ON THRESHOLD PHENOMENA IN CLASSICAL ELECTRODYNAMICS

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It is shown that for a wide class of problems in electrodynamics, the behavior of the amplitudes and phases at the threshold for the production of new proper waves can be determined from the conservation laws. The method proposed, which is analogous to the quantum theory of many-channel nuclear reactions, is employed for an explanation of the Wood anomalies.

In classical electrodynamics there is a wide class of problems, for example, the diffraction of waves from periodic structures, the reflection from the open end of a wave guide, etc., where the amplitude and the phase of the diffracted waves exhibit characteristic kinks for certain definite values of the parameters (frequency, angle of incidence, etc.). As noted in [1], this effect cannot be explained in a satisfactory manner in the usual theory.

On the other hand, as will be shown below, these singularities have much in common with the threshold anomalies in many-channel nuclear reactions, for whose description there exists a well-developed apparatus in quantum mechanics. [2] This similarity is not accidental and is due to the identity of the physical nature of the phenomena. Indeed, the threshold anomalies occur when a new reaction channel opens, which leads to characteristic kinks in the amplitudes in the channels which were already open. In the electrodynamic problems the kinks in the amplitudes and phases also occur when a new spectral line appears or at the threshold for the excitation of a new wave, i.e., practically also when a new reaction channel opens, although there is no such concept in classical electrodynamics.

Using the formal and physical analogy just mentioned, we consider below the two most characteristic examples of threshold effects in electrodynamics: the so-called Wood anomalies and the reflection and diffraction of light at a plane boundary between two media.

1. WOOD ANOMALIES

In 1902, R. W. Wood [3] called attention to the unusual behavior of the spectra obtained with diffraction gratings. He observed alternating bright and dark bands in the continuous spectra. The wave lengths corresponding to these bands were shifted when the angle of incidence of the light on the grating was altered. A qualitative consideration carried out by Rayleigh [4] and Wood [5] showed that in the diffraction of a monochromatic wave these anomalies occur at angles of incidence for which the condition

$$d(\cos\theta \pm 1) = n\lambda, \tag{1}$$

is satisfied, where d is the period of the grating, θ is the angle of incidence of the wave reckoned from the plane of the grating, $\lambda = 2\pi/k = 2\pi c/\omega$ is the wavelength, and n is an integer number. It is easy to see that condition (1) determines the threshold for the occurrence of a diffraction spectrum of order n.

Let us consider a plane homogeneous periodic grating with an arbitrary structure of the period. For simplicity we regard the grating as reflecting, although the results obtained are easily carried over to the case of transparent gratings. We place the y axis along the axis of periodicity of the grating and the x axis parallel to its grooves. The wave falls on the grating from the half-space $z > 0$. Then the field above the grating can be written as a superposition of waves of the form

$$\psi_n^{\pm} = \exp\left[iy\left(k_y - \frac{2\pi n}{d}\right) \pm iz\left[\frac{\omega^2}{c^2} - \left(k_y - \frac{2\pi n}{d}\right)^2\right]^{1/2}\right], \quad n = 0, \pm 1, \pm 2\ldots \tag{2}$$

Such a superposition is possible because owing to the periodicity of the grating, the component of the wave vector along the y axis can only change by a multiple of $2\pi/d$. Depending on the polarization of the wave, the functions $\psi_n^{\pm}$ can describe either the electric field component E_x or the magnetic field component H_x.

The functions (2) are undamped waves ("open reaction channels") if the inequality

$$k_y - \omega/c < 2\pi n/d < k_y + \omega/c \tag{3}$$

holds. For given values of ω this inequality determines the number N of undamped harmonics, i.e., the number of open channels. Energy transfer of the electromagnetic field at large distances from the grating can occur only in the open channels, which we number from n_1 to n_N. If a wave $\alpha_j\psi_j^-$, $n_1 \le j \le n_N$, falls on the grating, where α_j is an arbitrary amplitude, then the field far from the grating has the form

$$\alpha_j\psi_j^- + \sum_{n=n_1}^{n_N} \alpha_j R_{nj}\psi_n^+, \tag{4}$$

where R_{nj} is the coefficient for the transformation of the j-th incident wave into the n-th reflected wave. The energy current carried along the z axis by the wave $\psi_j^{\pm}$ is proportional to the z component of the wave vector

$$\varkappa_j = \left[\frac{\omega^2}{c^2} - \left(k_y - \frac{2\pi j}{d}\right)^2\right]^{1/2}, \tag{5}$$

which is easily seen by taking E_x or H_x for ψ and calculating the z component of the Poynting vector.

Taking account of the orthogonality of the functions ψ_j, the condition for the equality of the incident and reflected energies can be written in the form

$$\sum_j |\alpha_j|^2 \varkappa_j = \sum_{jj'n} \alpha_j \alpha_{j'}^* R_{nj} R_{nj'}^* \varkappa_n. \qquad (6)$$

Since the amplitudes α_j are arbitrary, we obtain the fundamental relation

$$\sum_n S_{nj} S_{nj'}^* = \delta_{jj'}, \qquad S_{nj} = \sqrt{\frac{\varkappa_n}{\varkappa_j}} R_{nj}, \qquad (7)$$

where the summation goes over the open channels. The equality (7) expresses the unitarity of the S matrix. In particular, if only one (the first) channel is open, then

$$|S_{11}|^2 = |R_{11}|^2 = 1. \qquad (8)$$

Let us now consider the behavior of the elements of the S matrix near the threshold for the opening of the (N + 1)-st channel, where the quantity κ_{N+1} is small. Below the threshold the quantity κ_{N+1} is pure imaginary and above the threshold it is real [cf. (5)]. Therefore, near the threshold the S matrix has the form

$$\begin{aligned} S &= S_0 + ia|\varkappa_{N+1}| \quad \text{below threshold,} \\ S &= S_0 \quad \text{at threshold,} \\ S &= S_0 + a\varkappa_{N+1} \quad \text{above threshold.} \end{aligned} \qquad (9)$$

We note here that $\kappa_{N+1} = 0$ at the threshold, which agrees with condition (1).

In order to determine the matrix a, we substitute each of these expressions in (7), keeping only terms of first order in $|\kappa_{N+1}|$. This yields

$$\begin{aligned} S_0 S_0^+ &= 1, \\ (S_0 + ia|\varkappa_{N+1}|)(S_0^+ - ia^+|\varkappa_{N+1}|) &= 1, \\ (S_0 + a\varkappa_{N+1})(S_0^+ + a^+\varkappa_{N+1}) + B\varkappa_{N+1} &= 1, \end{aligned} \qquad (10)$$

where

$$B_{jj'} = \frac{1}{\sqrt{\varkappa_j \varkappa_{j'}}} \overset{0}{R}_{N+1,\,j} \overset{0*}{R}_{N+1,\,j'}. \qquad (11)$$

From this we obtain

$$aS_0^+ - S_0 a^+ = 0, \qquad aS_0^+ + S_0 a^+ = -B, \qquad (12)$$

or

$$aS_0^+ = -B/2, \qquad S_0 a^+ = -B/2. \qquad (13)$$

Since B is a hermitian matrix, one of these equations is a consequence of the other. Hence

$$a = -BS_0/2. \qquad (14)$$

Substituting this relation in (9), we find

$$\begin{aligned} S &= S_0 - i|\varkappa_{N+1}|BS_0/2 \quad \text{below threshold,} \\ S &= S_0 - \varkappa_{N+1}BS_0/2 \quad \text{above threshold.} \end{aligned} \qquad (15)$$

Taking the corresponding matrix elements and taking account of the connection between S and R [cf. (7)], we obtain the behavior of all transformation coefficients $R_{jj'}$ near the threshold of a new reaction channel. Using (5), we easily see that all amplitudes and phases of the scattered waves near the threshold of a new channel characterized by the number j, have the form

$$C_1 + C_2 \left[\frac{\omega^2}{c^2} - \left(k_y - \frac{2\pi j}{d} \right)^2 \right]^{1/2}. \qquad (16)$$

The constants C_1 and C_2 are determined by the structure of the scattering system and by the channel number. Examples of such behavior determined numerically for concrete systems, can be found in the figures of the work of Vaĭnshteĭn.[1]

It is easy to see that the consideration of the diffraction of waves incident on the open end of a waveguide would be completely analogous. The role of the small parameter κ_{N+1} is in this case played by the propagation constant of the new wave along the waveguide.

2. FRESNEL FORMULAS

In order to illustrate our method by a sufficiently simple and exactly soluble example, we consider the diffraction and reflection of electromagnetic waves on the plane boundary between two uniform media. We choose the z axis normal to the plane of the boundary. The wave falls from a medium with dielectric constant ϵ_1 on the boundary with the medium with ϵ_2. Supplying the quantities referring to the incident, reflected, and diffracted waves with the indices 0, 1, and 2, respectively, we have for the tangential components of the wave vectors

$$\mathbf{k}_{\tau 0} = \mathbf{k}_{\tau 1} = \mathbf{k}_{\tau 2}, \qquad (17)$$

and for the normal components

$$k_{z1} = -k_{z0}, \qquad k_{z2} = \left[\frac{\omega^2}{c^2} \varepsilon_2 - k_{\tau 0}^2 \right]^{1/2}. \qquad (18)$$

If k_{z2} is real, the wave penetrates into the second medium; if it is imaginary, we have the case of total internal reflection.

According to our interpretation, there are two possible reaction channels in this case. The first channel corresponds to the reflected wave, the second to the diffracted wave, which opens, according to (18), for $k_{\tau 0}^2 = \omega^2 \epsilon_2 / c^2$ (i.e., when the angle of incidence is equal to the angle of total internal reflection).

Let us now consider the exact expressions for the Fresnel coefficients, restricting ourselves to the case where the electric field **E** is perpendicular to the incident plane:[6]

$$\begin{aligned} E_1 &= R_{10} E_0, \qquad E_2 = R_{20} E_0; \\ R_{10} &= \frac{k_{z0} - k_{z2}}{k_{z0} + k_{z2}}, \qquad R_{20} = \frac{2k_{z0}}{k_{z0} + k_{z2}}. \end{aligned} \qquad (19)$$

If the second reaction channel is closed (k_z pure imaginary) then $|R_{10}| = 1$ in agreement with (8). At the threshold for the second channel ($k_{z2} = 0$) the coefficients R_{10} and R_{20} are equal to 1 and 2, respectively. Near the threshold

$$R_{10} = 1 - \frac{2k_{z2}}{k_{z0}}, \qquad R_{20} = 2\left(1 - \frac{k_{z2}}{k_{z0}}\right). \qquad (20)$$

Without entering into a detailed analysis, we note merely that in this case all coefficients have characteristic root-type singularities [cf. (18)], and formula (20) is completely analogous to (16) obtained for a linear periodic system. Since there exists an exact solution, the values of the Fresnel coefficients at the threshold are known in this case.

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