THEORY OF TURBULENT ACCELERATION OF CHARGED PARTICLES IN A PLASMA

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Scattering, diffusion, and acceleration of nonrelativistic charged particles in a turbulent plasma are considered. The turbulence is assumed to be hydromagnetic and its main scale is assumed much greater than the Larmor radii of the accelerated particles. It is shown that if the pulsation spectrum $dh^2/dk \backsim k^{-\nu}$ then for any ν scattering is due to cyclotron resonance; for $\nu > 2$ acceleration is determined by the Fermi mechanism and for $\nu \leq 2$ the Fermi mechanism as well as cyclotron resonance are significant. In an unbounded plasma acceleration leads to growth of the turbulence internal scale and to a corresponding decrease of the number of accelerated resonance particles. A stationary distribution is established in a confined plasma and dissipation of turbulence energy is due to diffusion of the most energetic particles from the turbulent region. Turbulent diffusion and particle acceleration on the sun and in interstellar space are considered. The equilibrium spectrum of hydromagnetic pulsations $dh^2/dk \backsim k^{-2}$ is derived from the experimental data. The diffusion coefficient and energy spectra of the accelerated particles are calculated and the relation between the rate of turbulent energy dissipation and the parameters of the pulsation spectrum is estimated.

1. INTRODUCTION

COMPARISON of the results of an investigation of the generation of cosmic rays, fast protons, and nuclei in solar flares, as well as methods of turbulent plasma heating, suggests that a universal property of plasma turbulence in a magnetic field is the effect of turbulent acceleration, that is, the transfer of an appreciable fraction of the energy of hydromagnetic pulsations to vanishingly small fraction of particles. It is probable that the turbulent acceleration plays an important role in the dissipation of the pulsation energy. Thus, to explain the energetics of cosmic rays it is necessary to bring into play the most powerful galactic processes—supernova explosions. In laboratory experiments, turbulent acceleration sometimes hinders turbulent heating of plasma (transfer of the dissipated energy to the bulk of the particles). Investigations of fast nuclei of solar flares[1] show that turbulent acceleration has a collisionless nature even during the early stages: the chemical composition of the nuclei of the flares is constant and corresponds to the composition of the sun's atmosphere.

The hypothesis of particle acceleration in the annihilation of antiparallel magnetic fields is discussed widely in the astrophysical literature,[2-4] but so far no satisfactory mechanism of such a process has been indicated. In a number of papers (see the review of Tsytovich[5]) attempts were made to attribute the acceleration to Langmuir oscillations. The greatest difficulty with this hypothesis is the preferred acceleration of the ions. In addition, in solar flares, the time of damping of the Langmuir oscillations by collision (density $\sim 10^{10}$ cm^{-3}, temperature ~ 1 keV) is 10^{-2} sec, which is smaller by four orders of magnitude than the acceleration time. This leads to energy contradictions.

In the present paper we investigate the acceleration of particles by a large-scale hydromagnetic turbulence—an aggregate of interacting hydromagnetic waves with a broad spectrum and with a main scale which is much larger than the Larmor radii of the most energetic particles. As is well known, waves of this type exist in cases of turbulent acceleration. Inasmuch as we are considering particles whose velocity is much larger than the Alfven velocity, the Cerenkov resonance does not play an important role and the acceleration is due to the Fermi mechanism or to cyclotron resonance. The relation between these mechanisms is investigated in Sec. 2. It is assumed for simplicity that the angular distribution of the wave vectors of the pulsations has no maxima at large angles to the field. In this case cyclotron resonance is well described by the one-dimensional model (waves traveling along the field).

In Sec. 3 we present calculations of the acceleration by a stationary turbulence in an unbounded plasma, with allowance for the reaction of the particles on the waves, and obtain expressions for the spectrum and total number of accelerated particles, and also for the internal scale of the turbulence as a function of the pulsation spectrum, the energy dissipation velocity, and the time. In Sec. 4 we estimate the stationary distribution function in a bounded plasma with allowance for diffusion. The concluding section is devoted to experimental verification of the theory on the basis of the results of investigations of interplanetary medium. The measured pulsation spectra are used to plot the expected particle spectra, a number of effects are explained, and new phenomena are predicted.

2. TURBULENT DIFFUSION OF PARTICLES IN PHASE SPACE

The study of cyclotron resonance in a plasma is dealt with in many papers. We therefore confine ourselves only to a consideration of those problems which are connected with acceleration, scattering, and diffusion of particles. In this section we investigate these problems under the assumption that the spectrum of the turbulent pulsations is known and does not change in time. The plasma is assumed to be cold (pressure p $\ll H^2/8\pi$), and the fraction of the accelerated particles is assumed small. It is assumed that the main turbulence scale L is much larger than the Larmor radii of the fast particles, and therefore the amplitudes of the resonance pulsations increase with increasing particle energy. If the angular distribution of the wave vectors of the pulsations has a maximum along the force lines of the average field **H** or is isotropic (as will be assumed henceforth), then the main contribution to the scattering and to the acceleration will be made by waves traveling at small angles to the field (at a specified particle velocity, resonance with such pulsations corresponds to the longest wavelengths).

The quasilinear approximation equations for this case were derived in [6]. In our case, the equation from [6] for the fast-particle distribution function averaged over the pulsations is

$$\frac{\partial f}{\partial t}=\frac{e^2}{2M^2c^2}\sum_{i=1}^{2}\Big(\frac{v_z-\sigma_i u}{v_\perp}\frac{\partial}{\partial v_\perp}v_\perp-v_\perp\frac{\partial}{\partial v_z}\Big)\cdot$$
$$\times\Phi(|v_z-\sigma_i u|)\Big(\frac{\partial f}{\partial v_\perp}-\frac{v_\perp}{|v_z-\sigma_i u|}\frac{\partial f}{\partial v_z}\Big), \tag{1}$$

where e and M are the mass and charge of the proton: c is the velocity of light; $v_\perp$ and v_z are the particle velocities perpendicular and parallel to the average field **H**; $\sigma_1 = 1$, $\sigma_2 = -1$, and $\Phi(|v_z - \sigma i_u|)$ is the spectral function of the turbulence dh^2/dk with $k = \Omega_H/|v_z - \sigma i_u|$. The two signs of σ correspond to the two propagation directions of the waves, whose absolute velocity is $u = H/\sqrt{4\pi\rho}$ (ρ—density of the cold plasma).

Since we are considering particles with $v \gg u$, we can expand (1) in powers of u. This expansion is no longer valid for small $v_z \lesssim u$. We shall show somewhat later (see formulas (6) and (7)) that this circumstance introduces no appreciable error in the results. Performing this expansion, we obtain

$$\frac{\partial f}{\partial t}=\frac{e^2}{M^2c^2}\Big\{\Big(\frac{v_z}{v_\perp}\frac{\partial}{\partial v_\perp}-v_\perp\frac{\partial}{\partial v_z}\Big)\Phi(|v_z|)\Big(\frac{\partial f}{\partial v_\perp}-\frac{v_\perp}{|v_z|}\frac{\partial f}{\partial v_z}\Big)$$
$$+u^2\Big[\frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}\frac{v_\perp}{|v_z|}\Phi(|v_z|)\frac{\partial f}{\partial v_\perp}$$
$$+\Big(\frac{v_z}{v_\perp}\frac{\partial}{\partial v_\perp}-v_\perp\frac{\partial}{\partial v_z}\Big)\frac{\partial\Phi(|v_z|)}{\partial v_z}\frac{v_\perp}{v_z^2}\frac{\partial f}{\partial v_z}\Big]\Big\}. \tag{2}$$

Let us consider now different effects described by Eq. (2).

A. Scattering and diffusion. In the zeroth approximation in u^2, putting $v_\perp = v\sin\theta$ and $v_z = v\cos\theta$ (the angle θ is reckoned from **H**), we have

$$\frac{\partial f_0}{\partial t}=\frac{e^2}{M^2c^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\Big|\frac{\Phi(v\cos\theta)}{v\cos\theta}\Big|\frac{\partial f_0}{\partial\theta}. \tag{3}$$

Equation (3) describes pure scattering. Its solutions constitute a stationary isotropic distribution as well as a discrete set of exponentially damped angular harmonics. The first harmonic f_{01} and the damping time τ_S corresponding to it (which is maximal) characterize the isotropization of the angular distribution. If the pulsation spectrum has a power-law form ($dh^2/dk = h_0^2 L(kL)^{-\nu}$ where L is the main scale), we get

$$\tau_s=\frac{1}{\lambda_1\Omega_H}\Big(\frac{H}{h_0}\Big)^2\Big(\frac{v}{\Omega_H L}\Big)^{1-\nu}, \tag{4}$$

where λ_1 is the first eigenvalue of the operator

$$-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta|\cos\theta|^{\nu-1}\frac{\partial}{\partial\theta}.$$

When $\nu = 1$, the eigenfunctions are Legendre polynomials in $\cos\theta$. In particular, $f_{01} = \cos\theta$ and $\lambda_1 = 2$. When $\nu > 1$, the diffusion coefficient with respect to θ is proportional to $|\cos\theta|^{\nu-1}$ and tends to zero as $\theta \to \pi/2$ (i.e., at small longitudinal velocities). Therefore, generally speaking, it is necessary to distinguish between isotropization under initial distributions that are symmetrical and asymmetrical with respect to $\theta = \pi/2$. In the former case, the isotropization takes place independently in both hemispheres, and in the second it should be accompanied by transfers of particles from one hemisphere to other, with the diffusion coefficient tending to zero at the boundary $\theta = \pi/2$. This equation calls for an additional investigation and is of great importance for the calculation of the diffusion coefficient of particles along the force lines. In the present article we investigate spectra of pulsations with $\nu \le 2$ (see item B of this section), and we consider only the case with $\nu = 2$. It is clear that if effective exchange of particles between hemispheres takes place at this value of ν, then the exchange will be even more effective when $\nu < 2$.

Putting in (9) $\nu = 2$, $f_0 = \exp(-\lambda t/\lambda_1\tau_S)\psi$, and $\cos\theta = x$, we get

$$\frac{d}{dx}|x|(1-x^2)\frac{d\psi}{dx}+\lambda\psi=0. \tag{5}$$

in the region of small x ($|x| < x_0 \sim u/v$) there is a finite probability that the particles will pass through the point x_0, owing to the Fermi collisions. We shall show that in the presence of an arbitrarily small probability of such a transition, the isotropization time is close to (4) also for an asymmetrical initial distribution.

Assume that when $|x| > x_0$ the function ψ is described by Eq. (5), and when $|x| < x_0 \ll 1$ we have $x_0\psi'' + \lambda\psi = 0$. Let us construct the harmonics that are asymmetrical in θ. If $\psi_1^+(x)$ is the symmetrical harmonic with the smallest eigenvalue $\lambda = \lambda_1$, then when $|x| \gg x_0$ Eq. (5) is satisfied by the function

$$\psi_1^{-(0)}=\begin{cases}\psi_1^+(x), & x>0\\ -\psi_1^+(x), & x<0\end{cases}.$$

In the region $|x| \ll |x_0|$ the asymmetrical solution is of the form $\psi = ax$. Indeed, the damping time $\psi_1(-)$ is τ_S, and the time of establishment of the stationary mode in a layer of width $2x_0$, for a specified difference in values of ψ on the edges, is of the order of $x_0^2\tau_S/x_0 \ll \tau_S$ (the diffusion coefficient when $|x| < x_0$ is $x_0\tau_S$). Thus, it is necessary to join together the solution $\psi_1 = \psi_1^{-(0)}(x)$ at $|x| \gg x_0$ with $\psi_1 = ax$ at $x \le x_0$. The general solution of (5) for $|x| \ll 1$ is

$$J_0\Big(\frac{\sqrt{\lambda|x|}}{2}\Big)+\alpha N_0\Big(\frac{\sqrt{\lambda|x|}}{2}\Big),$$

where J_0 and N_0 are Bessel and Neumann functions of zero order. At large distances ($|x| \sim 1$), J_0 goes over into the solution $\psi_1^+(x)$ which is bounded at zero, and N_0

goes over into a certain solution of (5) $\psi_1^*(x)$, which has a logarithmic singularity at zero.

When $x > x_0$ the sought solution has the form $\psi_1 = \psi_1^+ + \alpha\psi_1^*(x)$, and at the point x_1 the function is

$$\psi_1 \approx A\left[J_0\left(\frac{\sqrt{\lambda_1 x_1}}{2}\right) + \alpha N_0\left(\frac{\sqrt{\lambda_1 x_1}}{2}\right)\right] = a x_1,$$

and its derivative is

$$\psi_1' \approx \frac{A}{4}\sqrt{\frac{\lambda_1}{x_1}}\left[J_1\left(\frac{\sqrt{\lambda_1 x_1}}{2}\right) + \alpha N_1\left(\frac{\sqrt{\lambda_1 x_1}}{2}\right)\right] = a.$$

From this we get the characteristic equation for α (the condition for the vanishing of the determinant of the system):

$$\alpha = -\frac{J_0(\xi) + {}^1/_2\xi J_1(\xi)}{N_0(\xi) + {}^1/_2\xi N_1(\xi)},$$

where $\xi = \sqrt{\lambda_1 x_1}/2$. Using the expansions of $J_{0,1}$ and $N_{0,1}$ at small values of ξ, we get

$$\alpha = \pi / \ln(4/\lambda_1 x_1) \tag{6}$$

and consequently

$$\psi_1^- = \psi_1^{-(0)} + \frac{\pi}{\ln(4/\lambda_1 x_1)}\psi^*(x). \tag{7}$$

When $x \sim 1$ the correction is logarithmically small, and when $x = x_0$ a transition layer with a sharp gradient is produced. The correction to λ is equal to α in order of magnitude, and is therefore insignificant. Consequently, when $\nu \leq 2$ the penetration of the particles through the barrier $\theta = \pi/2$ occurs at the same rate as the isotropization. However, at larger values of ν this conclusion does not hold; in particular, when $\nu \geq 3$ the rate of penetration decreases rapidly.

With the aid of τ_s and $f_1(\theta)$ we can easily calculate the coefficient of diffusion of the particles along the force lines. Using the method described in [7], we get

$$D_\parallel = v^2\tau_s\left(\int_0^\pi f_1^{(-)}\sin\theta\cos\theta\, d\theta\right)^2 \Big/ \int_0^\pi f_1^{(-)2}\sin\theta\, d\theta; \tag{8}$$

when $\nu = 1-2$ we get $D_\parallel \approx v\tau_s/3$. The transverse diffusion coefficient is $D_\perp = D_\parallel/[1 + (\Omega_H\tau_s)^2]$.

B. Acceleration. In the zeroth approximation in u_2/v_2, there is no acceleration. Therefore the energies of the particles with $v \gg u$ do not change appreciably during the isotropization time. This result has a clear-cut physical meaning: the scattering time τ_s is proportional to the reciprocal of the square of the Lorentz force, i.e., to $c^2/e^2v^2h^2$, and the acceleration time τ_ε is proportional to the reciprocal of the square of the electric force, $1/e^2E^2$. In hydromagnetic waves $E \sim uh/c$ and $\tau_s \sim u^2\tau_\varepsilon/v^2$. Consequently, in the investigation of the acceleration, the distribution function (which we shall denote by F) can be regarded as isotropic ($F = F(v, t)$). Substituting F in (2) and averaging the coefficients over the solid angle, we obtain the sought acceleration equation

$$\frac{\partial F}{\partial t} = \frac{1}{v^2}\frac{\partial}{\partial v}v^2 D(v)\frac{\partial F}{\partial v}, \tag{9}$$

where the diffusion coefficient in velocity space is

$$D(v) = \frac{e^2u^2}{M^2c^2v}\int_0^{\pi/2}\frac{\sin^3\theta}{\cos\theta}\Phi(v\cos\theta)\, d\theta. \tag{10}$$

When

$$\Phi(v\cos\theta) = \frac{dh^2}{dk}\Big|_{k=\Omega_H/v\cos\theta} = h_0^2 L\left(\frac{v_L}{\Omega_H}\right)^\nu |\cos\theta|^\nu$$

we get

$$D(v) = \Omega_H u^2\left(\frac{h_0}{H}\right)^2\left(\frac{v}{L\Omega_H}\right)^{\nu-2}\frac{2}{\nu(\nu+2)}$$
$$= \frac{2}{\nu(\nu+2)}\frac{u^2 v}{L}\left(\frac{h_0}{H}\right)^2\left(\frac{v}{L\Omega_H}\right)^{\nu-2}. \tag{11}$$

The solution of a diffusion equation of the type (9), with a power-law dependence of the diffusion coefficient on the coordinate and with an initial condition $F_{t=0} \propto \delta(v)$ is (see, for example, [8])

$$F(v, t) = \text{const}\cdot v^{3(\nu-1)/(3-\nu)}[D(v)t]^{-3/(3-\nu)}\exp\left\{-\frac{v^2}{(3-\nu)^2 D(v)t}\right\}. \tag{12}$$

Thus, a Maxwellian spectrum is produced asymptotically when $\nu = 1$, and the spectrum of the solar-flare protons, of the type $\exp(-v/v_0)$[9], is produced when $\nu = 2$. The values $\nu > 2$ are of no interest, since (see the next item) in this case the acceleration is determined not by the cyclotron resonance, but by the Fermi mechanism. It is important to note that if the constant ν remains unchanged, then the spectrum (12) is formed also for an arbitrary time dependence of H, L, u, and h_0: assuming $D(v) = D_0(v)\cdot\varphi(t)$ and introducing the variable

$$\tau = \int_0^t \varphi(t)\, dt,$$

we can reduce the equation for the nonstationary turbulence to the form (9). In particular, if $\varphi(t)$ fluctuates, then the acceleration is determined by the mean value of φ.

C. Relation between cyclotron resonance and the Fermi mechanism. As already noted, at particle velocities $v \gg u_a$ the role of the Cerenkov resonance is small. It can be shown that the Cerenkov diffusion coefficient in velocity space is smaller than (10) by approximately u^2v^2 times. Therefore the only competing process is the Fermi mechanism, i.e., acceleration upon adiabatic reflection of the particles from the waves with a scale larger than Av/Ω_H, where the adiabaticity parameter A denotes the ratio of the scale of the inhomogeneity of the field to the Larmor radius, beginning with which the magnetic moment is conserved. As shown by recent experimental investigations,[10] $A \sim 10$. The theory of the Fermi acceleration in a turbulent plasma was considered in [11], where it was shown, in particular, that the scattering of the particles in cyclotron resonance greatly increases the efficiency of the Fermi mechanism in a rarefied plasma. (as is well known, in the absence of scattering, the Fermi mechanism does not lead to an appreciable acceleration). It was also shown in [11] that if the turbulence spectrum is of the form $h(\lambda) \propto \lambda^\mu$ ($h(\lambda)$ is the average amplitude of the pulsations with scale λ), then the principal role in acceleration belongs when $\mu \leq 1/2$ to collisions with long waves, and when $\mu \leq 1/2$—with waves whose length is $\sim Av/\Omega_H$. Since $h(\lambda) = \sqrt{k\,dh^2/dk}\,|_{k=2\pi/\lambda}$, the exponent μ is connected with the value of ν introduced above by the relation $\mu = (\nu - 1)/2$ and $\mu > 1/2$ when $\nu > 2$. These results were obtained under the assumption that the investigated reflections are statistically independent. In the case $\nu > 2$, such a hypothesis is probable, since at large collision times, $\sim L/v$, the jaggedness of the crests of the large-scale waves and the rapid drift of the particles on the shock fronts come into play. When $\nu < 2$, the statistical independence of the Fermi collisions is ensured by scattering.

If $\nu > 2$, the Fermi diffusion coefficient in velocity space is $\approx u^2 v/L$ when $h_0 \sim H$ and $u^2 v h_0/LH$ when $h_0 \ll H$, and as can be seen from (11), it is approximately $Hh^{-1}(L\Omega_H/v)^{\nu-2}$ times larger than the cyclotron-resonance coefficient (under the assumption that $L \gg v/\Omega_H$). At the same time, as shown in [11], at not too large values of $\nu - 2$ the cyclotron resonance ensures the scattering necessary for the Fermi mechanism even at particle velocities exceeding u by several times, even though scattering is small during the reflection time. The diffusion of the particles along the field when $h_0 \sim H$ is determined by the Fermi collisions. The mean free path is on order of L and does not depend on the particle velocity.

When $\nu \leq 2$ the integral[11]

$$D(v) = 2u^2 v \int_0^{\theta_0} \sin\theta \cos^3\theta \frac{d\theta}{\lambda(h)}\bigg|_{h=H\,\mathrm{ctg}^2\theta} \tag{13}$$

diverges when $\theta_0 \to \pi/2$. Here $\lambda(h)$ is the average length of a wave with amplitude h. In the case of a power-law pulsation spectrum we have $\lambda(h) = 2\pi = 2\pi L(h/h_0)^{(\nu-1)/2}$. The limit of θ_0 is determined by the condition $\lambda(h = H\cot^2\theta) = Av/\Omega_H$. Substituting in (13) and assuming that $\pi/2 - \theta_0 \ll 1$, we obtain

$$D(v) = \frac{u^2 v}{4\pi L}\left(\frac{h_0}{H}\right)^2 \left(\frac{A}{2\pi}\right)^{\nu-2} \frac{\nu-1}{2-\nu}. \tag{14}$$

As $\nu \to 2$ we have

$$D(v) = \frac{u^2 v}{4\pi L}\left(\frac{h_0}{H}\right)^2 \ln\frac{L\Omega_H}{Av}. \tag{15}$$

Formulas (14) and (15) are valid when the collisions are statistically independent. It can be shown (with the aid of (13)) that if ν is not too close to 2, then the isotropization time in a solid angle element $\pi/2 + \theta_0 \gtrsim \theta \gtrsim \pi/2 - \theta_0$ is much smaller than the time of the Fermi collision of the particles with the same θ, and statistical independence obtains. As $\nu \to 2$, the scattering time in a small solid angle γ near $\theta = \pi/2$ is $\tau_S\gamma$ and is comparable with the collision time. In this case, the effective time of the Fermi collisions increases somewhat: the frequency is equal to $(1/\gamma\tau_S)\gamma = 1/\tau_S$. The factor γ determines the probability that the particle will stay in the solid angle γ if the angular distribution is isotropic. Recognizing that $\delta v = \delta v_Z v_Z/v$ and $\delta v_Z = 2u$, we obtain

$$D(v) = \frac{1}{2}\frac{\overline{(\Delta v)^2}}{\Delta t} = \frac{2}{\tau_s}u^2 \int_0^{\pi/2} \sin\theta\cos^2\theta\, d\theta = \frac{2}{3}\frac{u^2}{\tau_s}.$$

Thus, when $\nu \leq 2$ the Fermi mechanism becomes manifest primarily via collision of particles with small v_Z ($\ll v$) and short waves. The dependence of D(v) on the parameters of the problem is the same, apart from a numerical factor, as in cyclotron resonance (see (14) and (11)). It is very important that D(v) increases with increasing v, i.e., the wave-energy absorption rate per particle.

3. TURBULENT ACCELERATION IN AN UNBOUNDED PLASMA

In Sec. 5 we shall present a number of arguments to show that the parameter ν in the pulsation spectrum is close to 2. Consequently, the interaction between the particles and the waves is concentrated in the short-wave region of the spectrum. We shall show that in this case turbulent acceleration is a universal property of the turbulence in the plasma, and is due to the most fundamental singularities of the corresponding processes, namely: a) the resonant character of the scattering of the particles by the waves; b) the continuity of the flux of the turbulent energy in wave-number space; c) the increase of D(v) with v. Indeed, it follows therefore that the most effective dissipation is connected with the hard tail of the distribution function, and that as the acceleration proceeds the number of particles necessary to absorb the constant flux of the turbulent energy $\dot{\varepsilon}$ in the wave-number space should decrease. The dissipation will in this case be connected essentially with a certain group of fast particles, the velocities of which are bounded from below by the time-increasing quantity w. The wave number $k_0 = \Omega_H/w$ characterizes the internal scale of the turbulence. When $k > k_0$, the spectrum of the pulsations is abruptly cut off, and the particles with $v < w$ cease to accelerate, owing to the vanishing of the resonance waves (the Fermi acceleration is also stopped, owing to the absence of scattering).

If we assume that the boundary of the spectrum k_0 is sufficiently steep, we can consider this effect quantitatively. When $v > w(t)$ we have

$$\frac{\partial F}{\partial t} = \frac{1}{v^2}\frac{\partial}{\partial v} v^2 D(v)\frac{\partial F}{\partial v}. \tag{16}$$

When $v < w$, the function F is stationary and is determined from the continuity of F when $v = w$. The quantity w(t) is a self-consistent variable and is determined from the constancy of the number of particles

$$\frac{\partial}{\partial t}\int_v^\infty v^2 F(v)\, dv = 0 \tag{17}$$

for all $v < w$.

Finally, we introduce a condition that determines the number of particles with $v > w$. This condition is

$$\varepsilon = \frac{M}{2}\int_w^\infty v^2 \frac{\partial}{\partial v} D(v) v^2 \frac{\partial F}{\partial v}\, dv. \tag{18}$$

Here $\dot{\varepsilon}$ is the energy of the turbulent pulsations dissipated in 1 cm^3 per second, and the integral in the right side gives the rate of energy growth of the particles with $v > w$, which is determined from (16). The value of $\dot{\varepsilon}$ is governed by the turbulence properties in the long-wave part of the spectrum and does not depend on F. Equation (16) and the condition for the continuity of F when $v = w$ determine F accurate to a constant factor. This factor and w are determined from the conditions (17) and (18).

Let us consider in greater detail the case $\nu = 2$($D(v) \approx (u^2/L)(h_0/H)^2 v \equiv \Lambda v$). Equations (16) and (17) now take the form

$$\frac{\partial F}{\partial t} = \frac{\Lambda}{v^2}\frac{\partial}{\partial v} v^3 \frac{\partial F}{\partial v}, \quad \varepsilon = \frac{\Lambda M}{2}\int_w^\infty v^2 \frac{\partial}{\partial v} v^3 \frac{\partial F}{\partial v}\, dv$$

and (when $w \gg u$) they do not contain any scale parameters with dimensionality v or t. Therefore the solution is self-similar and, as shown in [11], the self-similar variable is $x = v/\Lambda t$. We seek a solution in the form $w = \Lambda t x_0$ (x_0 = const), $F = t^{-n}\psi(x)$. From (18) (with $\dot{\varepsilon}$ = const) we obtain $n = 4$. Then the equations take the form

$$x\psi'' + (3+x)\psi' + 4\psi = 0 \quad (x > x_0), \tag{19}$$

$$2\dot{\varepsilon}\Lambda^{-5}M^{-1} = \int_{x_0}^{\infty} x^2 \frac{d}{dx} x^3 \frac{d\psi}{dx}\, dx. \tag{20}$$

from the continuity of F when $x = x_0$ it follows that when $x < x_0$

$$\psi(x) = (x_0/x)^4 \psi(x_0). \tag{21}$$

The distribution function F at $x = x_0$, as seen from (21), is stationary and is equal to $F(v) = \Lambda^4 x_0^4 \psi(x_0) v^{-4}$.

The confluent hypergeometric equation (19) has an analytic solution

$$\psi(x) = (x-3)e^{-x}, \tag{22}$$

as can be readily verified directly. The second solution (see, for example [12]) has the form x^{-4} when $x \to \infty$, and leads to divergence of the particle energy and therefore must be discarded.

From (18), (21) and (22) we get $x_0 = 4$. Finally, putting $\psi(x) = Q(x-3)e^{-x}$ and substituting in (20) we get

$$\psi(x) = \frac{e^4}{11536}\dot{\varepsilon}\Lambda^{-5}M^{-1}(x-3)e^{-x}.$$

The distribution function is

$$F(v,t) = \begin{cases} \dfrac{1}{1848}\dfrac{\varepsilon}{\Lambda M}\dfrac{1}{v_0^4}\left(\dfrac{v}{v_0} - 3\right)e^{-v/v_0+4}, & v > w = 4v_0 \\ \dfrac{32}{231}\dfrac{\dot{\varepsilon}}{\Lambda M}\dfrac{1}{v^4}, & v < w \end{cases} \tag{23}$$

here $v_0 = \Lambda t$.

Inasmuch as in the main scale ($k = 2\pi/L$) we have $h^2(L) = h_0^2/2\pi$, and the energy density is $h^2(L)/4\pi = h_0^2/8\pi^2$, the parameter $\dot{\varepsilon}$ is equal in order of magnitude to $(h_0^2 u/16\pi^3 L)(h_0/H)^2$ (i.e., the energy in the main scale divided by the time of turbulent decay). Recognizing that $\Lambda \approx (u^2/L)(h_0/H)^2$, and introducing the density N of the cold plasma, we can transform the factor $\dot{\varepsilon}/\Lambda M$ into $(Nu/4\pi^2)(h_0/H)^2$.

The total number of particles with velocities larger than v is

$$n(>v) = \int_v^{\infty} F(v,t)v^2\, dv.$$

when $v \geq w$ and $v < w$ we have

$$n = \frac{1}{1848}\frac{Nu}{4\pi^2 v_0}\left(\frac{h_0}{H}\right)^2\left(\frac{v}{v_0}\right)^3 e^{-v/v_0+4}, \quad v \geqslant w,$$
$$n = \frac{2N}{231\pi^2}\left(\frac{h_0}{H}\right)^2\frac{u}{v}, \qquad v < w. \tag{24}$$

the process under consideration begins when v_0 is of the order of several times u (when the role of the Cerenkov resonance becomes small). As seen from (24), even when $v_0 \sim (3-5)u$ the number of accelerated particles ($v > w$) is only $\sim 10^{-3}$ of the total number, and decreases rapidly with increasing w. Thus, only a small fraction of the particles is actually accelerated.

It is of interest to calculate the integral intensity spectrum of the accelerated particles

$$S(>v) = \int_0^{\infty} v^3 F dv. \tag{25}$$

When $v > w$

$$S(>v) = \frac{1}{1848}\frac{Nu}{4\pi^2}\left(\frac{h_0}{H}\right)^2 (x^4 + x^3 + 3x^2 + 6x + 6)e^{-x+4} \quad \left(x = \frac{v}{v_0}\right), \tag{26}$$

and when $v < w$

$$S(>v) = \left(\frac{32}{231}\ln\frac{4v_0}{v} + \frac{199}{924}\right)\frac{Nu}{4\pi^2}\left(\frac{h_0}{H}\right)^2. \tag{27}$$

When $v < w$ the intensity depends on v very little, and when $v > w$ it decreases steeply. It follows therefore, in particular, that after the lapse of a certain time (on the order of $10u/\Lambda$) the intensity of the particles with velocities $\sim 10u$ reaches a value $\sim (Nu/200)(h_0/H)^2$ and subsequently remains practically unchanged.

In conclusion we note that a similar effect, the growth of the internal turbulence scale and the acceleration of a decreasing number of particles, should take place when electrons are accelerated in cyclotron resonance with pulsations of the high-frequency branch of rapid magnetosonic waves, and also in Cerenkov resonance with Langmuir waves. In the latter case, owing to the strong dispersion, each value of the particle velocity corresponds to a definite wavelength ($kv \sim \omega_0$). If the spectrum of the Langmuir waves is such that the amplitude decreases with increasing k, a similar acceleration effect will take place.

4. ACCELERATION IN A BOUNDED PLASMA. ROLE OF DIFFUSION

In a bounded plasma the growth of the particle energy on the tail of the distribution function is stopped by diffusion, and the distribution function becomes stationary. The dissipated energy goes to replenishment of the fast particles that diffuse from the turbulent region. In this case the process is approximately deproximately described by the equations

$$\frac{u^2}{L}\left(\frac{h}{H_0}\right)^2\frac{1}{v^2}\frac{\partial}{\partial v}v^3\frac{\partial F}{\partial v} - \frac{\pi^2}{32}\frac{L}{z_0^2}\left(\frac{H}{h_0}\right)^2 F = 0, \tag{28}$$

$$\frac{\pi^2}{64}\frac{ML}{z_0^2}\left(\frac{H}{h_0}\right)^2\int_0^{\infty} v^5 F(v)\, dv = \varepsilon. \tag{29}$$

In (28) we replaced the diffusion coefficient $D_{\|}\partial^2/\partial z^2$ by the damping decrement of the fundamental $\pi^2 D_{\|}/4z_0^2$ ($2z_0$ is the length of the turbulent region along **H**) and expressed $D_{\|}$ in terms v, L, and h_0 with the aid of the formulas of the Sec. 2. A numerical calculation for λ_1 at $\nu = 2$ yields $\lambda_1 \approx 2.6$. Equation (29) describes the energy balance. The integral in the left side gives the rate of leakage of the energy as a result of the particle diffusion.

From (28) we obtain

$$F = \frac{C}{v}K_1\left(\frac{v}{w}\right), \quad w = \frac{4\sqrt{2}}{\pi}u\frac{z_0}{L}\left(\frac{h_0}{H}\right)^2, \tag{30}$$

where K_1 is the Macdonald function. Substituting in (29), integrating, and putting $\varepsilon = (h_0^2 u/8\pi^2 L)(h_0/H)^2$, we get C and obtain

$$F = \frac{\pi^2\sqrt{2}}{4096}\left(\frac{L}{z_0}\right)^3\frac{NH^4}{h_0^4}\frac{1}{u^2 v}K_1\left(\frac{v}{w}\right). \tag{31}$$

The concentration of the fast particles in the plasma, q, is equal to

$$q = \frac{1}{N}\int_0^\infty v^2 F\,dv = \frac{\sqrt{2}}{128}\frac{L}{z_0} \quad (32)$$

and depends only on the ratio of the dimension of the turbulent region and the main scale. The time of plasma confinement

$$\tau = \frac{4}{\pi^2}\frac{1}{q}\frac{z_0^2}{D_\parallel} = \frac{128\sqrt{2}}{\pi^3}\left(\frac{z_0}{L}\right)^2\left(\frac{h_0}{H}\right)^2\frac{z_0}{w} \quad (33)$$

should, according to the conditions of the problem, be sufficiently large compared with the time necessary to accelerate the particles to a velocity $\sim w$. Since $w \approx (u^2/L)(h_0/H)^2 t$, the condition for the applicability of the result is $t \ll \tau$ or, after substituting the values of w,

$$z_0 \gg L. \quad (34)$$

The fast-particle concentration will always be small in this case.

The results of Secs. 3 and 4 show that the cause of the acceleration of a small fraction of particles is the growth of the amplitude of the pulsations with increasing wavelengths, from which it follows that the efficiency of the interaction of the particle with the waves increases with increasing acceleration. This effect can be used to accelerate particles to energies $\sim Mw^2/2$ (see (30)) with high efficiency. However, inasmuch as the particle velocity cannot exceed $L\Omega_H$, under laboratory conditions ($L \lesssim 10$ cm, $\Omega_H \lesssim 10^7$ rad/sec) the obtained energies are small ($\lesssim 50$ keV). The situation is entirely different in outer space, where, owing to the tremendous dimensions of the turbulent regions, the particles can acquire a very high energy.

5. APPLICATIONS TO COSMOPHYSICAL PROBLEMS

At the present time there exist certain data on the spectra of the plasma turbulence in the interplanetary medium, and much material has been accumulated on a variety of fast particles observed in this region. An investigation of plasma turbulence with the aid of automatic interplanetary stations is highly promising. Let us consider certain applications of the theory developed above to cosmophysical problems.

A. Propagation of fast particles of solar flares in the interplanetary medium. Experiment has shown that fast particles with energies up to ~ 1 GeV are generated in solar flares. The propagation of these particles in the interplanetary medium can be satisfactorily described by means of a spherical diffusion wave from an instantaneous point source:

$$S(>v) = \mathrm{const}\cdot(D_R t)^{-3/2}\exp(-R^2/4D_R t), \quad (35)$$

where D_R is the diffusion coefficient, t is the time from the instant of the flash, and R is the heliocentric distance. Since actually the diffusion is along the force lines, $D_R = D_\parallel (dR/ds)^2$, where ds is the differential of the arc of the force line, and $D_\parallel$ is determined by relations (4) and (8) of Sec. 2. The measurements are made, as a rule, near the earth's orbit, and $R = R_0 \approx 1.5\times10^{13}$ cm is fixed, while (35) describes the time dependence of the intensity. The maximum is reached at $t = t_0 = R^2/6D_R$. If we measure $S(>v, t)$ for a number of energies, we can determine $D_R(v)$ and from it construct the spectrum of the pulsations in the region of the earth's orbit.

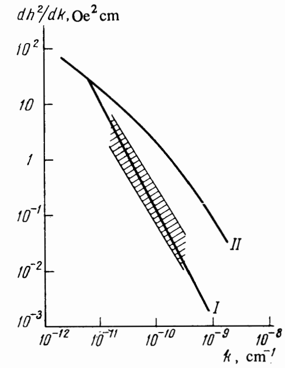

FIG. 1.

Results of investigations of a number of flares in 1961–1962, given in [13], show that if we replace t by $x = vt$ in (35), then in the energy interval from 3 to 400 MeV the diffusion curves $S(>v, x)$ coincide, apart from a normalization factor that depends on v and determines the injection spectrum. The maximum is reached at $x_0 = vt_0 = 1.5\times10^{14}$ cm. Since $D_R \propto v$, the pulsation spectrum is of the form $dh^2/dk \propto k^{-\nu}$ with $\nu \geq 2$. If we assume that $\nu = 2$, then we find with the aid of (4) and (8) that $dh^2/dk = 6\times10^{-22}\,k^{-2}\,[\mathrm{Oe}^2\cdot\mathrm{cm}]$ (for $x_0 = 1.5\times10^{14}$ cm). This spectrum is represented by curve I of Fig. 1. For comparison we present the spectrum measured experimentally on the automatic station "Mariner IV".[14] 1) The diffusion coefficient calculated from these spectra agrees well with the value 5×10^{21} cm^2/sec at energy 100 MeV, obtained in the analysis of a number of flares in [15].

However, this form of the spectrum corresponds to the quiescent state of the solar wind. The spectrum is appreciably altered in the case of disturbances on the sun. Thus, in October 1962, a spectrum corresponding to curve II in Fig. 1 was obtained on the "Mariner II" station.[16] As seen from the figure, in the region of short waves the spectrum also has the form $dh^2/dk \propto k^{-2}$. Apparently, the equilibrium spectrum of the large-scale hydromagnetic turbulence, corresponding to a time-invariant energy flux in the wave-number space, is of the form $dh^2/dk \propto k^{-2}$, and the spectrum II of Fig. 1 in the long-wave region has not yet time to assume its form, and retains its initial form. If the flare occurs at an instant when a pulsation spectrum of the type II is present, then the diffusion wave of the low-energy particles will be strongly stretched in time compared with the typical cases.

B. Acceleration in the interplanetary medium. The presence of turbulence in the solar wind should lead to particle acceleration.

At solar-wind velocities $U \sim (3-4)\times10^7$ cm/sec and at large-scale inhomogeneity dimensions 5×10^{12} cm near the earth's orbit, the spectrum of the turbulence and the average field can be regarded as stationary during the time $\tau \sim (1-2)\times10^5$ sec. If $dh^2/dk = Ak^{-2}$,

1)In the experiment, the time dependence of h^2 is measured and the frequency spectrum dh^2/df is plotted, where $f = (k/2\pi)(U + u)$, $u \sim 5\times10^6$ cm/sec is the Alfven velocity, $U = 3\times10^7$ cm/sec is the velocity of the solar wind, and f is the Doppler frequency.

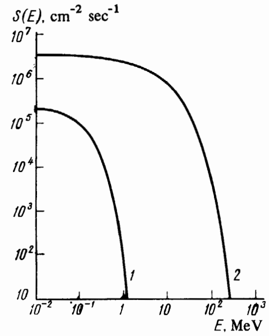

FIG. 2.

then the parameter v_0 (see Sec. 3) can be represented in the form $v_0 = Au^2t/H^2 = At/4\pi MN$. For $A = 6\times 10^{-22}$ (spectrum I, Fig. 1) we have $v_0 = 10^6$, which is 30 times smaller than the velocity of the solar wind and 5 times smaller than the Alfven velocity (typical parameters of the solar wind are $H = 5\times 10^{-5}$ Oe, $N = 5\ cm^{-3}$, and $u = 5\times 10^6$ cm). However, at larger turbulence (spectrum II), the acceleration becomes sufficiently effective, $v_0 = 10^8$ cm/sec. The spectrum of the particles corresponding to the parameters $H = 5\times 10^{-5}$ Oe, $N = 5\ cm^{-3}$ and $t = 2\times 10^5$ sec and to the pulsation spectrum II is shown in Fig. 2 (curve 1). The calculation was made on the basis of the formulas of Sec. 3. The value of ε is determined from the part of the spectrum with $k \sim (1-2)\times 10^{-10}\ cm^{-1}$, where, in accordance with our assumption, ε ceases to depend on k. As seen from the figure, the intensity of the particles with energies on the order of hundreds of keV are quite large and amount $\sim 10^4\ cm^{-2}\ sec^{-1}$.

A number of Soviet and American automatic stations revealed in 1964–1965 intensity flares of protons with energies ~ 500 keV – 1 MeV, in which the fluxes increased from a value corresponding to the background to a value $\sim 10^3\ cm^{-2}\ sec^{-1}$.[17–19] The spectra of these flares turned out to be very soft. Thus, at energies close to 30 MeV, there are noticeable decreases in the intensity of the cosmic rays, by several percent, accompanying the flares of protons with energies ~ 1 MeV. These decreases, as is well known, indicate the appearance of a disturbed plasma ([20], Sec. 26). It can therefore be assumed that the flares of low-energy protons are connected not with acceleration on the sun, but with the turbulence of the interplanetary medium.

It should be borne in mind that the experimental results presented above were obtained during years of low solar activity. It is possible that in maximum years (in the presence of powerful and frequent chromospheric flares) the turbulent pulsations are sometimes much more powerful. In this connection we indicate that when the pulsation amplitudes are increased by a factor of 3 compared with the pulsation spectrum II of Fig. 1 within a time $\sim 10^5$ sec, the particles are accelerated to $v_0 \approx 10^9$. The spectrum for $v_0 = 10^9$ cm/sec is also shown in Fig. 2 (curve 2). In this case powerful streams of protons appear, with energies of dozens of MeV. Similar phenomena were observed with the aid of sounding balloons in 1959–1966.[15, 21] Several days following the diffusion of proton wave of the solar flare (the maximum of which was observed within a few hours), a new maximum was produced, with a softer spectrum. This maximum is always connected with the arrival of disturbed plasma on earth. A number of difficulties that are encountered when attempts are made to explain this effect as being due to acceleration on the sun were indicated in [22]. It is probable that these effects are cases of the most powerful turbulent acceleration in interplanetary medium.

6. CONCLUSION

The foregoing investigation shows that in the case of large-scale hydromagnetic turbulence in a collisional plasma there occurs acceleration of a small fraction of particles to high energies. In spite of a certain indeterminacy in certain characteristics of the turbulence, the theory makes it possible to explain a number of effects connected with turbulent acceleration. In particular, an analysis of different phenomena on the basis of the present theory leads to an important conclusion concerning the form of the equilibrium spectrum of such a turbulence; this conclusion is confirmed by direct measurements.

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