STATIC SCALAR AND ELECTRIC FIELDS IN EINSTEIN'S THEORY OF RELATIVITY

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A model of a spherically-symmetric body (sphere) in the presence of gravitational, electromagnetic, and scalar fields is considered. Static solutions are found which differ from the solutions obtained by Fisher and Bergmann and Leipnik in that, contrary to the latter solutions, the present ones satisfy the general requirements imposed on static solutions in general relativity theory and thus can be naturally interpreted physically.

WE consider in this paper a spherically symmetrical statistical model of a body of limited dimensions (sphere) in the presence of an electromagnetic and scalar field besides the gravitational field. A similar problem, in which only a gravitational and an electromagnetic field took part, was considered a number of times (see, for example, the papers of Papapetrou[1] and Bonnor et al.[2]), and the possibility of such equilibrium solutions was demonstrated. This fact was also known to Einstein[3].

A rather detailed investigation of the problem in the presence of only a gravitational and a scalar field with a point source was carried out by Fisher in 1948[4]. We note, however, that his solution has in itself no physical meaning, since the "elementary Euclidean" condition[5] was not satisfied at the point r = 0 of the spherical coordinate system (that is, at this point the ratio of the length of an infinitesimally small circle to the diameter is not equal to π, as follows from the equation $e^{\lambda} = 0$). In addition, because of an error in the calculations, Fisher[4] arrived at the incorrect conclusion that the function e^{ν} has a non-Schwarzschild behavior as $r \to \infty$, and obtained an incorrect result (infinity) for the total energy of the system, owing to the use of a strongly singular transformation of the coordinates. Thus, the described system can actually not be realized. Moreover, it follows from our later conclusions that the statistical model of the particle (sphere) of "dustlike" matter (that is, without the presence of extraneous forces inside the particle) can likewise not be realized if only the gravitational and scalar fields take part (this is seen from the impossibility of satisfying a condition of the type (15) $\kappa^2 m^2 + \kappa G^2 = 0$; here G is the scalar constant and κ the gravitational constant). Physically this deduction is understandable, since both the gravitational and the scalar forces are attractive and therefore it is impossible to attain equilibrium in this system on their basis alone.

The problem of a pointlike source of gravitational and scalar massless fields was considered also by Bergmann and Leipnik[6] in 1957. Besides the "Fisher" solution, they present also other solutions. However, some of them do not satisfy the Galilean condition at infinity, as noted by the authors themselves, and the others do not correspond to the natural condition of positiveness (> 0) of the quantities $\kappa^2 m^2$ and κG^2 (or else of the sum $\kappa^2 m^2 + \kappa G^2$). In particular, the solution for which m = 0 and $G \neq 0$ fails furthermore to satisfy the Bianchi identity $(1/2)\nu\rho + jV' = 0$ (see formula (6) below) at the point r = 0, since at this point we would have $\rho = 0$ and $jV' \neq 0$.

It is physically clear that if we introduce an additional electrostatic field besides the gravitational and scalar fields, then we can hope to find an equilibrium solution, since the electrostatic forces are repulsive. We shall not introduce any phenomenological extraneous forces ("tensions") inside the body, that is, the picture considered corresponds to a "dustlike" structure of the body. For the Einstein equations $G_{\mu}^{\nu} = 8\pi\kappa T_{\mu}^{\nu}$ the material tensor T_{μ}^{ν} consists in this case of three parts. The matter tensor is of the form

$$ {}_{m}T_{\mu}{}^{\nu} = -\rho g_{\mu\sigma}\frac{dX^{\sigma}}{ds}\frac{dX^{\nu}}{ds}, $$

where ρ is the invariant mass density. The electromagnetic field tensor (in the Gaussian gauge) is

$$ {}_{el}T_{\mu}{}^{\nu} = \frac{1}{4\pi}\left(F_{\mu\sigma}F^{\nu\sigma} - \frac{1}{4}\delta_{\mu}{}^{\nu}F_{\sigma\lambda}F^{\sigma\lambda}\right), $$

where $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$. The tensor of the scalar field with mass is

$$ {}_{sc}T_{\gamma}{}^{\nu} = \frac{1}{4\pi}\Big[-\nabla_{\gamma}V\nabla^{\nu}V + \frac{1}{2}\delta_{\gamma}{}^{\nu}(\nabla_{\sigma}V\nabla^{\sigma}V - \mu^2V^2)\Big]. $$

The equations of the electromagnetic and scalar fields are of the form

$$ \nabla_{\mu}F^{\nu\mu} = 4\pi J^{\nu}, \quad \nabla_{\sigma}\nabla^{\sigma}V + \mu^2 V = -4\pi j, $$

Here j is the invariant density of the source of the scalar field.

In the spherically-symmetrical statistical case under consideration we have

$$ (ds)^2 = g_{\mu\nu}\,d\xi^{\mu}\,d\xi^{\nu} = -e^{\lambda}(dr)^2 - r^2(d\theta)^2 - r^2\sin^2\theta\,(d\varphi)^2 + e^{\nu}(dt)^2, $$
$$ \lambda = \lambda(r), \quad \nu = \nu(r) \quad (\xi^1, \xi^2, \xi^3, \xi^4 = r, \theta, \varphi, t). $$

Einstein's equations take the form

$$ -\frac{\lambda'}{r} + \frac{1-e^{\lambda}}{r^2} = -\varkappa(V')^2 - \varkappa\mu^2V^2e^{\lambda} - \varkappa e^{-\nu}(\varphi')^2 - 8\pi\varkappa\rho e^{\lambda}, \quad (1) $$

$$ \frac{\nu'}{r} + \frac{1-e^{\lambda}}{r^2} = \varkappa(V')^2 - \varkappa\mu^2V^2e^{\lambda} - \varkappa e^{-\nu}(\varphi')^2, \quad (2) $$

$$ \frac{\nu''}{2} + \frac{\nu'-\lambda'}{2r} + \frac{\nu'(\nu'-\lambda')}{4} = -\varkappa(V')^2 - \varkappa\mu^2V^2e^{\lambda} + \varkappa e^{-\nu}(\varphi')^2; \quad (3) $$

where φ denotes the scalar potential of the electromagnetic field $(-A_4)$ and the prime denotes the derivative d/dr.

The equation of the electromagnetic field takes the form

$$\varphi'' + \left(\frac{2}{r} - \frac{\nu' + \lambda'}{2}\right)\varphi' = -4\pi J_4 e^{\lambda} \tag{4}$$

Here J_4 is the electric charge density, and the scalar field equation is

$$V'' + \left(\frac{2}{r} + \frac{\nu' - \lambda'}{2}\right)V' - \mu^2 V e^{\lambda} = 4\pi j e^{\lambda}. \tag{5}$$

The inverted Bianchi identity ($\nabla_\sigma T^\sigma_1 = 0$) takes the form

$$\frac{\nu'}{2}\rho + jV' + \varphi' J^4 = 0, \tag{6}$$

thus leaving four unknowns in the system (1)—(5).

Let us consider first the case of a massless scalar field. The sum of Eqs. (2) and (3) with $\mu = 0$ leads to the equation

$$\frac{\nu''}{2} + \frac{3}{2}\frac{\nu'}{r} + \frac{(\nu')^2}{4} + \frac{1}{r^2} - \lambda'\left(\frac{1}{2r} + \frac{\nu'}{4}\right) - \frac{e^{\lambda}}{r^2} = 0, \tag{7}$$

which can be integrated to yield

$$e^{-\lambda} = \left(1 + \frac{r\nu'}{2}\right)^{-2} + Dr^{-2}e^{-\nu}\left(1 + \frac{r\nu'}{2}\right)^{-2}; \tag{8}$$

Here D is the integration constant.

Let us find first the asymptotic form of the solutions of the system for $r \to \infty$, and then use this form as the "external" solution. To construct the solution inside the body, on the other hand, we shall use the well known methods of "joining" the internal and external solutions.

Outside the masses (that is, at $\rho = j = J_4 = 0$), Eqs. (4) and (5) have the following solutions, which vanish at infinity:

$$\varphi' = -\frac{\varepsilon}{r^2}e^{(\lambda+\nu)/2}, \quad V' = -\frac{G}{r^2}e^{(\lambda-\nu)/2}, \tag{9}$$

ϵ is the electric charge. In this case the problem reduces to a completely defined system of equations

$$-\frac{\lambda'}{r} + \frac{1 - e^{\lambda}}{r^2} = -\frac{\varkappa G^2}{r^4}e^{\lambda-\nu} - \frac{\varkappa\varepsilon^2}{r^4}e^{\lambda}, \tag{10}$$

$$\frac{\nu'}{r} + \frac{1 - e^{\lambda}}{r^2} = \frac{\varkappa G^2}{r^4}e^{\lambda-\nu} - \frac{\varkappa\varepsilon^2}{r^4}e^{\lambda}. \tag{11}$$

Finding the solution of this system as $r \to \infty$, we use the Galilean conditions at infinity ($e^{\nu} \to 1$, $e^{\lambda} \to 1$) and the condition for the correspondence with the Newtonian approximation (for which purpose the integration constant is chosen such that $e^{\nu} \to 1 - 2\kappa m/r + \ldots$). As a result we obtain the following expansions:

$$e^{\nu} = 1 - \frac{2\varkappa m}{r} + \frac{\varkappa\varepsilon^2}{r^2} + \frac{1}{3}\frac{(\varkappa m)\varkappa G^2}{r^3} + \ldots, \tag{12}$$

$$e^{-\lambda} = 1 - \frac{2\varkappa m}{r} + \frac{\varkappa G^2 + \varkappa\varepsilon^2}{r^2} + \frac{(\varkappa m)\varkappa G^2}{r^3} + \ldots \tag{13}$$

Naturally, when $\epsilon = 0$ these expansions give the asymptotic form of Fisher's solutions. When G = 0 they go over into the solutions of the electrostatic problems (Nortstrom-Reissner[7]; $e^{\nu} = e^{-\lambda} = 1 - 2\kappa m/r + \kappa\epsilon/r^2$). When $\epsilon = G = 0$ they go over into the Schwarzschild solution.

Using the obtained asymptotic form, we can readily determine the constant D in the integral (8) of the initial system of equations

$$D = \varkappa^2 m^2 + \varkappa G^2 - \varkappa\varepsilon^2. \tag{14}$$

To satisfy the "elementary Euclidean" condition as $r \to 0$ it is necessary to have $e^{\lambda} = 1$. If the function e^{ν} is not too singular at zero, then it follows from (8) that the solution for e^{λ} of interest to us can be obtained only under the condition

$$D = \varkappa^2 m^2 + \varkappa G^2 - \varkappa\varepsilon^2 = 0. \tag{15}$$

We note that when G = 0 this gives the condition $\kappa^2 m^2 - \kappa\epsilon^2 = 0$ for electrostatic problems, as obtained in the models of Papapetrou and Bonnor. Thus, we have a solution of the problem outside the masses in the form (12) and (13), which we shall use as the "external" solution in constructing the model of a sphere of sufficiently large radius

$$R \gg \varkappa m, \ \varkappa\varepsilon^2, \ \varkappa G^2, \tag{16}$$

and an exact relation, which is valid both outside and inside the sphere:

$$e^{\lambda} = (1 + r\nu'/2)^2. \tag{17}$$

We shall seek the internal solution for $\nu(r)$ in the form

$$\nu(r) = Ar + B, \tag{18}$$

A and B are constants. The conditions for the joining of the quantities ν and ν' on the boundary of the sphere (see Synge[8]) dictate their continuity and yield

$$A = \frac{2\varkappa m}{R^2} + \frac{4\varkappa^2 m^2 - 2\varkappa\varepsilon^2}{R^3} + \ldots, \tag{19}$$

$$B = -\frac{4\varkappa m}{R} + \frac{-4\varkappa^2 m^2 + \varkappa\varepsilon^2 + 2\varkappa G^2}{R^2} + \ldots \tag{20}$$

The quantity e^{λ} becomes continuous automatically, by virtue of (17). The derivative λ' is not continuous (and experience a discontinuity together with ν'' and ρ). It is then necessary to join together the derivatives of the fields (φ' and V'). In view of the arbitrariness, we shall assume for $\varphi'(r)$ inside the sphere, everywhere, its limiting value

$$\varphi' = -\frac{\varepsilon}{R^2}\exp\left(\frac{\lambda(R) + \nu(R)}{2}\right) \cong -\frac{\varepsilon}{R^2}. \tag{21}$$

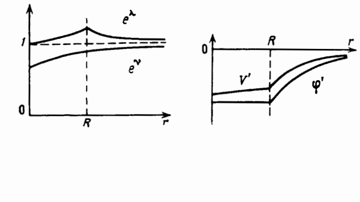
Then the scalar field is continuous automatically by virtue of Eq. (2), which yields inside the sphere

$$\varkappa(V')^2 = \varkappa e^{-\nu}(\varphi')^2 - \frac{A^2}{4} \cong \frac{\varkappa\varepsilon^2 - \varkappa^2 m^2}{R^4}. \tag{22}$$

Thus, the natural condition that $(V')^2$ be positive is satisfied by virtue of condition (15). Einstein's equation (1), upon substitution of (17)—(21), gives for the mass density

$$8\pi\varkappa r^2 e^{2\lambda}\rho \cong \frac{4\varkappa m r}{R^2} - \frac{2\varkappa\varepsilon^2 r^2}{R^4}. \tag{23}$$

Consequently, to ensure positiveness of $\rho(r)$ inside the sphere we get the condition $R > \kappa\epsilon^2/2\kappa m$, which obviously does not contradict conditions (15) and (16). The behavior of the functions e^{ν}, e^{λ}, φ', and V' is shown schematically in the figure.



Let us examine the behavior of the covariant densities of the electric and scalar charges in this model. For the former we have from formula (4)

$$8\pi r e^{2\lambda} J_4 = -\left(1+\frac{Ar}{2}\right)\left(4-\frac{A^2r^2}{2}\right)\varphi' \cong -4\varphi' > 0,$$

Since Ar $\ll$ 1 and $\varphi' < 0$. Consequently J_4 has a constant sign throughout (and is positive). For the density of the source of the scalar field we get from (5) and (15)

$$8\pi\varkappa r e^{\nu+\lambda} V' j \cong 4\frac{\varkappa\varepsilon^2 - \varkappa^2 m^2}{R^4} = 4\frac{\varkappa G^2}{R^4}.$$

Consequently, j has a constant sign throughout (and $j < 0$, since $V' < 0$). These formulas lead to the presence of (integrable) singularities in the densities j and J_4 at the origin ($\sim 1/r$). It is known from an analysis of similar models that this singularity is typical and is connected with our very simple choice ν' = const. If we were to choose at zero at least $\nu' \sim r$, and accordingly, values of φ' and V′ that vanish at zero as $r \to 0$, the singularity would disappear. Similar remarks can be made also with respect to $\rho(r)$.

The expression for the total electric charge of the system in our metric

$$I = 4\pi\int_0^\infty \sqrt{J_\mu J^\mu}\, r^2 e^{\lambda/2}\, dr = -\int_0^\infty (\varphi' r^2 e^{-(\nu+\lambda)/2})'\, dr \tag{24}$$

yields after substituting the constructed solution

$$I = -[\varphi' r^2 e^{-(\lambda+\nu)/2}]|_0{}^R = \varepsilon,$$

which agrees with the electric-charge conservation law. The total mass of the system, in accordance with the formula[5]

$$M = \frac{1}{2\varkappa}[r(e^\lambda - 1)]|_0{}^\infty \tag{25}$$

takes on in this case the value

$$M = \frac{1}{2\varkappa}\left[\underset{(r\to\infty)}{r}\left(\frac{2\varkappa m}{r}+\ldots\right)\right] - \frac{1}{2\varkappa}[\underset{(r\to\infty)}{r}\quad(1+\ldots-1)] = m.$$

Thus, we have completed the construction of the model.

Let us consider now one possibility of constructing a model of the sphere in the presence of a scalar field with mass μ. In place of relation (8) we obtain in this case

$$e^{-\lambda} = \left(1+\frac{r\nu'}{2}\right)^{-2} + r^{-2}e^{-\nu}\left(1+\frac{r\nu'}{2}\right)^{-2}\left[\bar{D} + 2\varkappa\mu^2\int_r^\infty V^2 r^2 (r^2 e^\nu)'\, dr\right], \tag{26}$$

where $\overline{D}$ is an integration constant. Outside the masses, the system (1)—(5) becomes fully defined. Finding the asymptotic form of its solutions as $r \to \infty$, we have for the electric-field potential, as before, the exact expression (9). For the remaining quantities, assuming additionally $\exp(-\mu r) \ll 1$, we get

$$V = \frac{G}{r(\mu r)^{\varkappa m\mu}}\exp\left(-\mu r - \frac{\alpha}{r}+\ldots\right)+\ldots,$$
$$2\alpha = \varkappa m + 3\varkappa^2 m^2\mu - \varkappa\varepsilon^2\mu, \tag{27}$$

$$e^{-\lambda} = 1 - \frac{2\varkappa m}{r} + \frac{\varkappa\varepsilon^2}{r^2} + \varkappa G^2\left(\frac{\mu}{r} + \frac{1-\varkappa m\mu}{r^2}+\ldots\right)$$
$$\cdot\exp\left(-2\mu r - 2\varkappa m\mu\ln\mu r - \frac{2\alpha}{r}+\ldots\right)+\ldots, \tag{28}$$

$$e^\nu = 1 - \frac{2\varkappa m}{r} + \frac{\varkappa\varepsilon^2}{r^2} - \frac{\varkappa G^2}{2}\left(\frac{1}{r^2}+\ldots\right)$$
$$\times\exp\left(-2\mu r - 2\varkappa m\mu\ln\mu r - \frac{2\alpha}{r}+\ldots\right)+\ldots, \tag{29}$$

where the last dots stand for terms with exponentials raised to higher powers. We note the characteristic difference between the asymptotic form of the scalar field and that of the planar case (factor $(\mu r)^{-\kappa m\mu}$) which was pointed out to us by L. G. Zastavenko. The expansions (27)—(29) can be readily compared with results of Stephenson[9], who obtained expansions of the same quantities in powers of κ.

Using the obtained asymptotic form, let us find the constant in the exact integral (26), $\overline{D} = \kappa^2 m^2 - \kappa\epsilon^2$. In order that there be no singularity of $e^{-\lambda}$ at zero, we now get in lieu of (15) the condition

$$\varkappa^2 m^2 - \varkappa\varepsilon^2 + 2\varkappa\mu^2\int_0^\infty V^2 r^2 (r^2 e^\nu)'\, dr = 0. \tag{30}$$

Further, to construct the model of the body, we use the asymptotic form as the "external" solution. This means that the following conditions

$$\frac{\varkappa m}{R},\ \frac{\varkappa\varepsilon^2}{R^2},\ \frac{\varkappa G^2}{R^2},\ e^{-\mu R} \ll 1$$

are superimposed on the radius of the sphere, and consequently the scalar field outside the masses will make an exponentially small contribution. Leaving just as small a scalar field inside the sphere, we obtain a model that is very close in its physical meaning with the model of Bonnor in which only the gravitational and electric fields take part. This is seen also from formulas (26) and (30), which yield directly the relations $\kappa^2 m^2 - \kappa\epsilon^2 \cong 0$ and $\epsilon^\lambda \cong (1 + r\nu'/2)^2$, which are close to those of Papapetrou and Bonnor. Choosing again for the internal solution $\nu = Ar + B$, where A and B are constants, and joining ν and ν' on the boundary of the sphere, we get

$$A = \frac{2\varkappa m}{R^2} + \frac{4\varkappa^2 m^2 - 2\varkappa\varepsilon^2}{R^3}+\ldots,\quad B = -\frac{4\varkappa m}{R} + \frac{3\varkappa\varepsilon^2 - 6\varkappa^2 m^2}{R^2}+\ldots$$

Continuing the quantities V and V′ smoothly inside the sphere by means of the simplest choice $V = V_1 + v_2 r$ (v_1 and v_2 are constants), we ensure all the remaining joining conditions (continuity of e^λ, φ', and ${}_{sc}T_1^1$). This leads to the following expression for V inside the sphere:

$$V = \frac{G}{R}(\mu R + \mu\varkappa m + 2 + \ldots)e^{-\mu R+\ldots} +$$
$$+\frac{G}{R}(-\mu R - \mu\varkappa m - 1 + \ldots)e^{-\mu R+\ldots}\frac{r}{R}.$$

For the mass density we obtain from (1) and (2)

$$8\pi\varkappa r e^\lambda \rho = \lambda' + \nu' - 2\varkappa r(V')^2 \cong 2A - 2\frac{\varkappa G^2}{R^2}\mu^2 r e^{-2\mu R} > 0,$$

that is, the positiveness of $\rho(r)$ is ensured. Analogously we demonstrate in this model the constancy of the sign of the density of the electric charge $J_4(r)$, and the equality of the total charge ϵ and of the total mass of the system m.

The question of the possible role of gravitation in the nature of elementary particles was raised by Einstein himself, and its different quantum and classical aspects are under discussion to this very day[10]. An analysis of such models may be of certain interest from the point of view of the problem of elementary-particle theory. I hope to tackle this problem in the near future, and primarily to the problem of the stability of the obtained static equilibrium solutions.

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