AN EXACTLY SOLVABLE MODEL OF A DERIVATIVE COUPLING INTERACTION

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A model is considered in which the interaction has the form of a bilinear term involving derivatives. Such an interaction can be solved exactly, but specific difficulties arise owing to the presence of the derivatives, which consist in determining the correct ordering of the differential operators which occur in complicated expressions.

1. INTRODUCTION

A general discussion of the S-matrix within the axiomatic framework leads to the conclusion that the arbitrariness involved in extending the S-matrix off the energy shell consists in the freedom of choosing the renormalization constant multiplying the propagator[1]. Such a renormalization of the propagator can be naturally interpreted as a renormalization of the Heisenberg field (or current) by a factor which is the square root of the renormalization constant of the propagator. But from perturbation theory we know that the field strength renormalization is achieved by means of introducing into the interaction Lagrangian terms which involve derivatives of the fields[2]. From the point of view of the axiomatic approach this means that the coefficient functions of the Wick polynomial in terms of which the S-matrix is represented, or the vacuum expectation values of the functional derivatives of the S-matrix, must contain, in the case of a scalar field, Klein-Gordon differential operators.

On the other hand, it is usually assumed that the interaction Lagrangian (in renormalizable theories) does not involve differential operators, and this fact simplifies the situation considerably. Indeed, in this case a hermitian Lagrangian always leads to a unitary S-matrix, the Dyson T-products coincide with the Wick T-products, etc.[3]. This means that the renormalization of the field operators perturbs this simple situation, and we run into all the complications implied by the presence of derivatives in the theory.

In the usual perturbation theory this question is not simple either. Thus, for instance, in considering the renormalization of external lines one encounters an expression of the type

$$\varphi'(k) = \varphi(k) + (Z-1)\frac{1}{k^2 - m^2 - i\varepsilon}(k^2 - m^2)\varphi(k),$$

which contains a well-known indeterminacy: the wave function $\varphi(k)$ effectively contains a delta function $\delta(m^2 - k^2)$, therefore, it would seem that the second term is either zero or $(Z-1)\varphi(k)$, depending on the "order of operation":

$$(Z-1)\frac{1}{k^2 - m^2 - i\varepsilon}(k^2 - m^2)\delta(m^2 - k^2) \tag{1.1}$$

$$= \begin{cases} (Z-1)\dfrac{1}{k^2 - m^2 - i\varepsilon}[(k^2 - m^2)\delta(m^2 - k^2)] = 0 \\ (Z-1)\left[\dfrac{1}{k^2 - m^2 - i\varepsilon}(k^2 - m^2)\right]\delta(m^2 - k^2) = (Z-1)\delta(k^2 - m^2) \end{cases}$$

The first alternative implies $\varphi' = \varphi$, the second one implies $\varphi' = Z\varphi$. For selfconsistency of the renormalization procedure it would be necessary that $\varphi' = Z^{1/2}\varphi$. This should be obtained as a result of summation of all terms of the perturbation expansion and a "correct application" of the hypothesis of adiabatic switching off of the interaction[4].

A formulation of rules of handling integrals involving derivatives of delta functions, constituting the equivalent in axiomatic theory of the adiabatic hypothesis of perturbation theory, was given in[5]. It consisted in requiring that an integral involving a Klein-Gordon operator acting on a delta function, in the case that the functions φ_1 and φ_2 do not decrease sufficiently fast at infinity, should be given the following interpretation:

$$\int dx_1 dx_2 \varphi_1(x_1)(-K_{x_1})\delta(x_1 - x_2)\varphi_2(x_2)$$
$$= \frac{1}{2}\int dx\{(-K_x\varphi_1(x))\varphi_2(x) + \varphi_1(x)(-K_x\varphi_2(x))\}, \tag{1.2}$$

where $-K_x = -\Box_x + m^2$. In solving problems arising in a theory with derivatives these rules should be applied repeatedly in the complicated expressions arising in iterative solutions. Therefore, our aim will be, first of all, to derive algorithms which operate automatically. Making use of these rules we show how the whole problem of renormalization of the field operators can be handled, resorting to methods from both perturbation theory and the axiomatic approach. As a result we obtain a consistent treatment of this problem in which the results can be exhibited in closed (summed) form.

As a start we consider a model problem in which the only interaction is the "interaction" which leads to the renormalization of the Heisenberg field operator. This is in fact a model of a quadratic interaction, and therefore its perturbation theory is simple in the combinatorial sense, since the interaction vertices corresponding to this model have only two lines each. Owing to this circumstance all perturbation expansions can be summed and yield final results in closed form.

The problem turns out to be nontrivial only owing to the presence of derivatives in the interaction Lagrangian. Thus, in this model the combinatorial complications of perturbation theory are separated from the difficulties related to the presence of derivatives, and it is the latter category that will be investigated here in

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"pure form."

Summing up, one can say that the renormalization of the field operators is technically complicated because of the need to take into account at all times the presence of derivatives, and therefore the rules for handling integrals are not quite usual. As will be seen, all renormalization constants have in fact operator character (in the classical sense of differential or integral operators), and are not numerical, and only under special conditions can they be replaced by numbers.

We solve the equations of motion in a definite model and determine the Heisenberg field and current operators. In particular, we discuss the problem of the well-known difficulty involving the so-called integrability condition:

$$\frac{\delta j(x)}{\delta\varphi(y)} - \frac{\delta j(y)}{\delta\varphi(x)} = i[j(x), j(y)]. \tag{1.3}$$

The difficulty consists in the fact that this condition must hold both for the renormalized and unrenormalized theories. But in the usual approach to renormalization, the current operator is multiplied by a factor $Z^{1/2}$, and this leads to a contradiction with (1.3), where the left hand side is linear in the current, whereas the right hand side is quadratic.

2. DESCRIPTION OF THE MODEL

It is known that the quasilocal terms in the interaction Lagrangian of the scalar theory which lead to the renormalization of the field operator will have the following form (according to perturbation theory [2]):

$$^1/_2(Z-1)(-K_{\zeta_1})\delta(\zeta_1-\zeta_2):\ \varphi(\zeta_1)\varphi(\zeta_2):$$

We consider a model in which this term will be the only interaction term. According to the rules of perturbation theory the corresponding S-matrix should be written in the form

$$S = T_W \exp\left\{-\frac{i}{2}(Z-1)\int d\zeta_1 d\zeta_2\,(-K_{\zeta_1})\delta(\zeta_1-\zeta_2):\varphi(\zeta_1)\varphi(\zeta_2):\right\}. \tag{2.1}$$

Here the operator $(-K_{\zeta_1})$ which acts on $\delta(\zeta_1-\zeta_2)$ should be understood in the sense of the rules for handling such integrals formulated in the preceding section.

The S-matrix (2.1) is causal because the Lagrangian is local. This S-matrix does not lead to a real interaction of the particles, a fact which will become clear from the structure of the Heisenberg field and current operators.

We define the current j(x) according to our usual rule [6]:

$$j(x) = i\frac{\delta S}{\delta\varphi(x)}S^+. \tag{2.2}$$

Since we are in possession of an explicit expression for the S-matrix, the functional derivative can be computed and yields for the current

$$j(x) = (Z-1)T_W\left\{\int dx_1\,(-K_{x_1})\delta(x-x_1)\varphi(x_1)S\right\}S^+. \tag{2.3}$$

Making use of the definition of the Heisenberg field operator

$$A(x) = T_W\{\varphi(x)S\}S^+, \tag{2.4}$$

we rewrite the expression for the current in the form

$$j(x) = (Z-1)\int d\zeta\,(-K_\zeta)\delta(x-\zeta)A(\zeta). \tag{2.5}$$

We introduce a special notation for this integral operator

$$(-\widehat{K\delta})f(x) = \int d\zeta(-K_\zeta)\delta(x-\zeta)f(\zeta). \tag{2.6}$$

Then

$$j(x) = (Z-1)(-\widehat{K\delta})A(x). \tag{2.7}$$

This expression of the current in terms of the field (in the present case as a linear transformation of the Heisenberg field) is the dynamical coupling, which is true only due to the definite model chosen by us. This is the dynamical principle of our model.

In addition to this relation between field and current we have another relation given by the Yang-Feldman equation, and which is not at all specific for the model under consideration, but, on the contrary, is valid for any theory:

$$A(x) = \varphi(x) - \int d\zeta D^{adv}(x-\zeta)j(\zeta). \tag{2.8}$$

Introducing again an integral operator

$$\hat{D}^{adv}f(x) = \int d\zeta\, D^{adv}(x-\zeta)f(\zeta), \tag{2.9}$$

we can rewrite the Yang-Feldman relation in the form

$$A(x) = \varphi(x) - \hat{D}^{adv}j(x). \tag{2.10}$$

Thus in our model we have two equations which express the current and field in terms of the free field $\varphi(x)$. These equations form a closed system, and by solving them one can express A(x) and j(x) in terms of $\varphi(x)$.

Indeed, substituting (2.7) into (2.10) and defining yet another integral operator

$$\hat{L}f(x) = \int d\zeta_1 d\zeta_2 D^{adv}(x-\zeta_1)(-K_{\zeta_1})\delta(\zeta_1-\zeta_2)f(\zeta_2), \tag{2.11}$$

we obtain directly

$$A(x) = \varphi(x) - (Z-1)\hat{L}A(x). \tag{2.12}$$

This equation can be solved formally:

$$A(x) = \frac{1}{1-(1-Z)\hat{L}}\varphi(x). \tag{2.13}$$

Such a solution involving an inverse operator should, of course, be interpreted as the sum of a power series

$$A(x) = \sum_{\nu=0}^{\infty}(1-Z)^\nu\hat{L}^\nu\varphi(x). \tag{2.14}$$

This circumstance relates the model under consideration with perturbation theory.

We note now that the operator $\hat{L}$ involves derivatives of the delta function. In forming powers of $\hat{L}$ which occur in the expansion (2.14) we shall have to make use repeatedly of the rules for handling such integrals formulated in the preceding section (this has the meaning of an "adiabatic method"). For this reason the rules must be formalized to such an extent as to make their application as automatic as possible.

In order to achieve this we introduce one more integral operator

$$\hat{I}f(x) = \int d\zeta\,\delta(x-\zeta)f(\zeta). \tag{2.15}$$

This is the identity operator so long as the expression is not subject to differentiation, but we need the operator in order to define correctly the order in which the differential operators act. The meaning of this operator is easily illustrated on the following simple example. We consider a $\varphi(x)$ such that $(-K_x)\varphi(x) = 0$, and try to find the meaning of $(-K_x)\hat{I}\varphi(x)$. According to the rules of the preceding section

$$(-K_x)\hat{I}\varphi(x) = \int d\zeta\,(-K_\zeta)\,\delta(x-\zeta)\,\varphi(\zeta) \equiv (-\widehat{K\delta})\,\varphi(x) = {}^1/_2(-K_x\varphi(x)) + {}^1/_2\varphi(x)\,(-K_x) = {}^1/_2\varphi(x)\,(-K_x). \quad (2.16)$$

The free differential operator in the right hand side must act on the test functions or on other functions of the same variable if $(-\widehat{K\delta})\varphi(x)$ is one of the factors in a more complicated expression. Thus, here $\hat{I}$ is no longer the unit operator. The reason for this lies, of course, in the fact that one of the integrations can no longer be removed by means of the delta function, before taking care of the action of the differential operators.

The operator $\hat{I}$ which has been introduced looks somewhat artificial, since it is expressed in terms of a delta function, which itself is defined only as a generalized function, associating to each (admissible) test function its value at the point where the argument of the delta function vanishes:

$$\int f(x)\,\delta(x-a)\,dx = f(a).$$

The artificial character of the rules of operation with $\hat{I}$ is caused by the fact (this is explained in detail in[5]) that we have written this operator in a limiting form which has only symbolic meaning, namely the meaning described by the rules of operation. If we would carry out all reasonings making use of smooth cutoff functions, no doubts would arise as to the significance of the operator $\hat{I}$, but this would complicate all calculations considerably.

There are some important relations among the differential operators that have been introduced, and we shall make use of these relations. Two relations follow directly from the definitions. These are

$$(-\underset{\rightarrow}{K})\,\hat{I} = \hat{I}\,(-\underset{\leftarrow}{K}) \equiv (-\widehat{K\delta}), \quad (2.17)$$

$$\hat{D}^{adv}\,(-\widehat{K\delta}) \equiv \hat{L}. \quad (2.18)$$

In addition, the properties of the function D^{adv} imply that

$$(-\underset{\rightarrow}{K})\,\hat{D}^{adv} \equiv \hat{D}^{adv}\,(-\underset{\leftarrow}{K}) = \hat{I}, \quad (2.19)$$

and finally, according to (1.2)

$$\{X\}\,(-\widehat{K\delta})\,\{Y\} = {}^1/_2\{X\}\,(-\underset{\leftarrow}{K})\,\{Y\} + {}^1/_2\{X\}\,(-\underset{\rightarrow}{K})\,\{Y\}. \quad (2.20)$$

Making use of these relations we can transform powers of the operator $\hat{L}$. Indeed,

$$\begin{aligned}\hat{L}^n f(x) &= \text{from (2.18)} = \hat{L}^{n-1}\,\hat{D}^{adv}\,(-\widehat{K\delta})\,f(x),\\ &= \text{from (2.20)} = {}^1/_2\hat{L}^{n-1}\hat{D}^{adv}\,(-\underset{\leftarrow}{K})\,f(x) + {}^1/_2\hat{L}^{n-1}\hat{D}^{adv}(-\underset{\rightarrow}{K})\,f(x),\\ &= \text{from (2.19)} = {}^1/_2\hat{L}^{n-1}\hat{I}\,f(x) + {}^1/_2\hat{L}^{n-1}\,\hat{D}^{adv}\,(-\underset{\rightarrow}{K})\,f(x).\end{aligned} \quad (2.21)$$

Noting that a lowering of the power of the operator $\hat{L}$ leads to the appearance of powers of the operator $\hat{I}$, we compute the general expression

$$\begin{aligned}\hat{L}^m\,\hat{I}^k f(x) &= \hat{L}^{m-1}\,\hat{D}^{adv}\,(-\widehat{K\delta})\,\hat{I}^k f(x)\\ &= {}^1/_2\hat{L}^{m-1}\,\hat{I}^{k+1} f(x) + {}^1/_2\hat{L}^m\,\hat{I}^{k-1} f(x).\end{aligned} \quad (2.22)$$

These expressions have the form of recursion relations for the quantities $H(m, k) = \hat{L}^m\hat{I}^k$ with respect to the two discrete variables m and k. For solving these recursion relations it is convenient to reduce them to the form of a difference equation of the diffusion type, equation which can be solved by means of Green's functions, since unlike the differential equation describing diffusion, the difference equation has characteristics. (The reason for the latter circumstance consists in the fact that the binomial coefficients used in constructing the Green's function of the equation with finite differences vanish for finite values of their variables, in distinction from the Gaussian function.) Following this procedure we are led to the following fundamental formula for the operator $\{1-(1-Z)\hat{L}\}^{-1}$:

$$\frac{1}{1-(1-Z)\,\hat{L}}\,f(x) = \frac{1}{1-(1-\sqrt{Z})\,\hat{I}}\,f(x) + (1-\sqrt{Z})\,\frac{1}{1-(1-\sqrt{Z})\,\hat{I}}\,\hat{D}^{adv}\,\frac{1}{1-(1-\sqrt{Z})\,\hat{I}}\,(-Kf(x)). \quad (2.23)$$

Making use of this formula, we now proceed to discuss other aspects of our model.

3. THE HEISENBERG OPERATORS OF THE MODEL

The field operator. Equation (2.23) is already a solution of the fundamental equation for the Heisenberg field operator. Since the operators entering this equation will be encountered repeatedly, we introduce for them the special notation:

$$\hat{N} = \{1-(1-\sqrt{Z})\,\hat{I}\}^{-1}; \quad (3.1)$$

$$\hat{\Delta}^{adv} = \hat{N}\,\hat{D}^{adv}\,\hat{N}. \quad (3.2)$$

In this notation Eq. (2.23) takes the form

$$\{1-(1-Z)\,\hat{L}\}^{-1} f(x) = \hat{N} f(x) + (1-\sqrt{Z})\,\hat{\Delta}^{adv}\,(-K_x f(x)). \quad (3.3)$$

Thus the Heisenberg field A(x) can be written in the form

$$A(x) = \hat{N}\varphi(x) + (1-\sqrt{Z})\,\hat{\Delta}^{adv}\,(-K\varphi(x)). \quad (3.4)$$

We stress the fact that A(x) contains two different parts: the first term does not vanish on the energy shell, whereas the second one does. If A(x) is not subjected to further differentiations we can set all operators $\hat{I}$ equal to one. Then the field A(x) becomes a simple function of the field $\varphi(x)$:

$$A(x) = \frac{1}{\sqrt{Z}}\,\varphi(x) + \frac{1-\sqrt{Z}}{Z}\,\hat{D}^{adv}\,(-K\varphi(x)).$$

On the energy shell the fields A(x) and $\varphi(x)$ differ only by the renormalization factor $Z^{-1/2}$, but off the energy shell they differ, in general, by the additional term. It is to be noted that the part of the field A(x) associated with the energy shell, and the part associated with the exterior of the shell are subject to different renormalizations.

It is easy to see that if we do not distinguish between the different orders in which the operators act, and carry over the action of, say, (−K) in the second term from the field φ to the function D^{adv}, we would obtain the result that the field is renormalized with the constant Z^{-1}. On the contrary, if the action of the operator (−K) in the first term in the derivation is carried over to the field φ we are led to the conclusion that on the energy shell the fields A(x) and $\varphi(x)$ are equal. Thus, if one does not distinguish the order of action of the operators (which is here automatically taken into account via the rule (1.2) and the operator $\hat{I}$, (2.15)) we would be led to the alternative which was discussed in the introduction, in connection with Eq. (1.1).

We also remark that A(x) locally commutes with $\varphi(x)$ and therefore these two fields belong to the same Borchers class[7]. This agrees with the fact that our model S-matrix differs from the identity operator only outside the energy shell.

The current operator. Making use of the expression for the operator A(x) we can now determine the corresponding current operator j(x), utilizing the fundamental formula (2.7). For this we shall need to determine the action of the operator $(-K\delta)$ on N. We first remark that according to (3.4) and (2.20)

$$\begin{aligned}(-K\delta)\hat{I}f(x) &= {}^1\!/_2\hat{I}f(x)(-K_x) + {}^1\!/_2(-K)\hat{I}f(x)\\ &= {}^1\!/_2\hat{I}f(x)(-K_x) + {}^1\!/_2(-\widehat{K\delta})f(x).\end{aligned} \tag{3.5}$$

But according to the definition of $\hat{N}$, we have $\hat{N} - 1 = (1 - Z^{1/2})\hat{I}\hat{N}$. Therefore, considering $\hat{N}f(x)$ as a new function, we find, according to (3.5)

$$(-\widehat{K\delta})\hat{I}\{\hat{N}f(x)\} = {}^1\!/_2\hat{I}\hat{N}f(x)(-K_x) + {}^1\!/_2(-\widehat{K\delta})\hat{N}f(x). \tag{3.6}$$

and then we let the operator $(-\widehat{K\delta})$ act on (3.6). Carrying out the computation, we obtain

$$(-\widehat{K\delta})\hat{N}f(x) = \frac{1}{1+\sqrt{Z}}\left\{\hat{N}f(x)(-K_x) + (-Kf(x))\right\}. \tag{3.7}$$

With the aid of this formula we easily compute that

$$\begin{aligned}j(x) = &-(1-\sqrt{Z})\hat{N}\{\varphi(x)(-K_x)\\ &+(-K\varphi(x))\} - (1-\sqrt{Z})^2\hat{\Delta}^{adv}(-K\varphi(x))(-K_x).\end{aligned} \tag{3.8}$$

It can be seen from (3.6) that in the same manner as the field, the current operator contains terms which vanish on the energy shell, as well as terms which do not vanish there, and the powers of the renormalization constant for these terms are different. We note that the current operator contains "free" Klein-Gordon operators. Since the current is a generalized function, this is not at all surprising. Any generalized function has to be interpreted in the sense of being integrated with functions belonging to the class of test functions, on which these operators are supposed to act. This is a consequence of the symbolical notation adopted by us.

We now compute the current commutator [j(x), j(y)]. We note that as a consequence of the exact equation $(-K_x)D(x-y) = 0$ only terms which do not vanish on the energy shell contribute to the commutator of the currents, i.e., only the first term in (3.8) contributes. After this remark it is easy to see that

$$[j(x), j(y)] = -i(1-\sqrt{Z})^2\hat{\Delta}(x-y)(\underset{\rightarrow}{-K_x})(\underset{\rightarrow}{-K_y}). \tag{3.9}$$

We also verify that our current, when substituted into the Yang-Feldman equation (2.10), leads to the same expression (3.4) for the field. Indeed,

$$\begin{aligned}A(x) = \varphi(x) - \hat{D}^{adv}j(x) = \varphi(x) &+ \hat{D}^{adv}\{(1-\sqrt{Z})\hat{N}[\varphi(x)(-K_x)\\ &+(-K\varphi(x))] + (1-\sqrt{Z})^2\hat{\Delta}^{adv}(-K\varphi(x))(-K_x)\}.\end{aligned} \tag{3.10}$$

The free operators $(-K_x)$ acting on $\hat{D}^{adv}$ yield $\hat{I}$, and making use of (3.6), we see in the right hand side we again obtain the expression (3.4).

Applying the operator (−K) to the Yang-Feldman equation we obtain

$$(-KA(x)) = (-K\varphi(x)) - \hat{I}j(x). \tag{3.11}$$

This is exactly the form of the Klein-Gordon equation for our model. The usual Klein-Gordon equation $(-KA(x)) = j(x)$ is generally not valid in this case. It will be satisfied only if we return to the energy shell, setting $(-K\varphi(x)) = 0$, and in addition requiring that $\hat{I} = 1$, i.e., if one renounces the possibility of taking further variations and differentiations.

One should not be surprised by this circumstance—after all we are interested in an off-shell theory for which the usual equations of motion should not be expected to hold. Indeed, for the free field the fact that $(-K\varphi(x)) = 0$ (the free-field limit of the equation $(-KA(x)) = j(x)$) does not hold off the energy shell is tautological to the words "outside the energy shell."

4. THE INTEGRABILITY CONDITION

We now consider the integrability condition (1.3) for our model. The retarded operator is defined as the functional derivative of the current $\delta j(x)/\delta\varphi(y)$. Causality guarantees that this operator is retarded, i.e., that it vanishes in the future cone.

Since we are in possession of a complete expression for j(x), we can construct an explicit expression for the retarded operator—the only thing being required is caution. According to the rules for variation of a functional which has the form of an integral over field operators multiplied by appropriate coefficient functions, one should successively eliminate one operator at a time, suppressing the corresponding integrations, and replacing in the coefficient functions the coordinate of the eliminated field operator by the coordinate of the field with respect to which the functional derivative is taken. At the same time it is important that the expression subjected to variation should not contain derivatives of the fields. All derivatives should first be transposed onto the coefficient functions. Then the operation of taking functional derivatives can be written in terms of delta-functions, for which we have to formulate a special condition in our symbolic calculus.

We shall denote such a delta function by boldface letters: $\boldsymbol{\delta}(\mathbf{x}-\mathbf{y})$, and its special character consists in the fact that it leads to the appearance of the operator $\hat{I}$. This delta function is in a way "transparent" for differentiation operators, and therefore it need not be retained. (In this respect $\boldsymbol{\delta}(\mathbf{x}-\mathbf{y})$ is a delta function with usual properties, whereas the kernel of the operator $\hat{I}$ exhibits unusual properties.)

In order to illustrate the difference between the two kinds of delta functions, we remark that

$$\frac{\delta\varphi(x)}{\delta\varphi(y)} = \boldsymbol{\delta}(\mathbf{x}-\mathbf{y}), \tag{4.1}$$

whereas

$$\frac{\delta}{\delta\varphi(y)}\int d\zeta\,\delta(x-\zeta)\varphi(\zeta) = \delta(x-y) = \int d\zeta\,\delta(x-\zeta)\boldsymbol{\delta}(\boldsymbol{\zeta}-\mathbf{y}). \tag{4.2}$$

It is clear from Eqs. (4.1) and (4.2) that we actually encountered the function $\boldsymbol{\delta}(\mathbf{x}-\mathbf{y})$ in the derivation of the expression (2.3) for the current, and we have immediately removed it according to (4.2). We stress once more the fact that the necessity of introducing the object $\boldsymbol{\delta}(\mathbf{x}-\mathbf{y})$ with the usual properties of the delta function is related to the fact that the fundamental delta function occurring in the interaction should be interpreted, strictly speaking, in a "smoothed sense," and should be considered a delta function only within the framework of the symbolic calculus utilized here (taking the corresponding precautions). On the other hand $\delta(\mathbf{x}-\mathbf{y})$ is an "innocent" delta function, for which all these precautions are superfluous[3].

Our machinery is exactly suitable for transposing the differentiations from the field operators onto the coefficient functions. Indeed, let us compute as an example the functional derivative of

$$\int d\zeta \Phi(\zeta)(-K\varphi(\zeta)).$$

Using the established procedure we write

$$\int d\zeta\, \Phi(\zeta)(-K\varphi(\zeta)) = 2\int d\zeta_1 d\zeta_2\, \Phi(\zeta_1)(-K\delta(\zeta_1-\zeta_2))\varphi(\zeta_2) - \int d\zeta(-K\Phi(\zeta))\varphi(\zeta). \tag{4.3}$$

The right hand side is represented in a form admitting a functional differentiation, according to our condition. Thus

$$\frac{\delta}{\delta\varphi(y)}\int d\zeta\, \Phi(\zeta)(-K\varphi(\zeta)) = 2\int d\zeta_1 d\zeta_2\, \Phi(\zeta_1)(-K\delta(\zeta_1-\zeta_2))\boldsymbol{\delta}(\zeta_2-y) - \int d\zeta(-K\Phi(\zeta))\boldsymbol{\delta}(\zeta-y) \tag{4.4}$$

Obviously the same procedure (4.3) can be used in the case of an integral operator of the general form

$$\frac{\delta}{\delta\varphi(y)}\hat{O}(-K\varphi(x)) = 2\hat{O}(-\widehat{K\delta})\,\boldsymbol{\delta}(x-y) - \hat{O}(-\underset{\leftarrow}{K})\,\boldsymbol{\delta}(x-y) = \hat{O}\boldsymbol{\delta}(x-y)(-\underset{\rightarrow}{K}). \tag{4.5}$$

Making use of the formulated rules and of the expression (3.8) for the current j(x), it is easy to obtain an expression for the retarded operator:

$$\frac{\delta j(x)}{\delta\varphi(y)} = -(1-\sqrt{Z})\hat{N}\boldsymbol{\delta}(x-y)\{(-K_x)+(-K_y)\} - (1-\sqrt{Z})^2\hat{\Delta}^{adv}(x-y)(-K_x)(-K_y). \tag{4.6}$$

The expression for the retarded operator contains terms of two kinds: a quasi-local term which is symmetric in x and y and is proportional to $(1-Z^{1/2})\hat{N}$, and a term whose support is localized in the past cone, which is not symmetric in x and y, and is proportional to $[(1-Z^{1/2})\hat{N}]^2$.

In forming the difference of the retarded and advanced operators, which occurs in the integrability condition (1.3), the symmetric (quasilocal) terms cancel, and we obtain

$$\frac{\delta j(x)}{\delta\varphi(y)} - \frac{\delta j(y)}{\delta\varphi(x)} = -(1-\sqrt{Z})^2\{\hat{\Delta}^{adv}(x-y) - \hat{\Delta}^{ret}(x-y)\}(-K_x)(-K_y) = (1-\sqrt{Z})^2\hat{\Delta}(x-y)(-K_x)(-K_y). \tag{4.7}$$

Comparing (4.7) with (3.9)—the expression of the current commutator—we see that

$$\frac{\delta j(x)}{\delta\varphi(y)} - \frac{\delta j(y)}{\delta\varphi(x)} = i[j(x), j(y)] \tag{4.8}$$

i.e., the integrability condition holds in our model.

It is interesting to analyze in detail how this happens. Indeed, at first sight it would seem that renormalization must violate this condition, since the renormalization constant of the current occurs with power two in the commutator, and only with power one in difference of the left hand side. As a matter of fact the situation is more complicated even in the simple model we have analyzed. The current, as well as the retarded operator contains both terms with the "renormalization factor" $(1-Z^{1/2})\hat{N}$, and with the factor $[(1-Z^{1/2})\hat{N}]^2$, but the quadratic factor multiplies those terms which vanish on the energy shell. Therefore they give a vanishing contribution to the commutator, as a consequence of $(-KD(x)) = 0$. As a result the commutator as a whole is proportional to $(1-Z^{1/2})\hat{N}^2$, and does not involve higher power terms. At the same time, as we have seen, in the difference between the advanced and retarded operators the terms proportional to the first power of $(1-Z^{1/2})\hat{N}$ cancel because of symmetry.

5. CONCLUSION

The field-theory model involving derivatives in the interaction term which we have considered corresponds to what one usually means by field operator renormalization (or current renormalization) and allows us to reach several important conclusions. Firstly, we have convinced ourselves that the so-called renormalization "constants" are in general integral operators (of the type of $\hat{N}$, cf. (3.1)). They become numbers only after one assumes that $\hat{I} = 1$, which can be done only if no further differentiations of the corresponding operator are envisaged. Otherwise this leads to incorrect results. In particular, the operators $\hat{I}$ and $\hat{N}$ allow one to obtain the result about the renormalization of external lines in perturbation theory diagrams.[4]

Another essential circumstance is related to the fact that the Heisenberg field and current operators contain even in the simplest model both terms which vanish on the energy shell, and terms which are related to extensions of the S-matrix off the energy shell. An important conclusion consists in the fact that both in the current and the field operator, the terms on and off the energy shell have different renormalization "constants." We stress the fact that one cannot neglect the off-shell terms as long as one intends to carry out further differentiations and variations (e.g., application of the reduction formula). In particular, these peculiarities of the Heisenberg operators are responsible for the consistency of the integrability condition (1.3) under renormalization.

We note that our model is constructed in such a manner that the S-matrix does not lead to real transitions among physical states. In this sense this is a theory of an "almost free field." However we cannot consider the S-matrix to be identically equal to one, since the current has nontrivial matrix elements and the propagators differ from the free-field propagators. In essence the

[3] An "innocent" delta function has other defects. Being a function of the difference x − y, it behaves improperly under differentiations:

$$[(-\underset{\rightarrow}{K}_x)\boldsymbol{\delta}(x-y)] = \boldsymbol{\delta}(x-y)(-\underset{\rightarrow}{K}_y) \neq [\boldsymbol{\delta}(x-y)(-\underset{\leftarrow}{K}_y)] = (-\underset{\leftarrow}{K}_x)\boldsymbol{\delta}(x-y).$$

This circumstance has prompted the introduction of the other (nonboldface) delta function, for which $[(-\underset{\rightarrow}{K}_x)\delta(x-y)] = [\delta(x-y)(-\underset{\leftarrow}{K}_y)]$.

S-matrix differs from the identity only off the energy shell[4)]. Therefore the field A(x) differs from $\varphi(x)$, although it belongs to the same Borchers class.

In the sequel we hope to apply the model to a more realistic case, when in addition to the renormalization terms the interaction contains also fundamental terms of the usual nature.

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[1] B. V. Medvedev and M. K. Polivanov, Lectures at the International Winter School for Theoretical Physics, Dubna, 1964; M. Polivanov, in High Energy Physics, Trieste Seminar 1965, IAEA, Vienna, 1965.

[2] N. N. Bogolyubov and D. V. Shirkov, Vvedenie v teoriyu kvantovannykh poleĭ (Introduction to the Theory of Quantized Fields), Gostekhizdat, 1957, Eng. Translation, Interscience, N. Y. 1959.

[3] B. V. Medvedev, Doctoral Dissertation, Math. Inst. of the Acad. Sci. USSR, 1964.

[4] F. J. Dyson, Phys. Rev. 76, 1736 (1949).

[5] B. V. Medvedev and M. K. Polivanov, DAN SSSR, 177, No. 6 (1967) [Soviet Physics-Doklady 12 (1968), in press].

[6] N. N. Bogolyubov, B. V. Medvedev, and M. K. Polivanov, Voprosy teorii dispersionnykh sootnosheniĭ (Problems of the theory of dispersion relations), Fizmatgiz, 1958.

[7] H. J. Borchers, Nuovo Cimento 15, 784 (1960).

4) We note that since the interaction Lagrangian is given, the extension of the S-matrix off the energy shell is determined, in distinction from the usual situation in axiomatic theory.

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