VIOLATION OF ISOTOPIC INVARIANCE AND THE PION FORM FACTOR

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A nonlinear integral equation for the pion form factor is obtained by assuming minimality of electromagnetic interaction and employing a certain approximation. An approximate analytic solution of the equation is presented. The form factor is found to depend only on the pion structure constant, the pion mass, and the pion mass difference. The mean-square radius is 0.3 Fermi.

1. INTRODUCTION

AT first glance the title of this paper sounds somewhat paradoxical. How can the form factor, related as it is to the isotopic-invariant strong interaction, depend on the violation of that invariance? Nevertheless these questions turn out to be rather closely related. In the first place the very concept of the form factor is inseparable from the existence of the electromagnetic interaction, which violates isotopic invariance. In the second place, one may mention the problem of the electromagnetic mass in classical electrodynamics. From the classical point of view we can represent the pions as balls—either neutral or positively or negatively charged. But then the difference in mass of these balls should be related to the radius—the charged ball will be heavier than the neutral one by $\Delta m = \frac{2}{3}\alpha/rc^2$. Thus, knowing the mass difference of the pions, it is possible to determine their "radius."

This relation between this mass difference and "dimension" of the pion is not confined to classical physics only. There exists in quantum field theory as well a formula relating this mass difference to the pionic form factor,[1, 2] although here we shall derive it in a certain approximation. However, in classical as well as in quantum field theory, we have so far dealt only with what might be called the global violation of isotopic invariance, i.e., with the simple mass splitting, connected with the total charge, in an isomultiplet. In such a situation we must not expect to learn a lot about the charge distribution from the one number available to us, namely Δm.

In local theories, with which we usually deal, the violation of isotopic invariance should also be local, i.e., should be of the form $\partial_\mu I_\mu(x) = \rho(x)$, which may throw some additional light on the form of the charge distribution, i.e., on the form factor. However, what should be put on the right and left side of this equality when one is talking about violation of isotopic invariance? The answer to this question has in fact been given in a number of papers [3–6] and reduces to the following.

In the absence of the electromagnetic interaction the theory is isotopic-invariant. This means that along with the vector for the electric current (more precisely its isovector part J_μ^0) there exist local operators $j_\mu^\pm$, for which

$$\partial_\mu j_\mu{}^Q(x) = 0 \quad (Q = +, -, 0),$$

and there exist space integrals whose time-dependent components represent the generators of the isotopic group. If we now turn on the electromagnetic interaction, and do so in a minimal way, then it turns out that

$$i\partial_\mu J_\mu{}^Q(x) = QeA_\mu(x)J_\mu{}^Q(x), \tag{1}$$

where J_μ^Q denotes the current when the electromagnetic interaction is turned on, and A_μ is the electromagnetic potential.

In Sec. 2 we shall give a somewhat different definition of the charged vector current than that given in the above-mentioned papers, and will find its divergence accurate to terms of order e^2. If we then apply the resultant expression to pions we obtain, within a certain approximation, a nonlinear integral equation for the pion form factor. Section 3 is devoted to the approximate analytical solution of this equation by means of expanding it in eigenfunctions of the group of motion in the Lobachevsky space and asymptotic solution of the resultant functional equation. At the end of this section we give an expression for the mean square radius which turns out to be 0.3 F.

2. THE CURRENT DENSITY, DIVERGENCE, AND EQUATIONS FOR THE PION FORM FACTOR

As already mentioned in the introduction, the definition of the charged current density in the absence of the electromagnetic interaction represents no difficulties. When the electromagnetic interaction is turned on the current becomes "dressed" by electromagnetic vertices. It is therefore natural to define such a "dressed" current by the "electromagnetic" S-matrix:

$$J_\mu{}^Q(x) = T(j_\mu{}^Q(x)S_{\rm em})S_{\rm em}^+$$
$$= \exp_>\left\{ie\int_{x_0}^{\infty} L_{\rm em}^{\rm int}(t)\,dt\right\} j_\mu{}^Q(x)\exp_<\left\{-ie\int_{x_0}^{\infty} L_{\rm em}^{\rm int}(t)\,dt\right\};$$

where the symbols > and < denote the corresponding time ordering. Expanding the right-hand side of the equation by perturbation theory it is not hard to obtain, accurate to terms of order e^2,

$$J_\mu{}^Q(x) = j_\mu{}^Q(x) + ie\int d^4y\,[\mathscr{L}_{\rm em}^{\rm int}(y), j_\mu{}^Q(x)]\theta(y_0 - x_0)$$
$$- e^2\int d^4y_1 d^4y\,[\mathscr{L}_{\rm em}^{\rm int}(y_1)[\mathscr{L}_{\rm em}^{\rm int}(y), j_\mu{}^Q(x)]]\,\theta(y_{10} - y_0)\,\theta(y_0 - x_0). \tag{2}$$

where $\theta(x)$ is the usual sign function.

Let us suppose now that the electromagnetic interaction has the usual "current × potential" form, i.e.,

$$\mathscr{L}_{\rm em}^{\rm int}(x) = a_\mu(x)j_\mu(x), \tag{3}$$

where a_μ is the electromagnetic field operator (which is not renormalized by the strong interaction), and j_μ is the electromagnetic current density, consisting of an isoscalar part and the third component of an isovector.

Let us further make an assumption about the character of the commutator $[j_\mu, j_0^Q]$, namely, we assume that it vanishes outside the light cone and has a singularity of the δ-function type at the vertex of the cone. [The quark model[7] or the model with the non-Abelian group of gauge transformations (see, for example, [8]) may serve to justify these assumptions.] This gives

$$[j_\mu(y),\ j_0{}^Q(x)]\,\delta(x_0-y_0)=Qj_\mu{}^Q(x)\,\delta(x-y). \qquad (4)$$

The Schwinger terms, which generally appear in commutation relations of space and time components, are irrelevant for our purposes, since they appear only in commutators with the same isotopic indices.

It is now sufficient to take the divergence of both sides of Eq. (2) and, making use of Eqs. (3) and (4), as well as the conservation of j_μ^Q, obtain an equality which will be valid for any matrix elements not containing photons:

$$\partial_\mu J_\mu{}^Q(x)=-\frac{1}{2}Qe^2\int d^4y\,(D^{\rm ret}(y-x)\,\{j_\nu(y),\,j_\nu{}^Q(x)\} + D^1(y-x)\,\theta(y_0-x_0)[j_\nu(y),\,j_\nu{}^Q(x)]), \qquad (5)$$

where $D^{\rm ret}$ and D^1 are the retarded photon function and the vacuum expectation of the anticommutator of the photon fields (see, for example, [9]). The same formula may be obtained from Eq. (1) by expanding J_μ^Q and A_μ on the right hand side in a series in e.

Let us consider now the question of relativistic invariance of (5). The invariance of the first term is obvious, and the invariance of the second term is based on the vanishing of the current commutator outside the light cone. However, matrix elements of products of currents entering the commutator and the anticommutator will be expanded in what follows in a complete set of intermediate states and then that series will be cut off. Whereas for the anticommutators such a procedure offers no danger, for the commutator such a cut-off will give rise to violation of local commutativity and, as a consequence, to the loss of relativistic invariance. In order to avoid this we shall make use of the Dyson representation[10] and represent the commutator in the form

$$\theta((x-y)^2)[j_\nu(y),\ j_\nu{}^Q(x)],$$

realizing that such a procedure of extracting $\theta((x-y)^2)$ can lead, as a consequence of lack of definition of the product of generalized functions, to a divergence, to the necessity for renormalization and, as a consequence, to the appearance of undetermined constants. However, as will become apparent from what follows, this indeterminacy is unimportant for our purposes.

We thus obtain as a result of the Dyson procedure a relativistically invariant factor.

$$\mathfrak{D}(y-x)=D^1(y-x)\,\theta(y_0-x_0)\,\theta((y-x)^2)=D^1(y-x)\,V_+(y-x) \qquad (6)$$

in front of the current commutator. We now transform Eq. (5), with conditions (6) taken into account, to the momentum representation. By proceeding in the usual fashion it is not hard to obtain

$$\langle \mathbf{p}', f|\mathscr{D}^Q(0)|\mathbf{p}, i\rangle = -\frac{e^2Q}{2}\int d^3q\sum_n\{D^{\rm ret}(q-p')[\langle \mathbf{p}', f|j_\nu(0)|\mathbf{q}, n\rangle\langle \mathbf{q}, n|j_\nu{}^Q(0)|\mathbf{p}, i\rangle + \langle \mathbf{p}', f|j_\nu{}^Q(0)|\mathbf{q}, n\rangle\langle \mathbf{q}, n|j_\nu(0)|\mathbf{p}, i\rangle] + \mathfrak{D}(q-p')\,[\langle \mathbf{p}', f|j_\nu(0)|\mathbf{q}, n\rangle\langle \mathbf{q}, n|j_\nu{}^Q(0)|\mathbf{p}, i\rangle - \langle \mathbf{p}', f|j_\nu{}^Q(0)|\mathbf{q}, n\rangle\langle \mathbf{q}, n|j_\nu(0)|\mathbf{p}, i\rangle]\}, \qquad (7)$$

where $\mathscr{D}^Q(0) = i\partial_\mu J_\mu^Q(0)$, and $D^{\rm ret}(k) = -(k^2 - i\varepsilon k_0)^{-1}$ (see [9]).

Let us proceed now to the calculation of $\mathfrak{D}(k)$. By simple integration it is not hard to find for the Fourier transform of the characteristic function of the upper cone V_+

$$V_+(p)=\int d^4x\,V_+(x)\,e^{ipx}=\frac{8\pi}{(p^2-ip_0\varepsilon)^2} = -8\pi\int_0^\infty d\alpha\,\alpha\exp[i(p^2-i\varepsilon p_0)\,\alpha],$$

and folding the resultant expression with the Fourier transform of D^1, we obtain

$$\mathfrak{D}(k)=\frac{1}{(2\pi)^4}\int d^4p\,D^1(p)\,V_+(k-p) = -\frac{i}{\pi(k^2-i\varepsilon k_0)^2}\int_0^\infty\frac{d\alpha}{\alpha}\exp[i\alpha(k^2-i\varepsilon k_0)]. \qquad (8)$$

However, the expression written by us is meaningless since the integral is divergent. In order to give to the function $\mathfrak{D}$ some sort of meaning we note that formally expression (8) satisfies the differential equation (here and in the following we omit instructions about how the pole is to be treated, since they are immaterial for us)

$$\frac{d\mathfrak{D}(k^2)}{dk^2}=-\frac{1}{k^2}\mathfrak{D}(k)-\frac{i}{\pi(k^2)^2},$$

whose solution is

$$\mathfrak{D}(k)=\frac{i}{\pi(k^2)^2}(\ln|k^2|+C),$$

where C is an arbitrary real constant. Such a procedure for redefining $\mathfrak{D}$, as is not hard to understand, is equivalent to the usual subtraction procedure. Thus as a result of the Dyson procedure an undetermined constant appears in the term with the anticommutator of the currents. However, as will be shortly seen, in the case of interest to us there arises a condition which makes it possible to eliminate this indeterminacy.

Let us pass now directly to the derivation of the equations for the pion form factor. To this end we take as the initial and final states in (7) one-pion states with charges s and s′, and in the sum over the intermediate states we also stick to one-pion states only. Then the matrix elements of the current on the right hand side can be replaced according to a well known formula by the form factor:

$$\langle \mathbf{p}', s'|j_\mu{}^Q(0)|\mathbf{p}, s\rangle=\frac{(p'+p)_\mu}{(2\pi)^3\sqrt{4p_0p_0'}}T_{s's}{}^QF(p, p'), \qquad (9)$$

where T^Q (Q = −, +, 0) are the isotopic matrices for the triplet. However, in the matrix element $\langle \mathbf{p}', s|\mathscr{D}^Q(0)|\mathbf{p}, s\rangle$ on the left hand side we cannot make use of isotopic invariance. In accordance with our approximation, we shall keep in that matrix element only terms of order e^2 and terms proportional to the mass difference. If one now makes use of the transformation properties of $\langle \mathbf{p}', s|\mathscr{D}^Q|\mathbf{p}, s\rangle$ with respect to the operations T, C, and complex conjugation, and also take into consideration the law of conservation of the electric current ($\mathscr{D}^0 = 0$) and the fact that the antiparticle of the π^+ meson is the π^- meson, then it is not hard to obtain

$$\langle \mathbf{p}' s'|\mathscr{D}^Q|\mathbf{p}, s\rangle = -\frac{Q}{(2\pi)^3\sqrt{4p_0p_0'}}(2m\Delta m\{T^0, T^Q\}_{s's}F(pp') + i\alpha[T^0, T^Q]_{s's}G(pp')), \qquad (10)$$

where $\Delta m = m_{\pi^\pm} - m_{\pi^0}$, and $\alpha = e^2/4\pi$ is the fine-structure constant; the functions F and G depend on a product of 4-vectors.

There is no difficulty in establishing the relation between the functions F and G and the invariant functions of the matrix element of J_μ^Q, which, accurate to terms whose contribution to the divergence is of order e^2, has the form

$$\langle \mathbf{p}', s|J_\mu^Q|\mathbf{p}, s\rangle = \frac{1}{(2\pi)^3\sqrt{4p_0p_0'}}(T^Q_{s's}(p+p')_\mu F(pp') + i\alpha|Q|T^Q_{s's}(p_\mu - p_\mu')\bar{G}(pp')).$$

From this we conclude that the function F(pp′) in Eq. (10) represents the mesonic form factor, and $G(m^2) = 0$.

Substituting now expressions (9) and (10) into Eq. (7) we see that on the right and left sides there appear two terms, which transform differently under rotations in isotopic space: one as the anticommutator $\{T^0, T^Q\}$, and the other as the commutator $[T^0, T^Q]$. Moreover, the first of them is purely real and the second purely imaginary. This makes it possible for us to equate such terms and obtain two equations:

$$F(pp') = \frac{\alpha}{32\pi^2 m\Delta m}\int\frac{d^3q}{q_0}\frac{(p+q)_\mu(p'+q)_\mu}{p'q - m^2}F(p'q)F(qp), \qquad (11)$$

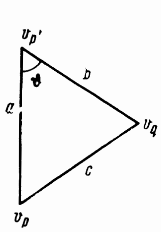
$$G(pp') = -\frac{1}{16\pi^3}\int\frac{d^3q}{q_0}\frac{(p+q)_\mu(p'+q)_\mu}{p'q-m^2}\left[\ln\left(\frac{p'q}{m^2}-1\right)+C\right]F(p'q)F(qp) \qquad (12)$$

with the additional conditions $G(m^2) = 0$ and $F(m^2) = 1$, the first of which makes it possible to determine the constant C.

In this fashion we have obtained a nonlinear integral equation for the pion form factor, whose study will be taken up in the following section. The second relation connects the radiative correction to the pion β decay with the form factor; it is not clear to us at this time whether any useful information can be extracted from it.

3. SOLUTION OF THE EQUATION FOR THE FORM FACTOR, MEAN-SQUARE RADIUS

Let us now consider the solution of Eq. (11). For this purpose it is convenient to transform to the velocity space (see, for example, [11]) which is the Lobachevski space in which the absolute velocity of the particle ($v_\mu = p_\mu/m$) is represented by a point and the scalar product of two momenta divided by the masses of the corresponding particles is represented by the hyperbolic cosine of the length of the arc between the points corresponding to the velocities of these particles. Thus, $pp'/m^2 = \cosh a$, $p'q/m^2 = \cosh b$, $pq/m^2 = \cosh c$, where the sides a, b, and c form a triangle in the Lobachevsky space (see the figure), i.e., $\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos$.



Rewriting Eq. (11) in terms of these variables we obtain after integration over the azimuth angle:

$$F(a) = \frac{\alpha m}{16\pi\Delta m}\int d\mu(b)\frac{\operatorname{ch} a + \operatorname{ch} b + \operatorname{ch} c + 1}{\operatorname{ch} b - 1}F(b)F(c) = \varepsilon'\{\Psi_1(a) + \Psi_2(a) + \Psi_3(a)\}, \qquad (13)$$

where $\varepsilon' = m/16\pi\Delta m$, $d\mu(b) = \sinh^2 b \sin\theta\, db\, d\theta$ is the invariant measure, and

$$\Psi_1(a) = \int d\mu(b)\frac{\operatorname{ch} b + 1}{\operatorname{ch} b - 1}F(b)F(c) = \int d\mu(b)(\Phi_1(b) + \Phi(b))F(c),$$

$$\Psi_2(a) = \int d\mu(b)\frac{\operatorname{ch} c}{\operatorname{ch} b - 1}F(b)F(c) = \int d\mu(b)\Phi(b)F_1(c),$$

$$\Psi_3'(a) = \frac{\Psi_3(a)}{\operatorname{ch} a} = \int d\mu(b)\frac{F(b)}{\operatorname{ch} b - 1}F(c) = \int d\mu(b)\Phi(b)F(c).$$

(The definition of the functions Φ, Φ_1, and F_1 is obvious from these equations.) We call attention to the fact that each of the functions Ψ represents a contraction over the group of motion in the Lobachevsky space, and it is therefore natural to expand in terms of "radial" eigenfunctions of this group[12, 13]

$$p_\rho(a) = \frac{P^{-1/2}_{-1/2+i\rho}(\operatorname{ch} a)}{\sqrt{\operatorname{sh} a}} = \sqrt{\frac{2}{\pi}}\frac{\sin\rho a}{\rho\operatorname{sh} a}. \qquad (14)$$

Thus, let

$$F(a) = \int_0^\infty f(\rho)p_\rho(a)\rho^2\,d\rho \qquad (15)$$

and conversely

$$f(\rho) = \int_0^\infty db\,F(b)p_\rho(b)\operatorname{sh}^2 b. \qquad (16)$$

These equations, and a special case of the "multiplication theorem" [13] for the functions p_ρ:

$$\int_0^\pi p_\rho(c)\sin\vartheta\,d\vartheta = \sqrt{2\pi}p_\rho(a)p_\rho(b),$$

where a, b, and c are the sides of the triangle shown in the figure, give us

$$\psi_1(\rho) = \sqrt{2\pi}[\varphi_1(\rho) + \varphi(\rho)]f(\rho),$$
$$\psi_2(\rho) = \sqrt{2\pi}\,\varphi(\rho)f_1(\rho), \qquad \psi_3'(\rho) = \sqrt{2\pi}\,\varphi(\rho)f(\rho). \qquad (17)$$

If in addition one makes use of the well-known identity[14]

$$p_\rho(a)\operatorname{ch} a = \frac{1}{2}\left(\frac{\rho+i}{\rho}p_{\rho+i}(a) + \frac{\rho-i}{\rho}p_{\rho-i}(a)\right),$$

then one may show that

$$\begin{pmatrix}\psi_3(\rho)\\ \varphi_1(\rho)\\ f_1(\rho)\end{pmatrix} = \frac{1}{2}\left[\frac{\rho+i}{\rho}\begin{pmatrix}\psi_3'(\rho+i)\\ \varphi(\rho+i)\\ f(\rho+i)\end{pmatrix} + \frac{\rho-i}{\rho}\begin{pmatrix}\psi'(\rho-i)\\ \varphi(\rho-i)\\ f(\rho-i)\end{pmatrix}\right], \qquad (18)$$

and

$$f(\rho) = \frac{1}{2}\left(\frac{\rho+i}{\rho}\varphi(\rho+i) + \frac{\rho-i}{\rho}\varphi(\rho-i)\right) - \varphi(\rho). \qquad (19)$$

If one now substitutes Eq. (17) and (18) into the obvious equality

$$f(\rho)=\psi_1(\rho)+\psi_2(\rho)+\psi_3(\rho),$$

one obtains in place of an integral equation the following functional equation:

$$2f(\rho)=\sqrt{2\pi}\,\varepsilon'\Big\{\frac{\rho+i}{\rho}[\varphi(\rho+i)+\varphi(\rho)][f(\rho+i)+f(\rho)]+\frac{\rho-i}{\rho}[\varphi(\rho-i)+\varphi(\rho)][f(\rho-i)+f(\rho)]\Big\}, \qquad (20)$$

to which it is moreover necessary to add Eq. (19) which relates φ and f.

It is seen that the resultant nonlinear functional system of equations is, unfortunately, no simpler than the initial integral equation as far as exact solutions are concerned. However, it has the advantage that it makes it possible to study various limiting cases, and in particular the case of large ρ, which according to Eqs. (14) and (15) corresponds to small momentum transfers.

Let us turn now to the asymptotic behavior of $f(\rho)$. A simple integration by parts of Eq. (16) leads us to the result that all the coefficients in the expansion in $1/\rho$ vanish. Therefore in order to determine the asymptotic behavior of $f(\rho)$ it is necessary to know the analytic properties of the function F(z) in the complex a-plane.[15] As is well known the pion form factor is analytic in the complex plane of $t = -2m^2(\cosh a - 1)$ except for the cut $[4m^2, \infty)$. In the a-plane this leads to singularities on the lines $\mathrm{Im}\, a = \pi \pm 2\pi n$, so that the maximum value through which one may shift the integration contour in the expression

$$f(\rho)=\sqrt{\frac{2}{\pi}}\frac{1}{\rho}\int_0^\infty F(a)\,\mathrm{sh}\,a\sin\rho a\,da=\sqrt{\frac{2}{\pi}}\frac{1}{2i\rho}\int_{-\infty}^{\infty}F(a)\,e^{i\rho a}\,\mathrm{sh}\,a\,da,$$

equals $i\pi$. This gives rise to the following asymptotic behavior as $\rho\to\infty$:

$$f(\rho)\sim O(e^{-\pi\rho}/\rho).$$

However, for the function

$$\varphi(\rho)=\sqrt{\frac{2}{\pi}}\frac{1}{\rho}\int_0^\infty\frac{F(a)\,\mathrm{sh}\,a}{\mathrm{ch}\,a-1}\sin\rho a\,da=\frac{1}{\sqrt{2\pi}i\rho}\int_{-\infty}^{\infty}\frac{F(a)\,\mathrm{sh}\,a}{\mathrm{ch}\,a-1}e^{i\rho a}da$$

the shifting of the contour of integration results in semirounding of the pole at the point a = 0, which gives rise to behavior of the type

$$\varphi(\rho)\sim\frac{\sqrt{2\pi}}{\rho}+O\Big(\frac{e^{-\pi\rho}}{\rho}\Big). \qquad (21)$$

Our approximation will consist in the following: After substituting Eq. (21) into Eq. (20), we discard the exponentially small terms and arrive at a linear equation for $f(\rho)$:

$$2\rho^2 f(\rho)=\varepsilon[(2\rho+i)f(\rho+i)+(2\rho-i)f(\rho-i)+4\rho f(\rho)], \qquad (22)$$

where $\varepsilon = 2\pi\varepsilon'$. In fact this ignoring of exponentially small terms is equivalent to the replacing of the function $\Phi(b) = F(b)/(\cosh b - 1)$ in the integral equation (12) by $2/b \sinh b$.

We look for a solution of Eq. (22) in the form

$$f(\rho)=\int_{-\infty}^{\infty}\eta(x)\,e^{-i\rho x}dx, \qquad (23)$$

where $\eta(x)$, as well as its first derivative, vanish as $x\to\infty$. As regards the limit $x\to-\infty$, in view of the reality of $f(\rho)$ the function $\eta(-x) = \eta^*(x)$ and, consequently, should also vanish. These boundary conditions follow immediately from the relation between η and F (see below) and the degree of decrease which is required by the integral equation for F in the above-mentioned approximation. Indeed, Eq. (15) may be rewritten in the form

$$F(a)=\sqrt{\frac{2}{\pi}}\frac{1}{\mathrm{sh}\,a}\frac{\partial}{\partial a}\int_{-\infty}^{\infty}d\rho\,e^{i\rho a}\frac{f(\rho)+f(-\rho)}{2}, \qquad (24)$$

from which, with Eq. (23) taken into account, it follows immediately that

$$F(a)=\frac{1}{\mathrm{sh}\,a}\frac{d}{da}\Big(\frac{\eta(a)+\eta(-a)}{2}\Big). \qquad (25)$$

(The numerical coefficient in front of the right hand side is immaterial for our purposes, since the function $\eta(a)$ is anyway only defined to within an arbitrary constant.)

In this fashion, one may conclude that

$$f(\rho)=\frac{1}{i\rho}\int_{-\infty}^{\infty}\eta'(x)\,e^{-i\rho x}dx=-\frac{1}{\rho^2}\int_{-\infty}^{\infty}\eta''(x)\,e^{-i\rho x}dx,$$

and upon substitution of the above into Eq. (22) arrive at the following differential equation for $\eta(x)$:

$$\eta''(x)-i\varepsilon[2\eta'(x)(\mathrm{ch}\,x+1)+\eta(x)\,\mathrm{sh}\,x]=0. \qquad (26)$$

Although no physical meaning can be attached to the asymptotic behavior of Eq. (26), since in accordance with our approximation its solutions can be believed only in the region of small x, it is nevertheless necessary to study its asymptotic behavior in order to fix the correct solution of Eq. (26) by an appropriate choice of one of the arbitrary constants. (The second arbitrary constant will be utilized for normalization of the form factor F.)

In order to choose the correct asymptotic solution and to understand its behavior in the region of small x, we make the substitution

$$\eta(x)=e^{i\varepsilon(\mathrm{sh}\,x+1)}\xi(x), \qquad (27)$$

since for the two linearly independent solutions of the equation for ξ

$$\xi''(x)+\varepsilon^2(\mathrm{ch}\,x+1)^2\xi(x) \qquad (28)$$

we may choose solutions with a definite parity under the replacement $x\to-x$:

$$\xi(x)=C_0\xi_e(x)+C_1\xi_o(x),$$

where $\xi_e(0) = 1$, and $\xi_o'(0) = 1$. With the substitution $\sigma = \sinh x$, it is not hard to see that for $x\to\infty$ the two linearly independent solutions have the form

$$\xi(x)\sim\frac{1}{\sqrt{\mathrm{ch}\,x}}\begin{cases}\sin(\varepsilon\,\mathrm{sh}\,x)\\ \cos(\varepsilon\,\mathrm{sh}\,x)\end{cases}$$

and taking into account the relation (27), we conclude that in order to ensure that $\eta'(x)$ decreases it is necessary to take

$$C_1=-iC_0. \qquad (29)$$

In this fashion the formulas (25), (27), and (28), the condition (29), and the normalization condition uniquely define the pionic form factor. However, in view of the fact that our approximation may be valid only for small x, as well as in view of the extremely poor and contradictory quality of experimental data on the pionic form factor, we shall confine ourselves to the calculation of the mean-square radius. This is simplest to do by substituting into Eq. (26)

$$\eta(x) = a_0 + a_1 x + a_2 \frac{x^2}{2!} + a_3 \frac{x^3}{3!} + a_4 \frac{x^4}{4!} \dots$$

and equating to zero the coefficients of equal powers of x, thus expressing a_n in terms of, say, a_0 and a_2, which may be taken as arbitrary. Making use of the substitution (27), it is not hard to establish that condition (29) now takes on the form $a_0 = a_2/4\varepsilon(1 - 2\varepsilon)$. With this condition taken into account

$$a_4 = \left(1 - 16\varepsilon^2 - \frac{\varepsilon}{1 - 2\varepsilon}\right) a_2,$$

and utilizing expression (26) we obtain the first two terms in the expansion of the form factor:

$$F(x) = 1 - \frac{x^2}{6}\left(1 - \frac{a_4}{a_2}\right),$$

from which we conclude that

$$\langle r \rangle = \left(\frac{\varepsilon}{1 - 2\varepsilon} + 16\varepsilon^2\right)^{1/2},$$

where $\varepsilon = \alpha m / 8\Delta m$. Upon substitution of numerical values $\alpha = 1/137$, m = 135 and $\Delta m = 4.6$ we obtain

$$\langle r \rangle = 0{,}23\, m^{-1} \approx 0{,}3 \text{ F}$$

4. CONCLUSION

The mean-square radius of the pion obtained in the previous section is based on three approximations. Let us discuss each of these in turn.

The first approximation consisted in discarding terms of order e^2 in the derivation of the expression for the divergence. The validity of this approximation is, apparently, not to be doubted, since a similar expansion has worked beautifully in quantum electrodynamics.

The situation is much more complex as regards the second approximation—discarding states other than the one-particle states—in the expansion of the product of currents in a complete set. Strictly speaking, this approximation has no justification even of the "nearest singularity" or "saturation condition" type. Apparently, the only justification is that one sets $pp' = m^2$ in Eq. (11), then one arrives at the formula which relates the mass difference to the form factor. If one substitutes into this formula the pionic form factor in the ρ-meson approximation, then a mass difference is obtained in satisfactory agreement with experiment. This makes it possible to hope that a reasonable value for the form factor will be obtained also when one moves a small distance away from the point $pp' = m^2$. The same is indicated by a comparison of the radius obtained by us with the "ρ-mesonic" pion radius, which, as is not hard to calculate, equals 0.46 F. Unfortunately, no further comparisons are possible since the existing experimental data[16] are, first, not sufficiently accurate and, second, as a rule, are related to assumptions of theoretical character of the type of the "pole diagram of Chew and Low."

The question naturally arises of testing the correctness of the similar equation on the nucleon form factor, which is now well known. However, the utilization of only the law of violation of isotopic invariance in the form (1), for example, does not allow one to write closed equations for the nucleon form factors, if for no other reason than because the right-hand side of such an equation contains the scalar form factor. To close the system of equations we need additional ones, which we do not know how to find (this, of course, provided we make no assumption of "similar" behavior for all nucleon form factors). Thus the only possibility for testing the validity of the one-meson approximation at this time consists in taking into account the three-meson states, even if only in the form of the ω meson.

Finally, the third approximation—discarding exponentially small terms in ρ as $\rho \to \infty$ (which corresponds to small momentum transfers)—may be easily overcome with the help of an electronic-computer numerical solution of the initial integral equation, taking as the starting point the solution obtained in this paper. However, this seems to us premature until the question of the validity of the second approximation is clarified.

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[1] Riazuddin, Phys. Rev. **114**, 1184 (1959).

[2] L. D. Soloviev and A. V. Shchelkachev, Nucl. Phys. **76**, 684 (1966).

[3] M. V. Terent'ev, Nonconservation of the Vector Current and the Universality of the Electromagnetic Interaction, ITÉF Preprint No. 463 (1966).

[4] M. Veltman, Phys. Rev. Letters **17**, 553 (1966).

[5] C. Bouchiat, G. Flamand, and J. M. Kaplan, A Derivation of Veltman's Equation of Classical Field Theory, Preprint (1966).

[6] S. L. Adler, Phys. Rev. **139**, B1638 (1965).

[7] M. Gell-Mann, Physics **1**, 63 (1964).

[8] L. S. Grown, Gauge Invariance, Lorentz Covariance and Current Correlation Functions, Preprint (1966).

[9] N. N. Bogolyubov and D. V. Shirkov, Vvedenie v teoriyu kvantovykh poleĭ, Gostekhizdat, 1957 [Introduction to the Theory of Quantized Fields, Wiley, 1959].

[10] F. Dyson, Phys. Rev. **110**, 1460 (1958) (Russ. Transl. PSF 1, 1959).

[11] N. A. Cherinkov, Mezhdunarodnaya zimnyaya shkola po teoreticheskoĭ fizike (International Winter School in Theoretical Physics), Dubna, **3**, 151 (1964); Nauchn. dokl. vyssh. shk., ser. fiz.-mat. nauk **2**, 158 (1958).

[12] N. Ya. Vilenkin, Ya. A. Smorodinskiĭ, Preprint JINR, Dubna, E-1503, (1964).

[13] N. Ya. Vilenkin, Spetsialnye funktsii i teoriya predstavleniĭ grupp (Special Functions and the Theory of Group Representations) Fizmatgiz, 1965.

[14] I. M. Ryzhik and I. S. Gradshteĭn, Tablitsy integralov, summ, ryadov i proizvedeniĭ (Tables of Integrals, Sums, Series, and Products) Gostekhizdat, 1951, p. 384 [transl. Academic Press, 1966].

[15] M. A. Fevgrafov, Asimptoticheskie otsenki i tselye funktsii (Asymptotic Estimates and Entire Functions), Fizmatgiz, 1963, p. 28.

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83