DISTRIBUTION OF CAPTURED PARTICLES IN A POTENTIAL WELL IN THE ABSENCE OF COLLISIONS

A. V. GUREVICH

P. N. Lebedev Physical Institute, Academy of Sciences, U.S.S.R.

Submitted February 17, 1967

Zh. Eksp. Teor. Fiz. **53**, 953–964 (September, 1967)

The collisionless capture of electrons in a region of finite motion owing to a nonstationarity of a longitudinal electric field is investigated. A kinetic equation is formulated which describes the distribution function of the captured particles for a slow (adiabatic) variation of the field. Particular solutions of this equation are obtained and analyzed. Capture in a rapidly increasing field is also considered. It is shown that the concentration of the captured electrons increases with increasing depth of the potential well no more rapidly than $(e\varphi/T)^{1/2}$, i.e., much more slowly than in the case of a Boltzmann equilibrium distribution. However, for $(e\varphi/T) \gg 1$ the concentration of the captured particles is always greater than that of the uncaptured ones. These results are applied in an analysis of nonstationary motions in a plasma. A general kinetic equation is obtained which describes the self-similar motion of a rarefied plasma. The structure of a solitary wave (soliton) is investigated, taking account of the collisionless electron capture during the build-up of the field.

1. INTRODUCTION

PARTICLES captured into a region of finite motion in a longitudinal electric field have an important, and often a decisive, influence on the nonlinear motions in a plasma, Bernstein, Green, and Kruskal [1] have shown that by varying the number of particles captured by the field of the wave, one can construct stationary nonlinear longitudinal waves of arbitrary form, amplitude, and velocity. In the case of a solitary wave, considered by Sagdeev,[2] the number of captured electrons determines not only the properties of the wave but also the very possibility of its existence.[1)]

The captured particles are of great importance for the relatively slow motions of a rarefied plasma with $v \ll (T_e/m)^{1/2}$. This is immediately seen from the equations

$$\frac{\partial f_i}{\partial t} + v\frac{\partial f_i}{\partial \mathbf{r}} - \frac{e}{M}\frac{\partial \varphi}{\partial \mathbf{r}}\frac{\partial f_i}{\partial \mathbf{v}} = 0, \tag{1}$$

$$\int f_i d^3v = N_{inf}(\varphi) + N_{fin}(\varphi), \tag{2}$$

which describe the motions of a collisionless plasma whose characteristic velocity is much smaller than the thermal velocity of the electrons and whose characteristic spatial dimension is much larger than the Debye radius. Here f_i is the distribution function of the ions and N_{inf} and N_{fin} are the concentrations of the electrons carrying out infinite and finite motions in the electric field φ, respectively.[2)] In onedimensional problems the concentration of the electrons with infinite motion decreases proportional to $(e\varphi/T)^{1/2}$ with increasing potential of the field φ (we assume that $\varphi \to 0$ for $r \to \infty$). Therefore the particles captured by the field at large values of φ constitute the majority. Hence they determine the potential of the field and the character of its effect on the motion of the plasma. The captured particles also play a very important role for small values of φ.[1,5]

The region of finite motion is clearly defined if the energy of the particles is conserved. The energy distribution of the particles carrying out an infinite motion is then determined by the conditions at infinity, whereas the distribution of the particles with finite motion is arbitrary. Actually, the energy is not conserved owing to collisions and nonstationarities of the field. The collisions lead to a diffusion of the particles in energy space. Therefore the regions of infinite and finite motion are effectively mixed. The characteristic time for such a mixing [5,6] is $\tau \sim e\varphi_m/\nu T$, where ν is the collision frequency, and φ_m is the maximal value of the potential. Let the time $\tau_1 = 1/\gamma$ characterize the nonstationary process (for example, γ may be the increment of the damping decrement of the wave). The regions of infinite and finite motion of the particles are effectively mixed when $\tau_1 \gg \tau$. This condition restricts the amplitude of the field:

$$\varphi_m \ll \frac{T}{e}\frac{\nu}{\gamma}. \tag{3}$$

For sufficiently rare collisions the ratio ν/γ is very small, and then the restriction (3) is very strong.

If the condition (3) is not satisfied, the collisions cannot have an important effect on the distribution of the particles in the region of finite motion. Then the basic role is played by the change of energy of the particle caused by the nonstationarities of the electric field. This process also leads to the capture of particles with infinite motion (we shall call it collisionless capture) and to a change of the distribution function of the captured particles. The study of the collisionless

---

[1)] For small numbers of the captured particles, a solution of the form of a solitary wave cannot exist. This is easily seen by considering an arbitrary distribution function for the captured electrons with respect to the energies and seeking a solution in the form of a solitary wave, as in [2].

[2)] In [3,4] the system of equations (1) was considered for the case of a monotonically decreasing negative potential, with $N_{fin} = 0$, $N_{inf} = N_0 \exp(e\varphi/T_e)$.

capture in a nonstationary field is the subject of the present paper.

Slowly Varying Field

Let us assume that a gas of non-interacting electrons is located in a nonstationary longitudinal field with the potential $\varphi(x, t)$ ($\varphi \to 0$ for $|x| \to \infty$).[3] We assume at first that the field varies slowly, or more precisely, that the potential of the field varies little during a period of vibration of an electron carrying out a finite motion:

$$\frac{\gamma}{\omega} = \frac{1}{\omega}\left|\frac{1}{\varphi}\frac{\partial \varphi}{\partial t}\right| \ll 1. \tag{4}$$

Here

$$\omega = \frac{\pi\sqrt{2}}{\sqrt{m}}\left(\int \frac{dx}{\sqrt{E + e\varphi}}\right)^{-1} \tag{5}$$

is the frequency of the vibrations of the captured electron, and E is its energy. The integration in (5) goes over all values of x for which $E + e\varphi \geq 0$.

With the condition (4), the distribution function of the captured particles can be sought in the form of a series in powers of the small parameter γ/ω. In first approximation the average distribution function then depends only on the energy of the particle E and on the time t: $f = f(E, t)$. The equation for the function f_1 (the index 1 will be omitted in the following) is obtained by averaging the kinetic equation over a time τ which is large compared to $1/\omega$ and small compared to the characteristic time of variation of the field $1/\gamma$:

$$\left\langle \frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{e}{m}\frac{\partial \varphi}{\partial x}\frac{\partial f}{\partial v}\right\rangle_\tau = 0, \tag{6}$$

where $\langle \ldots \rangle_t$ denotes the average over the time τ. Using in this equation $f = f(E, t)$ and $E = mv^2/2 - e\varphi(x, t)$, we obtain instead of (6)

$$\left\langle \frac{\partial f}{\partial t} - e\frac{\partial \varphi}{\partial t}\frac{\partial f}{\partial E}\right\rangle_\tau = 0. \tag{7}$$

The average over the time τ can be replaced by the average over a microcanonical ensemble. Multiplying (7) by $\delta(E - mv^2/2 + e\varphi)$ and integrating over the whole phase volume $d\Gamma = dxdv$, we therefore find

$$\frac{\partial f}{\partial t} + Q(E, t)\frac{\partial f}{\partial E} = 0. \tag{8}$$

Here $Q(E, t)$ is the average change of the energy of the captured particle per unit time due to the nonstationarities of the field:

$$Q(E, t) = -e\int \frac{\partial \varphi}{\partial t}\frac{dx}{\sqrt{E + e\varphi}}\Big/ \int \frac{dx}{\sqrt{E + e\varphi}}. \tag{9}$$

The characteristics of the kinetic equation (8) are determined by the equation

$$dE/dt = Q(E, t). \tag{10}$$

It is easy to see that the adiabatic invariant [7]

$$I(E, t) = \int p\, dx = \sqrt{2m}\int \sqrt{E + e\varphi}\, dx \tag{11}$$

is an integral of the motion for the equation of the characteristics (10). Hence the solution of the kinetic equation (8) is a function of the adiabatic invariant (11):

$$f = f(I[E, t]).$$

The specific form of the function $f(I)$ is determined by the boundary and initial conditions.

The boundary conditions for Eq. (8) are given on the boundary $E = e\varphi_k$ separating the regions of infinite and finite motion. We assume for simplicity that the region of finite motion consists of a single interval [then φ_k is the smaller of the two maxima of the function $\varphi(x, t)$ bounding the region under consideration] and that the boundary φ_k is constant in time. Then the total current of particles from the region of infinite motion to the region of finite motion through the boundary $E = e\varphi_k$ is equal to

$$S_{inf} = -\int_a^b v f_{inf}(v)\, dv - \int_{-b}^{-a} v f_{inf}(v)\, dv = -\frac{f_+ + f_-}{m}E_0, \tag{12}$$

$$a = (2e\varphi_k/m)^{1/2}, \quad b = [2(e\varphi_k + E_0)/m]^{1/2},$$

$$f_+ = f_0(\sqrt{2e\varphi_k/m}), \quad f_- = f_0(-\sqrt{2e\varphi_k/m}).$$

Here f_{inf} is the distribution function for the particles carrying out an infinite motion. As in a stationary field, the function f_{inf} is determined by the conditions at infinity (or on the surfaces enclosing the plasma). The field at infinity is zero, so that $f_{inf}(x \to \infty) = f_0(v)$. Then for arbitrary values of x

$$f_{inf} = f_0(\sqrt{2E/m}).$$

It is essential that one value of the energy E corresponds to two values of the function f_{inf}: $f_0(+\sqrt{2E/m})$ and $f_0(-\sqrt{2E/m})$. One of these describes the particles coming, from the left, the other describes the particles coming from the right. The two terms in the expression (12) for S_{inf} give the corresponding currents of the captured particles. If the heights of the maxima bounding the region of finite motion are not the same, then $f_+ \neq f_-$. Furthermore, $e\varphi_k$ in (12) is the minimal, and $e\varphi_k + E_0$ the maximal energy of the particles of infinite motion captured by the field. The energy E_0 is proportional to $\partial\varphi/\varphi t$; it is small. This allowed us to take the function f_0 outside the integral in the last expression (12).

For the calculation of E_0 we consider the change of energy of a particle with infinite motion:

$$\frac{dE}{dt} = \frac{d}{dt}\left\{\frac{m}{2}\left(\frac{dx}{dt}\right)^2 - e\varphi(x, t)\right\} = -e\frac{\partial \varphi}{\partial t}. \tag{13}$$

The energy E_0 is equal to the total change of the energy of the particle with infinite motion with $E = e\varphi_k$ during its motion between the maxima of the function $-\varphi(x, t)$. Taking account of (13), we find

$$E_0 = -e\int \frac{\partial \varphi}{\partial t}dt = -\sqrt{\frac{em}{2}}\int \frac{\partial \varphi/\partial t}{\sqrt{\varphi_k + \varphi}}dx. \tag{14}$$

The last expression has been obtained under the assumption that the integral converges.

Thus the total current of the particles of infinite motion captured by the field is determined by the expressions (12) and (14). It can also be written in the form

$$S_{inf} = \frac{f_+ + f_-}{m}\frac{dI_k}{dt}, \tag{15}$$

[3] In this section, the electric field is regarded as given. It can be external or self-consistent, but the reverse effect of the particles on the field is not taken into account. The fluctuations of the field and pair collisions are neglected. The results of this section are valid not only for electrons but also for ions in an electric field and for particles in a gravitational field.

where I_k is the adiabatic invariant (11) at the boundary to the region of infinite motion:

$$I_k = \sqrt{2me}\int\sqrt{\varphi_k+\varphi}\,dx. \tag{16}$$

The current of particles in the region of finite motion is according to (8),

$$S_{fin} = -Qf_{fin}\int\delta\left(E-\frac{mv^2}{2}+e\varphi\right)dx\,dv$$

$$= \sqrt{\frac{2}{m}}ef_{fin}\int\frac{\partial\varphi}{\partial t}\frac{\partial x}{\sqrt{E+e\varphi}}.$$

For $E \to e\varphi_k$ the two currents must be equal. This leads to the boundary condition for the function f_{fin}:

$$f_{fin}\Big|_{E=e\varphi_k} = {}^1/_2(f_+ + f_-). \tag{17}$$

If $f_+ = f_-$, i.e., $f_{inf} = f_{inf}(E)$ for $E = e\varphi_k$, then the boundary condition (17) is the condition of continuity for the distribution function on the boundary between the regions of infinite and finite motion. The expressions (15) and (16) for the number of captured particles and the boundary condition (17) are not altered even in the case when the boundary φ_k varies with time. This is understandable: a change in the boundary does not give rise to an immediate change in the current of the particles in energy space; only the volume of the phase space taken up by the particles of finite motion changes.

Besides the boundary condition (17), the equation (8) must further be supplemented by an initial condition for the distribution function in the region of finite motion:

$$f(0, E) = f_0(E). \tag{18}$$

Let us now consider some solutions of the kinetic equation (8). If the potential well is absent or very small (i.e., has a small depth or a small spatial extension) at the initial moment, then the initial distribution function does not play an important role in the problem. In this case the distribution function of the captured electrons at any time is determined by the boundary values (17) at previous times. If moreover the boundary value (17) does not change with time, then the function f_{fin} is stationary in the entire region of finite motion and is equal to $f_{fin}\,|\,E = e\varphi$:

$$f_{fin}(E, t) = {}^1/_2(f_+ + f_-). \tag{19}$$

This simple solution is of interest for a number of problems in plasma physics (solitary waves, self-similar motion); it will be used below.

Let us now consider another case—the increasing wave. The boundary of the region of finite motion φ_k is then determined by the amplitude of the wave φ_m. The boundary value for a Maxwellian distribution of the particles with infinite motion at the temperature T is according to (17)

$$f|_{E=e\varphi_k} = \sqrt{m/2\pi T}N_0\exp\{-e\varphi_m/T\}. \tag{20}$$

The energy of the captured particles varies within the limits $-e\varphi_m \le E \le e\varphi_m$. Let us assume that the potential is $\varphi(x, t) = \varphi_m(t)\psi(x)$. Then, according to (9)

$$Q(E, t) = e\frac{d\varphi_m}{dt}\left[y - \int\sqrt{y-\psi(x)}\,dx\Big/\int\frac{dx}{\sqrt{y-\psi(x)}}\right] = e\frac{d\varphi_m}{dt}[y - Q_1(y)].$$

In Eq. (8) it is convenient to replace the variable E and t by the new variables $y = E/e\varphi_m(t)$ and $\varphi_m(t)$. Then it takes the form

$$\varphi_m\frac{\partial f}{\partial\varphi_m} - Q_1(y)\frac{\partial f}{\partial y} = 0. \tag{21}$$

The solution of (21) with the boundary condition (20) is

$$f(\varphi_m, y) = N_0\sqrt{\frac{m}{2\pi T}}\exp\left\{-\frac{e\varphi_m}{T}\exp\left[\int_1\frac{dy}{Q_1(y)}\right]\right\}.$$

Rapidly Increasing Field

We have assumed above that the potential of the field increases slowly. Let us now consider the opposite case of a rapidly changing field. Let the potential φ increase during the time t_0 in some limited region of x values. The velocity of the electron before the turning-on of the field is v_0. If

$$\int_0^{t_0} e\frac{\partial\varphi}{\partial x}dt \ll mv_0, \quad t_0v_0\left|\frac{1}{\varphi}\frac{\partial\varphi}{\partial x}\right| \ll 1 \tag{22}$$

the velocity and the coordinate of the particle change slowly during the time when the field is present. The kinetic energy of the electron does not change during this time, but the potential energy increases. If $E = mv_0^2/2 - e\varphi(x_0, t_0) < e\varphi_k$, where φ_k is, as before, the smaller of the maxima of the function, the particle is captured by the field.

The distribution function of the captured electrons during the time when the field is present cannot change very much—it coincides with the initial distribution function in the unperturbed plasma. In the following time it changes, approaching a stationary distribution which depends only on the energy E. The latter is easily obtained by averaging the initial distribution function $f_0(v)$ over a surface of constant energy. In particular, we find for a Maxwellian function $f_0(v)$

$$f_{fin}(E) = \frac{mN_0\omega}{2\pi^{3/2}T^{1/2}}e^{-E/T}\int\frac{e^{-e\varphi/T}dx}{\sqrt{E+e\varphi}}, \tag{23}$$

where $\omega = \omega(E)$ is the frequency of oscillations of the captured electron (5).

Concentration of the Captured Particles

If the distribution function of the captured particles depends only on their energy, then the concentration is equal to

$$N_{fin}(\varphi) = \sqrt{\frac{2}{m}}\int_{-e\varphi}^{e\varphi_k}\frac{f_{fin}(E)\,dE}{\sqrt{E+e\varphi}}. \tag{24}$$

According to (19), (23), and (21), the distribution function f_{fin} remains equal to the value f_{inf} at $E = e\varphi_k$, or decreases in comparison to it, as the energy changes. This means that the concentration of the captured electrons increases no more rapidly than $\sqrt{|\varphi|}$ as the amplitude of the potential φ increases, i.e., it increases more slowly than is prescribed by an equilibrium (Boltzmann) distribution law. However, for $e|\varphi|/T \gg 1$ the concentration of the captured particles is much larger than that of the particles with infinite motion. The difference in the total concentration for slow and rapid switching-on of the field is small for periodic potentials. In the case of potentials of the form of solitary waves and at large values of

$e\varphi_m/T$, the concentration of the captured particles is much larger if the field is switched on slowly than when it is switched on rapidly.

The Mixing Process. The Change of the Entropy

We have assumed above that in the course of time, the distribution function of the captured particles approaches an averaged function which depends only on the energy. Let us consider this process in more detail. First we assume that the field $\varphi(x)$ is stationary. Then the distribution function averaged over a surface of constant energy is independent of the time:

$$f(E)=\frac{\sqrt{m}}{\sqrt{2}}\frac{\omega}{\pi}\int\frac{f(\sqrt{2(E+e\varphi)/m},x,t)}{\sqrt{E+e\varphi}}dx$$

$$=\frac{\sqrt{m}}{\sqrt{2}}\frac{\omega}{\pi}\int\frac{f(\sqrt{2(E+e\varphi)/m},x,0)}{\sqrt{E+e\varphi}}dx.$$

Here $f(v, x, t)$ is the distribution function at the time t, $f(v, x, 0)$ is the initial distribution function, and ω is the frequency of the oscillations of the captured particle (5).

Let us consider the difference

$$\Delta f(E, x, t)=f(\sqrt{2(E+e\varphi)/m}, x, t)-f(E).$$

For a given energy E the function Δf changes periodically in time with the frequency ω. In the absence of collisions these oscillations do not change their character in the course of time. It is essential, however, that the frequency of the oscillations depends on the energy E. Owing to this the function Δf also becomes oscillating with respect to the variable E as the time goes along. The frequency of the oscillations of the function Δf with respect to E increases with time.

Indeed, let us consider the function Δf for two close-lying energy levels E_1 and E_2. It oscillates at each of these levels with the nearly equal frequencies $\omega_1 = \omega(E_1)$ and $\omega_2 = \omega(E_2)$. Let us assume that Δf was a smooth function of the energy at the initial moment, i.e., that for given x it stayed almost constant in the transition from E_1 to E_2. However, owing to the difference in the frequencies ω_1 and ω_2 the phase difference between the oscillations at the levels E_1 and E_2 increases in the course of time. Therefore the difference between the values of $\Delta f(E_1)$ and $\Delta f(E_2)$ at one and the same point x increases. After the time $t_1 = \pi|\omega_1-\omega_2|^{-1} = \pi[(E_1-E_2)\partial\omega/\partial E]^{-1}$ the oscillations at the levels E_1 and E_2 have opposite phase, and the difference between the values of Δf is maximal. After the time $t_2 = 2\pi[(E_1-E_2)\partial\omega/\partial E]^{-1}$ the oscillations are again in phase. It follows from this that the frequency of the oscillations of the function Δf with respect to energy is $q_E = t\partial\omega/\partial E$. It increases linearly with time.

The increase of the frequency of the oscillations of the function with respect to E causes it to decrease. Indeed, let us take account of the fact that the efficiency of the diffusion in energy space due to the Coulomb collisions is proportional to the second derivative of the distribution function with respect to the energy. Since $\partial^2\Delta f/\partial E^2 \sim q_E^2\Delta f$, the smoothing of the oscillations of the distribution function owing to the collisions is p times faster than the change of the function $f(E)$, where

$$p\sim\left(E\frac{\partial\omega}{\partial E}t\right)^2\sim(\omega t)^2.$$

It follows from this that the function $f(v, x, t)$ relaxes to the average (smoothed) function $f(E)$ for $\omega t \gg 1$. Although we have been dealing only with a stationary field $\varphi(x)$, the same considerations hold also for a slowly (adiabatically) varying field.

The smoothing of the distribution function of the captured particles is accompanied by a change in the entropy s:

$$\Delta s=\int[f(v,x,t)\ln f(v,x,t)-f(E)\ln f(E)]\,dv\,dx. \tag{25}$$

It is easy to show that $\Delta s > 0$, i.e., the entropy increases during the mixing, as it should. It follows from (15), (17), and (25) that in the case of adiabatic capture, the change of the entropy is described by the expression

$$\frac{ds}{dt}=\frac{1}{m}\frac{dI_k}{dt}\left[f_+\ln f_+ + f_-\ln f_- -(f_++f_-)\ln\frac{f_++f_-}{2}\right]. \tag{26}$$

Here I_k is the adiabatic invariant on the boundary of the region of finite motion (16), and f_+ and f_- are the values of the distribution function of the particles with infinite motion coming from the left and from the right on the same boundary (12). The first and second terms in (26) describe the entropy before the capture, and the third term describes the entropy after the capture and mixing. If the values of f_+ and f_- do not differ strongly then

$$f_+-f_-=2\sqrt{\frac{2e\varphi_k}{m}}\frac{\partial f_0}{\partial v}(0)$$

and expression (26) simplifies:

$$\frac{ds}{dt}=\frac{2e\varphi_k}{m^2}\frac{dI_k}{dt}\frac{1}{f_0(0)}\left[\frac{df_0}{dv}(0)\right]^2. \tag{27}$$

The distribution function of the "infinite" electrons $f_0(v)$ is here defined in the coordinate system connected with the potential well. The change of the entropy is according to (27) proportional to the 3/2 power of the amplitude of the field.

During the process of mixing, the energy of the captured electrons may also change. Let, for example, u be the velocity of the potential well in the coordinate system moving with the average velocity of the electrons. In this coordinate system the change of the energy of the electrons during the mixing in the case of adiabatic capture is described by the expression

$$\frac{dW}{dt}=-\frac{1}{m}\frac{dI_k}{dt}\left[f_+\frac{m(u+\sqrt{2e\varphi_k/m})^2}{2}\right.$$

$$\left.+f_-\frac{m(u-\sqrt{2e\varphi_k/m})^2}{2}-\left(\frac{mu^2}{2}+e\varphi_k\right)(f_++f_-)\right]. \tag{28}$$

The first and second terms in the square brackets give the energy of the particles before the capture, and the third term is the energy after the capture and mixing.

If the values of f_+ and f_- do not differ strongly, then

$$\frac{dW}{dt}=-\frac{4e\varphi_k u}{m}\left(\frac{\partial f_0}{\partial v}\right)_u\frac{dI_k}{dt}. \tag{29}$$

It is seen from this that for $(\partial f_0/\partial v)_u < 0$ the energy of the electrons increases. This energy is transferred from the field, so that the field must be damped. For $(\partial f_0/\partial v)_u > 0$, on the other hand, the energy is transferred from the electrons to the field, which therefore increases. This process of particle-field interaction

is thus analogous to Landau damping. In the case of a stationary sinosoidal field it has been investigated in

3. ONE-DIMENSIONAL SELF-SIMILAR FLOW OF A RAREFIED PLASMA

As an application of the results obtained above, we consider first the one-dimensional self-similar motions of a collisionless plasma. An example for such a motion is the problem of the flow-out of a plasma into the vacuum. It has been considered by Pariĭskaya, Pitaevskiĭ, and the author.[9] The kinetic equation for the distribution function of the ions (1) in the case of a self-similar motion is written in the form

$$(u-\tau)\frac{\partial f_i}{\partial \tau}-\frac{e}{2T_i}\frac{d\varphi}{d\tau}\frac{\partial f_i}{\partial u}=0. \tag{30}$$

Here $u = v(2T_i/M)^{-1/2}$ and $\tau = x/t(2T_i/M)^{1/2}$ are the dimensionless velocity and coordinate, and φ is the potential of the electric field. Expressing $d\varphi/d\tau$ through the concentration of the ions with the help of the equation of quasineutrality (2), we obtain a closed nonlinear equation for $f_i(u, \tau)$. For a monotonically decreasing potential of the field the distribution of the electrons is Boltzmann-like. In this case, which is realized, for example, in the problem considered in [9], the expression for the derivative $d\varphi/d\tau$ has the simple form

$$\frac{d\varphi}{d\tau}=\frac{T_e}{eN}\frac{dN}{d\tau}, \quad N=\int f_i du. \tag{31}$$

A wide class of initial value problems leads, however, to a nonmonotonic behavior of the concentration and hence, of the field potential. For example, in the case of a plasma current with the average initial velocity u_0 flowing into a plasma at rest, a compaction occurs in the region of values τ such that $u_0 \gtrsim \tau \gtrsim 0$. The behavior of the concentration $N(\tau)$ in this case is shown qualitatively in Fig. 1.

In the same figure we have indicated the corresponding behavior of the potential $\varphi(\tau)$. In the region of compaction the potential is positive. Here the potential well for the electrons is formed. The well broadens slowly [in the sense of the criterion (4)] with time. The boundary value of the potential φ_k and the corresponding value of the distribution function of the "infinite" electrons do not change during the course of time. Hence the distribution function of the captured electrons is determined by expression (19).

For the concentration of the electrons in the region of the potential well we obtain according to (19) and (24)

$$N=N_{inf}+N_{fin}=N_k e^{\varphi^*}[1-\Phi(\sqrt{\varphi^*})]+N_k\frac{2\sqrt{\varphi^*}}{\sqrt{\pi}},$$

$$\Phi(x)=\frac{2}{\sqrt{\pi}}\int_0 e^{-t^2}dt, \tag{32}$$

where $\varphi^* = e(\varphi - \varphi_k)/T_e$, and N_k is the value of the concentration for $\varphi = \varphi_k$, i.e., on the boundary of the potential well. For example, in the case shown in Fig. 1, $N_k = N_{\tau \to -\infty}$. From this we find for $d\varphi/d\tau$:

$$\frac{d\varphi}{d\tau}=\frac{T_e}{eN_{inf}}\frac{dN}{d\tau}. \tag{33}$$

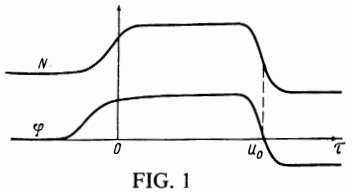

FIG. 1

Thus the general equation describing the one-dimensional self-similar motion of a rarefied plasma has the form

$$(u-\tau)\frac{\partial f_i}{\partial \tau}-\frac{T_e}{2T_i N_{inf}}\frac{dN}{d\tau}\frac{\partial f_i}{\partial u}=0, \quad N=\int f_i du, \tag{34}$$

where N is the total concentration, and N_{inf} is the concentration of the "infinite" electrons. In the region of monotonic behavior $N_{inf} = N$, and in the region of the potential well the dependence of N_{inf} on N is given implicitly by (32). Numerically, it is the following:

φ^*:	0	0.10	0.25	0.50	1.00	1.50	2.00	2.50	3.00
N/N_k:	1	1.081	1.203	1.322	1.558	1.755	1.936	2.095	2.250
N_{inf}/N_k:	1	0.724	0.639	0.524	0.428	0.372	0.336	0.310	0.293
N/N_{inf}:	1	1.49	1.88	2.52	3.64	4.72	5.76	6.76	7.68

For $N/N_k \to 1$ we have

$$\frac{N_{inf}}{N_k}=1-\frac{2}{\sqrt{\pi}}\sqrt{\frac{N}{N_k}-1},$$

and for $N/N_k \gg 1$

$$\frac{N_{inf}}{N_k}=\frac{2}{\pi}\frac{N_k}{N}.$$

It is seen from a comparison of formulas (31) and (33) that the influence of the electric field on the motion of the ions is increased by the factor N/N_{inf} in the region of the potential well. This value has been quoted above.

4. SOLITARY WAVES IN A COLLISIONLESS PLASMA

Let us consider a plasma with cold ions, $T_i/T_e \to 0$. The solitary waves in such a plasma have been investigated by Sagdeev.[2] He assumed that the distribution of the electrons in the field of the wave has the equilibrium (Boltzmann) form. However, in a collisionless plasma the distribution of the captured electrons may differ very much from an equilibrium distribution, which influences the properties of the wave. The number of captured electrons depends on the character of the increase of the wave. We must therefore analyze the process of increase of the wave.

The velocity of the solitary waves is of the order of the velocity of ionic sound, and their spatial extension is of the order of the Debye radius. Therefore the characteristic time for their variation $1/\gamma$ is of the order $1/\Omega_0$, where Ω_0 is the ion plasma frequency. The frequency of the oscillations of the electrons in the field of the wave ω is of the order of the plasma frequency for electrons. Hence

$$\omega/\gamma \gtrsim \sqrt{M/m} \gg 1. \tag{35}$$

This relation is fulfilled for any field amplitude. Therefore the variation of the solitary wave is always a process which is slow compared to the velocity of the captured electrons [cf. formula (4)]. For sufficiently rare collisions, when the inverse of condition (3) holds, collisionless capture of the electrons occurs in the field of a wave described by Eq. (8).

The boundary of the region of finite motion is $E = 0$. According to (19) the distribution function of the

captured electrons is in this case equal to

$$f_{fin} = N_0\sqrt{m/2\pi T}. \tag{36}$$

Here we have neglected the velocity of the solitary wave in comparison with the thermal velocity of the electrons. The change of the solitary wave is, of course, a process which is slow also compared to the time a stationary distribution of the "infinite" electrons. Hence the field of the solitary wave is at each moment described by the stationary Poisson equation:

$$d^2\varphi/dx^2 = -4\pi e(N_i - N_e). \tag{37}$$

Here N_i and N_e are the concentrations of the ions and electrons, respectively. In a plasma with cold ions we have

$$N_i(\varphi) = N_0 u/\sqrt{u^2 - 2e\varphi/M}, \tag{38}$$

where u is the velocity of the solitary wave relative to the ions. The concentration of the electrons is, using (32) and (36),

$$N_e(\varphi) = N_0\left\{\exp\left\{\frac{e\varphi}{T_e}\right\}\left[1-\Phi\left(\sqrt{\frac{e\varphi}{T_e}}\right)\right]+\frac{2}{\sqrt{\pi}}\sqrt{\frac{e\varphi}{T_e}}\right\}. \tag{39}$$

Let us substitute (38) and (39) in (37) and integrate it once. We also use the fact that in the solitary wave the field intensity $d\varphi/dx$ tends to zero together with the potential φ. For convenience we go over to the dimensionless variables $\varphi^* = e\varphi/T_e$ and $\xi = x/D$, where $D = (T/4\pi e^2 N_0)^{1/2}$ is the Debye radius. Instead of (37) we then obtain

$$\frac{1}{2}\left(\frac{d\varphi^*}{d\xi}\right)^2 = -\frac{Mu^2}{T_e}\left[1-\left(1-\frac{2T_e}{Mu^2}\varphi^*\right)^{1/2}\right] + e^{\varphi^*}[1-\Phi(\sqrt{\varphi^*})] - 1 + \frac{2\sqrt{\varphi^*}}{\sqrt{\pi}} + \frac{4\varphi^{*3/2}}{3\sqrt{\pi}}. \tag{40}$$

The velocity of the wave u is determined by the condition $d\varphi/dx = 0$ at $\varphi = \varphi_m$. We find

$$u^2 = \frac{T_e}{2M}\frac{F^2(\varphi_m^*)}{F(\varphi_m^*) - \varphi_m^*},$$

$$F(x) = e^x[1-\Phi(\sqrt{x})] - 1 + \frac{2x^{1/2}}{\pi^{1/2}} + \frac{4x^{3/2}}{3\pi^{1/2}}. \tag{41}$$

The dependence of the velocity of the wave on its amplitude is shown in Fig. 2.

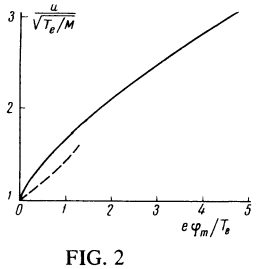

FIG. 2

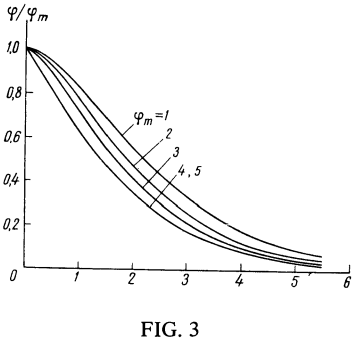

FIG. 3

The dashed curve in the figure represents the same dependence in the case of an equilibrium distribution of the electrons.[2] The difference between the curves is very large. It is important that a solution in the form of a solitary wave is possible only if

$$\varphi_m^* < F(\varphi_m^*) \leqslant 2\varphi_m^*.$$

This condition leads to a restriction of the amplitude ($\varphi_m^* \leq 4.79$) and of the velocity ($1 < u/\sqrt{T_e/M} < 3.1$) of the solitary wave. For $\varphi_m^* \ll 1$ we have

$$u^2 = \frac{T_e}{M}\left[1+\frac{16}{15\sqrt{\pi}}\sqrt{\varphi_m^*}+\frac{2}{3}\varphi_m^*\left(1+\frac{128}{75\pi}\right)\right].$$

The profile of the wave can be determined by integrating (41). For small amplitudes $\varphi_m^* \ll 1$ we have

$$\varphi = \varphi_m \operatorname{ch}^{-4}\left(\frac{1}{\sqrt{15}}\left(\frac{\varphi_m^*}{\pi}\right)^{1/4}\frac{x}{D}\right). \tag{42}$$

The form of the solitary waves for different values of the amplitude is shown in Fig. 3. For small φ_m^* the effective width of the wave is proportional to $(\varphi_m^*)^{-1/4}$.

The author is grateful to L. P. Pitaevskiĭ for a useful discussion and to L. V. Pariĭskaya for carrying out the numerical work.

[1] I. B. Bernstein, J. M. Green, and M. D. Kruskal, Phys. Rev. 108, 546 (1957).

[2] R. Z. Sagdeev, in Voprosy teorii plasmy (Problems of Plasma Theory), M. A. Leontovich, ed., No. 4, Atomizdat, 1964, p. 45.

[3] A. V. Gurevich, in Iskusstvennye sputniki Zemli (Artificial Earth Satellites), No. 7, 1960, p. 101.

[4] Ya. L. Al'pert, A. V. Gurevich, and L. P. Pitaevskiĭ, Iskusstvennye sputniki v rasrezhennoi plazme (Artificial Satellites in a Rarefied Plasma), Izd. Nauka, 1964.

[5] A. A. Vedenov, E. P. Velikhov, and R. Z. Sadgeev, Nuclear Fusion 1, 82 (1961).

[6] V. E. Zakharov and V. I. Karpman, Zh. Eksp. Teor. Fiz. 43, 490 (1962) [Sov. Phys.-JETP 16, 351 (1963)].

[7] L. D. Landau and E. M. Lifshitz, Mekhanika Fizmatgiz, 1958, p. 193 [Engl. Transl. Mechanics, Addison-Wesley, 1960].

[8] T. O'Neil, Physics of Fluids 8, 2255 (1965).

[9] A. V. Gurevich, L. V. Pariĭskaya, and L. P. Pitaevskiĭ, Zh. Eks. Teor. Fiz. 49, 647 (1965) [Soviet Phys.-JETP 22, 449 (1966)].

Translated by R. Lipperheide
109