CLASSICAL AND QUANTUM RESTRICTIONS ON THE DETECTION OF WEAK DISTURBANCES OF A MACROSCOPIC OSCILLATOR

V. B. BRAGINSKIĬ

Moscow State University

Submitted May 3, 1967

Zh. Eksp. Teor. Fiz. 53, 1434–1441 (October, 1967)

The weakest force whose effect on a macroscopic oscillator can be observed can depend on the coefficient of friction coupling the oscillator mass with the laboratory, and on the time required for the measurement. If the coefficient of friction is sufficiently small so that the relaxation time of the oscillator is greater than the time spent on the measurement, then it is possible to detect reliably a change of energy of the oscillator induced by an external force, which is smaller than the equilibrium energy κT. The relaxation time of mechanical oscillators under laboratory conditions may exceed 10^7 sec. As an example, an experiment is discussed in which an energy change of $\kappa \cdot 60^\circ K = 7.8 \times 10^{-15}$ erg can be determined reliably at a temperature $T_{equil} = 4500^\circ K$, where the time of measurement is equal to 23 sec and the relaxation time is 2400 sec.

The role of quantum fluctuations in the optical Fabry-Perot resonator used for the registration of the small vibrations of a mechanical oscillator is discussed. Approximate expressions for the smallest detectable forces are derived, taking account of the finite time spent during measurement. It is shown that a decrease in the spectral density of the fluctuation power modulation coefficient of the optical source and a narrowing of the natural line width lower the level of the smallest detectable forces. Estimates of the sensitivity level attainable in principle are given for experiments designed for the verification of the equivalence principle, for the detection of rare particles with fractional charges, or for the observation of gravitational radiation.

A great number of experiments, in which the observation of the physical effect was reduced to the registration of a small force acting on a macroscopic test body, have yielded fundamental physical information. Such experiments are, for example, the experiments of Einstein and de Haas, of Millikan, of Eötvos-Dicke, of Shubnikov and Lazarev (nuclear paramagnetism) and also a whole series of proposed and partially executed experiments (mechanical experiments testing parity nonconservation, searching for rare particles with fractional charges, attempting to detect gravitational radiation, etc.). The development of present-day experimental techniques has made it possible to decrease significantly (by several orders of magnitude as compared to the experiments already performed) the friction coupling the test body with the laboratory apparatus, and hence also the fluctuating forces acting on the test body. Evidently, a strong decrease in this fluctuating force should be expected in experiments carried out with test bodies in space (weightlessness and near-perfect vacuum).

In view of this development, it is reasonable to discuss, in connection with experiments using test bodies, the classical and quantum restrictions on weak disturbances on a macroscopic mechanical oscillator, taking account of the reverse effect of the measuring apparatus on the oscillator and of the restrictions on the time which may be spent during the measurement. Moreover, it is interesting to compare the presently attainable level of resolution in such experiments (sensitivity) with the sensitivity level which can in principle be achieved in some gravitational and nuclear experiments.

It is known that in the classical approximation, the smallest force whose effect on a macroscopic oscillator can be observed depends essentially on the coefficient of friction which couples the oscillator to the laboratory. As the coefficient of friction H decreases, the fluctuating forces decrease and so does therefore the smallest observable force. With decreasing H the time constant of the oscillator $\tau^* = 2m/H$ (m is its mass) increases, and the response to the action of a small force during the time $\hat{\tau}$ can be investigated by observing the change in the amplitude of the vibrations of the oscillator after the time $\hat{\tau}$. If this change is larger than that expected from fluctuations, we may say that the force $F(\tau)$ is detectable. In the particular case where the force $F(\tau)$ has the form of a train of sinusoidal oscillations of length $\hat{\tau}$, with amplitude F_0 and frequency ω, equal to the eigenfrequency of the oscillations of the oscillator ω_{mech}, one may observe F_0 if

$$F_0 \geqslant \theta\sqrt{2\kappa TH/\hat{\tau}} = \theta\sqrt{4\kappa Tm/\hat{\tau}\tau^*}, \tag{1}$$

where κ is the Boltzmann constant, T is the temperature, and θ is a dimensionless factor of the order of unity which depends on the chosen confidence level for the observation (cf. [1] for more details on θ). The relation (1) also holds for $\hat{\tau} \ll \tau^*$. We note that one may recommend a statistical method of evaluation where it is not necessary to know the values of T and τ^*,[1] in the case when the measurement can be repeated a few times (not less than twice). In the derivation of (1) we have assumed that the apparatus which registers the small oscillations does not react back on the oscillator, which is permissible in the classical approximation.

It is important that the force to be observed with the amplitude F_0 [relation (1)] gives rise to an addition or a subtraction of a reliably observable portion of the energy ΔW_0 which is smaller than the equilibrium energy

κT. If the initial amplitude of the vibrations of the oscillator is smaller than or of the order of $(\kappa T\hat{\tau}/m\omega_{mech}^2\tau^*)^{1/2}$ then $\Delta W_0 \approx \kappa T\hat{\tau}/\tau^*$, and if the initial amplitude is of the order of $(\kappa T/m\omega_{mech}^2)^{1/2}$, then $\Delta W_0 \approx \kappa T(\hat{\tau}/\tau^*)^{1/2}$.[1] The quantity τ^* (or the quantity H) depends essentially on the degree of sophistication of the experiment. Values of τ^* of the order of one year have already been obtained with the help of follow-up magnetic suspensions.[2]

A macroscopic test body whose position is fixed by non-uniform light currents represents an oscillator[1)] with the time constant τ^*[3]

$$\tau^* = 2m / H_{em} = mc^2 / N_0. \qquad (2)$$

where N_0 is the power of the light current incident on the body and c is the velocity of light. If $N_0 = 10^2$ erg/sec and m = 1 g, then $\tau^* = 10^{19}$ sec (!). It is clear that relation (1) ceases to be valid at the latest, when $\Delta W \approx \kappa T\hat{\tau}/\tau^* \approx \hbar\omega_{mech}$ or $\kappa T(\hat{\tau}/\tau^*)^{1/2} \approx \hbar\omega_{mech} = \Delta W$.

Before we consider the quantum mechanical restrictions on the observation of small forces, we illustrate by means of an example how energy increments smaller than κT can be observed reliably under laboratory conditions. In Fig. 1 we show the variation of the amplitude of the torsional oscillations of a pendulum with $\tau^* = 2400$ sec during the course of 100 sec. The pendulum consisted of a light aluminum-coated glass beam ($m = 10^{-2}$ g) of length 1 cm and cross section 0.3 cm^2, suspended in the vacuum on a tungsten thread of diameter 6 μ. The torsional rigidity of the thread of this pendulum was $K_\varphi = 2.4 \times 10^{-3}$ dyne · cm, the period was $\tau_0 = 2.3$ sec.

A sensitive photoelectric transducer registered small angular deviations with an accuracy of better than 8×10^{-7} radians. This setup is one of the variants of ponderomotoric devices for measuring the energy and the power of light currents.[4] Since the pendulum was not mounted on a special antiseismic platform the mean square value of the angle of deviation of the pendulum (measured after the time $3\tau^*$) was relatively large: $(\overline{\Delta\varphi^2})^{1/2} = 1.6 \times 10^{-5}$ radians. This value corresponds to a relatively high equivalent noise temperature $T_{equiv} = K_\varphi\overline{\Delta\varphi^2}/\kappa = 4500°$K. But even with this relatively large value of T_{equiv}, it was possible after $\tau \approx 23$ sec to detect reliably an energy change of the pendulum due to the action of an external force corresponding to an energy $\Delta W = \kappa \cdot 60°\text{K} = 7.8 \times 10^{-15}$ erg.

In Fig. 1 we show 38 values of the angular amplitude of the oscillations of the pendulum written down one after the other. At the instants b and d the pendulum was set in motion and was brought to rest by the light pressure of a short (0.5 sec) impulse with energy 0.9×10^3 erg which was incident upon the end of the beam.[2)] The horizontal lines in Fig. 1 represent the average values of the angular amplitudes of the oscillations and the confidence limits for them (with a confidence level of 0.95) in the time intervals a – b, b – c, c – d, and d – e. In the intervals b – c and c – d the average values of the amplitude are statistically indistinguishable. The difference would be significant if the averages differed by a half-width of the confidence interval. In our case $\Delta\varphi_{conf} = 1.8 \times 10^{-6}$ radians. This means that if the oscillations of the pendulum are built up from small amplitudes to values $\Delta\varphi = 1.8 \times 10^{-6}$ radians during the time $\hat{\tau} = 23$ sec, one can detect an energy increase of $\Delta W = K_\varphi(\Delta\varphi_{conf})^2 = 7.8 \times 10^{-15}$ erg $= \kappa \cdot 60°$K. This value is in agreement with the estimate which can be obtained starting from $T_{equiv} = 4500°$K and the formula $\Delta W \approx \kappa T_{equiv}\hat{\tau}/\tau^*$. Substituting here $\hat{\tau} = 23$ sec and $\tau^* = 2400$ sec, we obtain $\Delta W = \kappa \cdot 46°$K $= 6.5 \times 10^{-15}$ erg. We give two more numbers characterizing the setup just described. The quantity $\Delta\varphi_{conf} = 1.8 \times 10^{-6}$ radians corresponds to the threshold value of the amplitude for a force $F_0 = 1 \times 10^{-9}$ dyne (observation time 23 sec). This force can be exerted by the light pressure of a current of power 20 erg/sec.

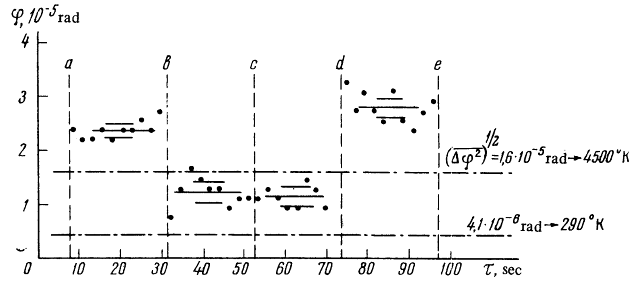

FIG. 1.

Let us consider the quantum restrictions on the measurement of small forces acting on a macroscopic mechanical oscillator. The exact solution for the probability p_{0n} for the transition of an oscillator from the ground state to the n-th state after the action of the force $F(\tau)$ during a finite length of time is known:

$$p_{0n} = e^{-y}\frac{y^n}{n!}, \quad y = \frac{1}{2\hbar\omega_{mech}m}\left|\int_{-\infty}^{+\infty} F(\tau)e^{-i\omega_{mech}\tau}d\tau\right|^2, \qquad (3)$$

where m is the mass of the oscillator and ω_{mech} is its eigenfrequency. If $F(\tau)$ has the form of a train of sinusoidal oscillations with the amplitude F_0, frequency ω_{mech}, and length τ, then

$$y = F_0^2\hat{\tau}^2 / 2\hbar\omega_{mech}m. \qquad (4)$$

We may say that the force with amplitude F_0 is detectable if $\sum_{i=1}^{\infty} p_{0i} = (1 - \alpha)$ is sufficiently close to unity (α has the meaning of a statistical error of the first kind). Therefore (3) can be rewritten in the form

$$(F_0)_{1-\alpha} = \frac{\sqrt{y}}{\hat{\tau}}\sqrt{2\hbar\omega_{mech}m}. \qquad (5)$$

For y = 2, 3, and 4 the quantity $1 - \alpha$ is equal to 0.86, 0.95, and 0.98, respectively.

From the known relations for p_{nm} obtained by the methods of perturbation theory,[6] one can show that for $F(\tau) = F_0\sin(\omega_{mech}\tau)$ during the time $\hat{\tau}$ the probability $p_{n,n+1}$ is close to unity if

[1)] A nonuniform light current incident on a reflecting body gives rise to a differential mechanical rigidity K_{mech} in the r direction: $K_{mech} \approx 2c^{-1}\,\partial N_0/\partial r$.

[2)] In addition, this allows one to verify the sensitivity of the photoelectric transducer by an independent measurement of the energy of the light impulse (with the help of a microcalorimeter).

$$F_0 \approx \frac{1}{\hat{\tau}}\sqrt{\frac{2\hbar\omega_{\mathrm{mech}} m}{n}}. \tag{6}$$

Hence, the larger n, the smaller can be the force to be detected. However, the relations (5) and (6) do not tell us what is the smallest detectable force with amplitude F_0 acting during the time τ_{meas}, since the information on whether the oscillator absorbed one or several quanta can be obtained only with the help of the spontaneous radiation. It is difficult to determine the time of spontaneous radiation, in particular, because it is not clear whether the gravitational field can be quantized (the rigidity of the oscillator may be of gravitational origin, as in some of the experiments of Eötvos [7]). In other words, the value of the quantity $[F_0]_{\min}$ for a time τ_{meas} spent during the measurement can only be determined by considering the oscillator together with the actual apparatus (which is also subject to quantum fluctuations).

It follows from (6) that the larger the initial amplitude of the oscillations, the smaller is the effect of the discreteness of the energy levels of the oscillator, and the larger is the effect of the fluctuations in the apparatus which registers the vibrations of the oscillator. In the problem considered below we therefore regard the oscillator as classical and take only account of the quantum fluctuations in the actual measuring device. We assume that a Fabry-Perot resonator is used as a measuring device. One of the mirrors of the resonator is attached to the mass of the oscillator (Fig. 2), and the source of coherent optical radiation with power N_0 and frequency ν_0 excites oscillations of the fundamental mode in the resonator. The motion of the mirror attached to the mass m leads to a modulation of the light current leaving the resonator, which is registered by a photo detector.

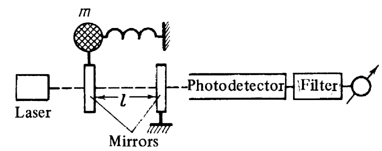

FIG. 2.

In order that one obtain the greatest sensitivity, this resonator must be somewhat (by $\Delta\nu \approx \nu_0/2Q_{\mathrm{opt}}$, where Q_{opt} is the quality) out of tune with the frequency ν_0. Then the mechanical vibrations of the mirrors leads to the largest percentage modulation of the light current.[3)] Under these conditions the smallest (classical) displacements of the mirrors which can be registered must give rise to a modulation of the current which is larger than the modulation due to the fluctuations of the number of photons emitted from the resonator.

The level of the amplitude fluctuations in the yield of the resonator will be determined by the fluctuations of the frequency and the amplitude of the optical generator. If we use the known expressions for the spectral density of the deviation of the frequency and for the coefficient of the amplitude modulation,[9] we may obtain a simple analytical expression for the smallest displacement $[x(\tau)]_{\min}$ which can be registered by a photo detector with a quantum yield close to unity:

$$[x(\tau)]_{min} = \frac{(1-R)c}{2\pi}\sqrt{\frac{2h\Delta f}{N_0\nu_0}}A, \qquad \frac{N_0}{h\nu_0\Delta f} \gg 1. \tag{7}$$

In (7), R is the reflection coefficient of the mirrors (the attainable value is $R \approx 0.995$), Δf is the frequency band characteristic for $x(\tau)$, and A is a dimensionless factor which takes account of the rigidity of the limit cycle of the nonlinear characteristics of the active medium in the optical generator, and of the Q of the resonator. The factor A is unity for a generator whose natural line width is given by the approximate formula[10]

$$\Delta\nu_{\mathrm{nat}} \approx \frac{8\pi h\nu_0(\Delta\nu_{\mathrm{res}})^2}{N_0}, \tag{8}$$

and for a spectral density of the modulation coefficient which is the same as for a source with independently emitted photons ($M_f^2 = 2h\nu_0/N_0$). For the presently existing helium-neon lasers $A \approx 10^2$ to 10^3.[11]

If we are interested in the effect of a sinusoidal train $F(\tau) = F_0 \sin(\omega_{\mathrm{mech}}\tau)$ of length $\hat{\tau}$ on the mass m, we must consider the case where the condition

$$\Delta x = F_0\hat{\tau}(2m\omega_{\mathrm{mech}})^{-1} \gtrsim [x(\tau)]_{min} = \frac{(1-R)c}{2\pi}\sqrt{\frac{2h}{N_0\nu_0\hat{\tau}}}A. \tag{9}$$

is fulfilled. On the other hand, the fluctuations of the light pressure on the walls of the resonator must be smaller than F_0:

$$F_0 \gtrsim \frac{(1+R)\,(\overline{\Delta N^2})^{1/2}}{2(1-R)c} \approx \frac{1}{(1-R)c}\sqrt{\frac{2h\nu_0 N_0}{\hat{\tau}}}A. \tag{10}$$

It is seen from a comparison of (9) and (10) that the requirements on the optical source are contradictory. The larger the power N_0, the smaller are the displacements $[x(\tau)]_{\min}$ and the forces which must be distinguished [cf. (9)]. On the other hand, with increasing N_0, the absolute value of the fluctuations of the pressure on the mirror increases, and so does therefore the threshold for the force to be detected (10). This means that there exists an optimal power $[N_0]_{\mathrm{opt}}$ for which the quantities F_0 calculated with the help of (9) and (10) agree. This amplitude of the force will be the smallest. Combining (9) and (10), we obtain

$$[F_0]_{min} = \frac{2}{\hat{\tau}}\sqrt{\hbar\omega_{\mathrm{mech}} m A}, \tag{11}$$

$$[N_0]_{opt} = m\omega_{\mathrm{mech}}(1-R)^2c^2/\pi\hat{\tau}\nu_0. \tag{12}$$

We recall that $A = 1$ for an optical source with independently emitted photons and a natural line width determined by (8). In this case (11) agrees with (5) up to a numerical factor. However, in (11), in contrast to (5), the quantity $\hat{\tau}$ agrees with the time spent during the measurement, τ_{meas}. As is seen from a comparison of (11) and (6), the neglect of the discreteness of the energy levels of the oscillator, used in the derivation of (11), is justified for A close to unity and $n \gtrsim 10^2$. Thus, when (12) is satisfied, one can use (11) for an estimate of the smallest observable amplitudes of the force during a given time of measurement.

The relation (11) shows that an improvement of the "quality" of the source (decrease of A) permits a

[3)] The presently attainable stability of the frequency of helium-neon lasers permits, according to Javan,[8] the registration of quasistatistical displacements of the order of 10^{-13} cm.

lowering of the threshold value $[F_0]_{min}$, but evidently, the latter cannot be smaller than the value of F_0 calculated from (6). The factor A has an interpretation analogous to that of the depression coefficient in the formula for the fluctuations of an electric current (shot effect). Evidently, one may assume that the development of nonlinear optics will lead to the appearance of optical sources with values of A much smaller than unity.

In summarizing our calculations, we should emphasize that our simple approach to the problem (classical oscillator—quantum mechanical optical registration of the vibrations) leads to two important recommendations with regard to the "quality" of the source (the quantity A) and the optical power $[N_0]_{opt}$, which allow one to determine the necessary requirements on the optical sources used in the registration of small vibrations. A more rigorous treatment of this problem (taking account of the quantum fluctuations in the oscillator itself) is no doubt desirable; such a treatment would be analogous to that of Schwinger,[13] but should take account of the necessity of introducing a measuring device and of the finiteness of the time spent during the measurement.

In conclusion, we consider some examples of physical experiments and estimate the approximate threshold values for the sensitivity which may be achieved in principle.

1. Test of the principle of equivalence. It follows from (11) that the amplitude of the smallest periodic acceleration in the oscillator $[a_0]_{min} = [F_0]_{min}/m$ which can be registered, is equal to

$$[a_0]_{min} \approx \frac{2}{\hat{\tau}} \sqrt{\frac{\hbar\omega_{mech}}{m}}. \tag{13}$$

The factor A = 1 in (13). The principle of equivalence can be tested in the gravitational field of the sun over the earth—sun distance (as was done in the experiments of Dicke[14]) with an accuracy up to a_0/g, where g is the acceleration of free fall in the field of the sun. Setting $\omega_{mech} = 10^{-3}$ sec^{-1}, $\hat{\tau} = 10^5$ sec, and $m = 10^2$ g, we obtain $a_0/g \approx 5 \times 10^{-21}$, which is 10 orders of magnitude smaller than the resolution achievable under terrestrial laboratory conditions.[12]

2. Observation of rare elementary particles with fractional charge. The observation of a single quark in a body with mass m in an experiment similar to the Millikan experiment reduces to the observation of a force eE/3 acting on the mass m, where E is the electric field intensity and e is the charge of the electron. Setting $eE/3 = [F_0]_{min}$, E = 10 cgs, $\hat{\tau} = 10^4$ sec, $\omega_{mech} = 10^{-2}$ sec^{-1}, and calculating $[F_0]_{min}$ with the help of (11), we obtain $m = 10^{19}$ g (!). We note that in the example mentioned above, the smallest amplitude of the force which could be registered during a time of about 20 sec is 1×10^{-9} dyne, with $m = 10^{-2}$ g. This means that the sensitivity (10^{-17} quarks per nucleon) achieved in the experiments[13,14] is very far from the limit even for laboratory conditions (in the experiments[15,16] $F_0 \sim 1 \times 10^{-9}$ dyne and $m \approx 10^{-8}$ g).

3. Highest threshold for the observation of gravitational radiation. A gravitational wave of sinsuoidal form gives rise to a difference of the forces on two separated test bodies (cf. the review article[17]):

$$F^\mu \approx -mc^2 R^\mu_{0\alpha 0} l^\alpha, \tag{14}$$

where l^α is the distance between the masses, which is small compared to the wave length, and $R^\mu_{0\alpha 0}$ are the Fourier components of the Riemann curvature tensor. Setting $F^\mu \approx [F_0]_{min}$, $l = 10^4$ cm, $\omega_{mech} = 10^{-3}$ radians/sec, $\hat{\tau} = 10^6$ sec, and $m = 10^5$ g, we obtain $R^\mu_{0\alpha 0} \approx 6 \times 10^{-49}$ cm^{-2}.

Using the connection between $R^\mu_{0\alpha 0}$ and the current density of the gravitational radiation t,[18] we obtain for the data quoted above $t = 1 \times 10^{-11}$ erg/sec cm^2 (this is an order of magnitude smaller than the possible current density of the gravitational radiation from i Bootes).

[1] V. B. Braginskiĭ, Vestnik MGU 2, ser. III, 65 (1965).

[2] J. W. Beams, R. D. Boyle, and P. E. Hexner, Rev. Sci. Instr. 32, 645 (1961).

[3] V. B. Braginskiĭ and A. B. Manukin, Zh. Eksp. Teor. Fiz. 52, 986 (1967) [Soviet Phys.-JETP 25, 653 (1967)].

[4] V. B. Braginskiĭ, I. I. Minakova, and P. M. Stepunin, PTÉ No. 3, 183 (1965).

[5] I. I. Gol'dman and V. D. Krivchenkov, Sbornik zadach po kvantovoi mekhanike (Collection of Problems in Quantum Mechanics) Gostekhizdat, 1957, p. 105, Engl. Transl. Pergamon Press, 1961.

[6] A. A. Sokolov and I. M. Ternov, Zh. Eksp. Teor. Fiz. 25, 698 (1953).

[7] R. Eötvos, D. Pekar, and E. Fekete, Ann. Physik 68, 11 (1922).

[8] A. Javan, Lasers and Applications, Columbus, 1963.

[9] I. L. Bershteĭn, I. A. Andronova, and Yu. I. Zaĭtsev, Izv. vyssh. ych. zav., Radiofizika 10, 59 (1967).

[10] C. H. Townes, Advances in Quantum Electronics, N. Y. 1961, p. 3.

[11] Yu. I. Zaĭtsev, Zh. Eksp. Teor. Fiz. 50, 525 (1966) [Soviet Phys.-JETP 23, 349 (1966)].

[12] W. R. Bennett, Gaseous Masers, Appl. Opt. Suppl. No. 1, 24 (1962).

[13] J. Schwinger, J. Math. Phys. 2, 407 (1961).

[14] R. H. Dicke, Scient. American 205, 84 (1961).

[15] G. Gallinaro and G. Morpurgo, preprint of the 52nd Conf. of the Italian Phys. Soc., October, 1966.

[16] V. B. Braginskiĭ, Ya. B. Zel'dovich, V. K. Martynov, and V. V. Migulin, Zh. Eksp. Teor. Fiz. 52, 29 (1967) [Soviet Phys.-JETP 25, 17 (1967)].

[17] V. B. Braginskiĭ, Usp. Fiz. Nauk 86, 433 (1965) [Soviet Phys.-Usp. 8, 513 (1965)].

[18] L. D. Landau and E. M. Lifshitz, Teoriya polya (The Theory of Fields), Fizmatgiz, 1963, Engl. Transl. Pergamon Press, 1959.

Translated by R. Lipperheide
166