3^2) G_{l_4}^{(4)}(k_4^2) G_{l_2}^{(2)}((q-k_4)^2) G_{l_5}^{(5)}((q-k_3-k_4)^2)}{(j+1-l_1-l_3)(j+2-l_3-l_4-l_5)(j+1-l_2-l_4)}, \quad (44b)$$

(all quantities in these expressions have been defined earlier).

The denominators in (44a) and (44b) correspond to the energy denominators of nonrelativistic perturbation theory with different intermediate states. In contrast to the graphs considered earlier, the graph of Fig. 8a has two types of intermediate states (two three-reggeon branch points), leading to two contributions: $f_j'(q^2)$ and $f_j''(q^2)$.

In analogy to the graph of Fig. 8a, one can also calculate more complicated graphs, for example, the graph of Fig. 9. This graph has six types of intermediate states corresponding to six expressions of the type (44a) and (44b). These six expressions are related to the six graphs of Fig. 10a. These graphs differ by the temporal sequence of emission and absorption of reggeons or, what is the same, by the various possibilities of vertical cuts. Each of the six expressions may be written

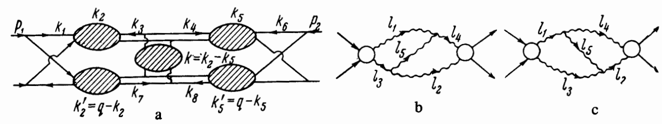


FIG. 8.

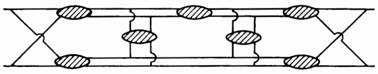


FIG. 9.

down without difficulty in analogy to (44a) and (44b).

If we perform the integrations over l_3, l_4, and l_5 in (44a) and over l_1, l_2, and l_5 in (44b), we obtain the expressions

$$f_j'(q^2) = \int dl_1 dl_2 d^2k_1 d^2k_2 N_{l_1 l_3} \gamma_{l_1 l_3} G_{l_1}^{(1)}(k_1) G_{l_3}^{(3)}(q-k_1)$$
$$\times G_{l_2}^{(2)}(k_2) G_{l_4}^{(4)}(q-k_2) G_{l_5}^{(5)}(q-k_1-k_2) r_{l_2 l_5} \gamma_{l_2 l_5} r_{l_1 l_5} N_{l_2 l_4}, \quad (45a)$$
$$l_3 = j+1-l_1 \quad l_4 = j+1-l_2, \quad l_5 = j+2-l_1-l_2;$$

$$f_j''(q^2) = \int dl_3 dl_4 d^2k_3 d^2k_4 N_{l_1 l_3} \gamma_{l_1 l_3} G_{l_1}^{(1)}(q-k_3) G_{l_3}^{(3)}(k_3)$$
$$\times G_{l_2}^{(2)}(q-k_4) G_{l_4}^{(4)}(k_4) G_{l_5}^{(5)}(q-k_3-k_4) r_{l_5 l_4} \gamma_{l_5 l_4} r_{l_3 l_5} N_{l_2 l_4}, \quad (45b)$$
$$l_1 = j+1-l_3, \quad l_2 = j+1-l_4, \quad l_5 = j+2-l_3-l_4.$$

The expressions (45a) and (45b) are in agreement with the rules formulated in the preceding section, which are now also applicable to the graphs of Fig. 8b and 8c with the indicated directions of the reggeon lines.

In a similar manner we can write down the graphs corresponding to Fig. 10. Corresponding to the graphs of Fig. 10a to f is one graph of the form (45).

We cannot, of course, consider all possible graphs. However, the character of the calculations above and in the Appendix, leads one to think that the rules established here are also valid in the general case. If we apply these rules, we are in the possession of a nonrelativistic reggeon graph technique for the account of the contributions of branch points in the scattering amplitude.

6. BRANCH POINTS DUE TO THE POMERANCHUK POLE

As is well known,[1, 3, 11] the Pomeranchuk pole, whose trajectory $\alpha(t)$ passes through the point j = 1 at t = 0, generates branch points $j_n(t)$ which accumulate in the point j = 1:

$$j_n = n\alpha(t/n^2) - n + 1, \quad n = 2, 3, \ldots \quad (46)$$

For small t the pole and the branch point are close to one another near j = 1. Therefore the asymptotic form of the total cross section, and the cross sections for elastic scattering and other processes proceeding via the exchange of a state with the quantum numbers of the vacuum, must be determined by the simultaneous effects of the poles and branch points. It is essential to take account of the influence of the pole on the branch points and of the influence of the branch points on each other. An important role in the whole analysis is played by the

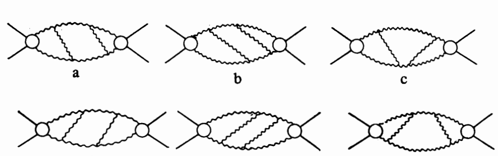


FIG. 10.

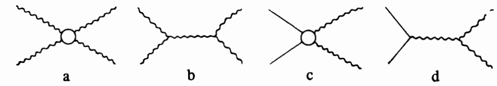


FIG. 11.

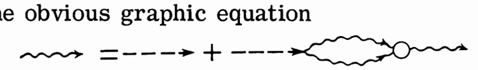
concept of enhancement, formulated in [7]. According to this idea, the most important contribution to the partial wave for small t and j close to unity comes from the completely enhanced graphs, i.e., from the graphs containing only three-reggeon interactions, cf. Fig. 7. The proof for this rests on the circumstance that if some graph contains internally a bubble of the type of Fig. 11a, then there exists another graph identical to the original one in all respects except that the bubble of Fig. 11a is replaced by the graph of Fig. 11b. This new graph gives a large contribution. The wavy line corresponds to the exact Green's function of the reggeon. In exactly the same way the bubble of Fig. 11c at the end of a graph can be replaced by the graph of Fig. 11d.

This can only be correct, of course, if the amplitude for the transformation of a single reggeon into two others is not small for small t and j close to unity. Taking account of our previous analysis, we see no reason for this amplitude to be small under such conditions. It is also clear that the special role of the enhanced graphs depends on the absolute value of their contributions to the partial wave amplitude. For example, the complicated enhanced graphs may give a smaller contribution than the simple unenhanced ones and vice versa, the complicated unenhanced graphs may give a larger contribution than the simple enhanced ones. These questions will be discussed below; for the present we consider the set of completely enhanced graphs.

For this set of graphs the amplitude for the partial wave in the t channel can be written in the form

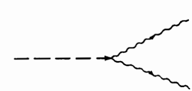
$$f_j(t) = g_1(t) G_j(t) g_2(t), \quad (47)$$

where $G_j(t) = G(j, k^2)$ is the exact Green's function of the Pomeranchuk reggeon including the contribution of the enhanced branch points. The function $G(j, k^2)$ satisfies the obvious graphic equation



(48)

where $G(j, k^2)$ is represented by a wavy line. The dotted line corresponds to the pole term $G_0(j, k^2) = [j - \alpha(k^2)]^{-1}$. The left-hand vertex corresponds to $\gamma_{l_1 l_2} r_{l_1 l_2}$:



$= \gamma_{l_1 l_2} r_{l_1 l_2}$ (49a)

The right-hand vertex represents the exact vertex part, for which, as usual, there exists no closed equation, but only a series expansion:

(49b)

where, as before, the vertices of the type of Fig. 7a correspond to $r_{l_1 l_2} \gamma_{l_1 l_2}$ and the vertices of the type of Fig. 7b to $r_{l_1 l_2}$.

We are interested in the case of branch points connected with the Pomeranchuk pole, where j is close to unity and all l_1, l_2, ... in the integrals are, by condition, close to unity (otherwise the unenhanced graphs would also be important). Therefore we may assume that $r_{l_1 l_2} = r$ is a constant which depends on l_1, l_2, and k_1^2, and $\gamma_{l_1 l_2} = \gamma_{11} = -1$ [cf. (16)]. The equation $\gamma_{l_1 l_2} = -1$ implies that in such a theory the effective coupling constant for three reggeons is pure imaginary and equal to ir.

It is convenient to introduce instead of j the variable $\omega = j - 1$ and to replace $\alpha(k^2)$ by $\alpha(k^2) - 1$. If we take into account that according to Froissart,[10] there can be no singularities to the right of j = 1 for negative t, we can, in the integrals determining the contribution of the branch points, integrate over ω along the imaginary axis with a definite rule for passing the singularities. This rule is given uniquely if we set

$$G_0(j, k^2) = G_0(\omega, k) = [\omega - \alpha(k^2) + \varepsilon]^{-1}, \tag{50}$$

where ε is an arbitrarily small positive number.

With these definitions equations (48) and (49) can be written in the form

$$G(\omega, k) = G_0(\omega, k) - \frac{r}{2} G_0(\omega, k) \int_{-i\infty}^{i\infty} d\omega_1 \int d^2k_1 G(\omega_1, k_1)$$
$$\times G(\omega - \omega_1, k - k_1) \Gamma(\omega_1, k_1;\ \omega - \omega_1,\ k - k_1) G(\omega, k), \tag{51}$$

$$\Gamma(\omega_1, k_1;\ \omega_2, k_2) = r - r^3 \int_{-i\infty}^{i\infty} d\omega_3 \int d^2k_3 G(\omega_3, k_3)$$
$$\times G(\omega_1 - \omega_3,\ k_1 - k_3) G(\omega_1 + \omega_2 - \omega_3,\ k_1 + k_2 - k_3) + \ldots;$$
$$d\omega \to d\omega / 2i\pi, \quad d^2k \to d^2k / (2\pi)^2. \tag{52}$$

In front of the integral in (51) we have placed the coefficient ½ to account for the identity of the reggeons. These equations are easily obtained with the help of the formulas of the preceding sections. Equation (51) can also be written in the form

$$G^{-1}(\omega, k) = G_0^{-1}(\omega, k) - \Sigma(\omega, k),$$
$$\Sigma(\omega, k) = -\frac{r}{2} \int d\omega_1 d^2k_1 G(\omega_1, k_1)$$
$$\times G(\omega - \omega_1, k - k_1) \Gamma(\omega_1, k_1;\ \omega - \omega_1,\ k - k_1). \tag{53}$$

Let us now estimate the absolute value of contribution of the separate graphs. This value depends essentially on the character of the trajectory $\alpha(k^2)$. If we assume that for small k

$$\alpha(k^2) = -\beta k^2, \tag{54}$$

i.e., $G_0^{-1}(\omega, k) = \omega + \beta k^2 + \varepsilon$ then we have in first approximation

$$\Sigma_0(\omega, k) = -\frac{r^2}{2} \int d^2k \frac{1}{\omega + \beta k_1^2 + \beta (k - k_1)^2 + \varepsilon}. \tag{55}$$

The integral (55) diverges logarithmically. The divergence arises from the fact that we have regarded the quantity r as a constant and have taken it outside the integral sign. This is justified only for $-k_1^2 \ll \mu^2$, where μ is a characteristic mass, for example the mass of the π meson. However this in itself is not essential, since the divergent part has no singularities in ω and must be included in the renormalization of the pole trajectory.

We easily find from (55) that

$$\Sigma_0(\omega, k) = \frac{r^2}{16\pi\beta} \ln \frac{\omega + \beta k^2/2}{\beta \mu^2}. \tag{56}$$

We have found that for small ω and k^2 the self-energy part $\Sigma_0(\omega, k)$ is much larger than $G_0^{-1}(\omega, k)$, by the factor r^2/ω or r^2/k^2. If we were to calculate the next correction to Σ, we would obtain a quantity of order r^4/ω, r^4/k^2, i.e., larger than $G_0^{-1}(\omega, k)$ by the factor r^4/ω^2 or r^4/k^4, etc. Under these conditions $G(\omega, k)$ may have very little to do with $G_0(\omega, k)$, and our entire discussion, starting from a pole, may become questionable. Moreover, in a situation where each successive approximation is larger than the previous one, the incompletely enhanced graphs of higher order will be larger than the completely enhanced graphs of lower order.

However, the situation is not necessarily that bad. The point is that the pole trajectory must have a singularity for $k^2 = 0$, since the positions of the pole and of the branch point coincide for $k^2 = 0$ (bound state at the boundary of the continuous spectrum). This singularity, if sufficiently strong, can lead to a large suppression of the contribution of the branch points. Moreover, in calculating $\Sigma(\omega, k)$ in the first approximation we have replaced $\Gamma(\omega_1 k_1,\ \omega_2 k_2)$ by r. However, Γ is given by a series with ever increasing terms. It is entirely possible that the sum of this series is small for small ω and k; then the value of Σ will also be small. In other words, words, the bad convergence character of the series for Σ and Γ may be connected with the circumstance that we have chosen a bad zeroth approximation. Equations (53) and (52) may in reality have a meaningful solution.

If such a solution exists, then all completely enhanced graphs are of the same order for the correct value of $G(\omega, k)$, and the incompletely enhanced graphs are small compared with the former. Hence, the choice of graphs is correct in this case, and the problem consists in finding the solution to (53) and (52).

The writing of this paper was preceded by an investigation of the properties of reggeon branch points which was carried out together with I. Ya. Pomeranchuk and K. A. Ter-Martirosyan, and has been published in part in [3, 7]. In preparing the present paper, and in particular in the analysis of the question of the enhancement of branch points, the author has received invaluable stimulation from the ideas of Isaak Yakovlevich Pomeranchuk during the preceding collaboration.

I am grateful to K. A. Ter-Martirosyan for numerous useful comments.

APPENDIX

We shall calculate here the asymptotic form of the graph of Fig. 8a. The notation for the momenta is indicated in the figure. Each momentum is written in the form (1). The amplitudes corresponding to the dashed bubbles have the form (8). The integration over the variables $k_{1\perp}$, α_1, α_2, and β_1 are performed in the same way as before (Sec. 2). We have

$$k_{1\perp}^2 \sim m^2, \quad \alpha_1, \alpha_2 \sim m^2/s, \quad \beta_2 < \beta_1 < 1.$$

In exactly the same fashion we perform the integration in the right-most bubble of the graph of Fig. 8a over the variables $k_{6\perp}$, β_6, β_5, and α_6, where

$$k_{6\perp}^2 \sim m^2, \quad \beta_5, \beta_6 \sim m^2/s, \quad \alpha_5 < \alpha_6 \lesssim 1.$$

There remain the integrations over $k_{2\perp}$, β_2, $k_{5\perp}$, α_5 and k_3, k_4, k_7, k_8.

In order for the amplitudes corresponding to the bubbles to be in the asymptotic region (energies large, masses and momentum transfers of order unity), we must require

$$\beta_1\alpha_3 s \gg m^2,\quad \beta_4\alpha_6 s \gg m^2,\quad (1-\beta_1)\alpha_7 s \gg m^2, \tag{A.1}$$
$$\beta_8(1-\alpha_6)s \gg m^2,\quad (\alpha_3+\alpha_4+\alpha_7+\alpha_8)(\beta_3+\beta_4+\beta_7+\beta_8)s \gg m^2;$$

$$k_{i\perp}^2 \sim m^2,\quad \alpha_i\beta_i s \sim m^2,\quad \alpha_4\beta_3 s \lesssim m^2,\quad \alpha_3\beta_4 s \lesssim m^2,$$
$$\alpha_7\beta_8 s \lesssim m^2,\quad \alpha_8\beta_7 s \lesssim m^2,\quad \alpha_5\beta_i s \lesssim m^2,\quad \beta_2\alpha_i s \lesssim m^2. \tag{A.2}$$

The last of equations (A.1) can be fulfilled in two cases:

$$\alpha_3+\alpha_4 \gg \alpha_7+\alpha_8,\quad \beta_7+\beta_8 \gg \beta_3+\beta_4; \tag{A.3}$$
$$\alpha_7+\alpha_8 \gg \alpha_3+\alpha_4,\quad \beta_3+\beta_4 \gg \beta_7+\beta_8. \tag{A.4}$$

In the first case we have, according to the last equations (A.2)

$$\alpha_5 \ll \alpha_3+\alpha_4,\quad \beta_2 \ll \beta_7+\beta_8. \tag{A.5}$$

in the second case

$$\alpha_5 \ll \alpha_7+\alpha_8,\quad \beta_2 \ll \beta_3+\beta_4. \tag{A.6}$$

The character of the integration over β_2 and α_5 depends on the structure of the singularities of the integrals over k_3, k_4 and k_7, k_8. Let us consider first the first case (A.3), (A.5) and the integration over β_2. We write down the denominators entering in the upper part of the graph of Fig. 8a, i.e., in the integral over k_3, k_4:

$$\begin{gathered} k_{3\perp}^2+\alpha_3\beta_3 s-m^2,\quad (k_{5\perp}-k_{4\perp})^2+(\alpha_4-\alpha_5)\beta_4 s-m^2,\\ (k_{2\perp}-k_{3\perp})^2+\alpha_3(\beta_3-\beta_2)s-m^2,\\ (k_{3\perp}+k_{4\perp})^2+(\alpha_3+\alpha_4)(\beta_3+\beta_4)s-m^2,\\ k_{4\perp}^2+\alpha_4\beta_4 s-m^2,\\ (k_{2\perp}-k_{3\perp}+k_{5\perp}-k_{4\perp})^2+(\alpha_3+\alpha_4)(\beta_3+\beta_4-\beta_2)s-m^2. \end{gathered} \tag{A.7}$$

Here we have used $\alpha_2, \beta_5 \sim m^2/s$ and $\alpha_5 \ll \alpha_3+\alpha_4$.

It is easy to see by considering these denominators, that the integral over β_3 differs from zero only if α_3 and α_4 have different signs and $|\alpha_4| > |\alpha_3|$. [This implies that α_5 can also be omitted in the fourth equation (A.7).] If these conditions are fulfilled, the integral over β_3 has, as a function of β_2, singularities in the upper and lower half-planes for β_2 of the order of β_4, $m^2/\alpha_4 s$, and $m^2/(\alpha_3+\alpha_4)s$. The quantity β_2 also enters in the denominators corresponding to the lower part of the graph of Fig. 8a. However, because of condition (A.5), β_2 drops out of the integral over k_7 and k_8 just the same as α_5 dropped out of the integral over k_3 and k_4. Therefore the integral over β_2 can be closed around the singularities of the integral over β_3, and it is different from zero. Analogously, the integral over α_5 is determined by the singularities of the lower part, i.e., the integral over k_7 and k_8.

The integrations over the variables corresponding to the different parts of the graph of Fig. 8a are now connected only via the amplitudes corresponding to the dashed bubbles. If these amplitudes are written in the form (5), (8), one can successively carry out all integrations.

The integration over the variables $k_{1\perp}$, α_1, β_1, and α_2 yields the quantity $N_{l_1 l_2'}(q, k_2)$, which coincides with (11). The integration over $k_{3\perp}$, β_2, β_3, and α_3 leads to the integral

$$\frac{1}{4(4\pi)^{1/2}}\int\frac{d^2k_{3\perp}\,d\alpha_3\,d\beta_3\,d\beta_2}{(2\pi)^4}|s|^2(-\alpha_3)^{l_2}(-\alpha_3+\alpha_4)^{l}\lambda^2 g_2 g$$
$$\times\{(k_3^2-m^2)[(k_2-k_3)^2-m^2][(k_3+k_4)^2-m^2][(k_5-k_4)^2-m^2]\}^{-1}. \tag{A.8}$$

As noted earlier, α_3 and α_4 have different signs, and $|\alpha_4| > |\alpha_3|$. It can be shown that the two regions $\alpha_3 < 0$, $\alpha_4 > 0$ and $\alpha_3 > 0$, $\alpha_4 < 0$ compensate each other exactly if the product of the signature of the three amplitudes $f(k_1, k_3, k_2)$, $f(k_4, k_6, k_5)$, and $f(k_3+k_4, k_7+k_8, k_2-k_5)$ is equal to -1, and add up if the product of the signatures is equal to $+1$. Assuming the latter, we consider the case $\alpha_3 < 0$, $\alpha_4 > 0$.

In performing the next integration over $k_{4\perp}$, α_4, and β_4, we note that all zeros of the denominators (A.7) depending on β_4 lie in the lower half-plane of the complex β_4 plane. Therefore the integral can be closed in the upper half-plane of β_4 around the singularities corresponding to the left-hand cut of $f(k_6, k_4, k_5)$. As a result the absorptive part of $f(k_6, k_4, k_5)$ appears in the answer.

If instead of α_3 we introduce $\gamma = -\alpha_3/\alpha_4$ and instead of α_4 the variable $x = -\alpha_4\beta_4 s$, then the integral over $k_{3\perp}$, β_3, α_3, β_2; $k_{4\perp}$, α_4 will have the form

$$r_{l_5 l_2,\, l}(-\beta_4)^{l_5-l_2-l}, \tag{A.9}$$

where $r_{l_5 l_2, l}$ is defined by (40).

Repeating the same arguments, we find that the integration over $k_{8\perp}$, α_5, α_8, β_8; $k_{7\perp}$, β_7 gives

$$r_{l_5' l,\, l_2'}(-\alpha_7)^{l_2'-l-l_5'}. \tag{A.10}$$

The integration over $k_{6\perp}$, α_6, β_6, β_5 gives $N_{l_5 l_5'}$. The region of integration over $-\beta_4$, $-\alpha_7$ is defined by the conditions $\beta_4\alpha_7 s \ll m^2$, $m^2/s \ll -\beta_4$, $-\alpha_7 \ll 1$, since $\beta_4\alpha_4 s \sim m^2$, $\alpha_7 \ll \alpha_4$.

Substituting all these results in the expression corresponding to the graph of Fig. 8a and performing the integration over α_7 and β_4, we obtain

$$\begin{aligned} F(q^2,s) = \frac{\pi}{2}\int & dl_2\,dl_2'\,dl_5\,dl_5'\,dl\,d^2k_2\,d^2k_5'\,\xi_{l_2}\xi_{l_5'}\xi_l N_{l_2 l_2'} G_{l_2}(k_2)G_{l_2'}(q-k_2)\\ & \times G_l(q-k_2-k_5')G_{l_5}(q-k_5')G_{l_5'}(k_5')r_{l_5' l,\, l_2'} r_{l_5 l_2,\, l} N_{l_5 l_5'}\\ & \times\{[(l_5'+l_5-l_2'-l_2)(l_2'+1-l_5'-l)]^{-1}[s^{l_5+l_5'-1}-s^{l_2+l_2'-1}]\\ & -[(l_2'+1-l_5'-l)(l_5+1-l-l_2)]^{-1}[s^{l_5+l_5'-1}-s^{l+l_2+l_5'-2}]\}. \end{aligned} \tag{A.11}$$

Going over to the partial wave amplitude, we find the expression given in the text and corresponding to the graph of Fig. 8b. The expression corresponding to the graph of Fig. 8c is obtained by considering the region (A.4).

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Translated by R. Lipperheide
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