EXCITATION OF LONGITUDINAL AMBIPOLAR SOUND IN A GAS DISCHARGE PLASMA IN A MAGNETIC FIELD AT LOW PRESSURE

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Submitted May 15, 1967

Zh. Eksp. Teor. Fiz. 53, 1453–1460 (October, 1967)

It is shown that in systems where removal of ions at the ends in a longitudinal ambipolar electric field is dominant, excitation of modes of longitudinal ambipolar sound, or waves whose phase velocity is of the order of the removal rate of ions in the ambipolar field, should be possible. In our opinion, these results permit us to interpret the well-known results of Vlasov [1] on the excitation and modification of ion-sound modes.

1. Recently Vlasov [1], in an investigation of discharge plasma from a heated cathode in the presence of a constant longitudinal magnetic field, discovered an interesting phenomenon, which consisted of the excitation and modification of the modes of ion sound with frequency

$$\omega \approx \frac{p}{L}\sqrt{\frac{T_e}{M}} \tag{1}$$

under conditions when the relation

$$\frac{p}{s} = \mathrm{const}\,\frac{L}{a}(T_e M)^{1/2}\frac{|\varkappa|}{H} \tag{2}$$

is satisfied, where M is the mass of the ions, $\kappa = d\ln n/dr$, p and s are the number of modes of oscillation along the length and the azimuth, which determine the number of nodes of the potential of the wave, L is the length of the plasma column, and a is the radius of the column.

Relation (2) was obtained empirically. Here the frequency and conditions of excitation are almost independent of the discharge voltage. It will be shown below that the excitation of longitudinal ion sound under conditions (2) can be connected with longitudinal ambipolar sound,[1)] the mechanism of excitation of which is not connected with an external electric field.

2. In the choice of the model on the basis of which the calculations were carried out, we started from the following considerations. In the experiments of Vlasov, the plasma column, negatively charged relative to the walls of the plasma chamber, was produced as the result of a stationary discharge in a strong magnetic field. We therefore assumed that the escape of ions in the stationary state takes place principally along the discharge axis at the ends of the system, in the positive (relative to the ends) longitudinal ambipolar electric field, the profile of which is symmetric relative to the plane passing through the middle of the column (z = 0), perpendicular to the direction of the magnetic field. The density profile in the stationary state is also symmetric relative to this plane (Fig. 1). The escape of the ions takes place on both sides of this plane without collisions with neutral atoms. We note that the conclusions of this research are applicable in those cases in which the escape of the ions occurs only at one of the ends.

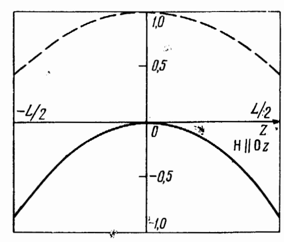

FIG. 1. Stationary potential profile $e\varphi/T_e$ (solid curve) and the stationary density profile n/n_0 (dashed).

In the process of escape of the ions in an ambipolar field to the ends, their energy increases to $e\varphi_0 \approx T_e$, where φ_0 is the value of the ambipolar potential at the ends. This leads to the possibility of the existence of waves, the longitudinal component of the phase velocity of which is of the order of ion sound; $v_{ph} \approx \sqrt{e\varphi_0/M} \approx \sqrt{T_e/M}$, which is also confirmed by calculations, as will be shown below. We have called these waves longitudinal ambipolar sound. It will be shown below that the oscillation of these waves is due to the drift of the ions in the magnetic field, when $\kappa \neq 0$; it is assumed here that the electrons are distributed in the Boltzmann fashion ($v_{ph} \ll \sqrt{T_e/m_e}$), and that there is no current in the plasma.

3. In the calculation of the stationary profiles of the density, potential and in the analysis of the stability, we start out from the kinetic equation for the ions

$$\frac{\partial f_i}{\partial t} + \mathrm{div}_r \mathbf{v} f_i - \frac{e}{M}\frac{\partial f_i}{\partial \mathbf{v}}\nabla\varphi + \frac{\partial f_i}{\partial \mathbf{v}}[\mathbf{v}\omega_{iH}] = J\delta(\mathbf{v}) \tag{3*}$$

and the equation for the electrons

$$n_e = n_0 \exp\left(\frac{e\varphi}{T_e}\right), \quad n_e' = n_0\frac{e\varphi'}{T_e}, \tag{4}$$

where J is the number of ions produced per unit time in a unit volume; the ions are produced in the ionization of neutral atoms with zero velocity, which is taken into account by a delta function. It is assumed that in the stationary state, $f_{i0} = f_{i0}(v_z)\delta(v_r)\delta(v_\varphi)$. It is also assumed that even in the stationary state the electrons have a

[1)]The theory of transverse ambipolar sound was developed in [2]

*$[\mathbf{vw}] \equiv \mathbf{v} \times \mathbf{w}$.

Boltzmann distribution, which is valid for small electron currents at the ends. The prime symbol refers to excited quantities.

In what follows, we limit ourselves to a consideration of the region $0 \le z \le L/2$, inasmuch as both halves of the column are symmetric relative to the plane $z = 0$.

4. The solution of the problem of the axial distribution of quantities in the stationary state reduces to the integration of the characteristics of the equation

$$v_z\frac{\partial F_i}{\partial z}-\frac{e}{M}\frac{d\varphi}{dz}\frac{\partial F_i}{\partial v_z}=J\delta(v_z),\quad F_i=\int f_{i0}\,dv_\varphi\,dv_r. \tag{5}$$

The equations of the characteristic have the form

$$\frac{dz}{v_z}=dv_z\Big/\frac{e}{M}\frac{d\varphi}{dz}=\frac{dF_i}{J\delta(v_z)}, \tag{6}$$

whence it follows that

$$\frac{Mv_z^2}{2}=e\varphi(z')-e\varphi(z);\quad \frac{dF_i}{dz}=J\frac{\delta(v_z)}{v_z}, \tag{7}$$

where z' corresponds to the point of creation of the particles. We find further that

$$F_i=J\int_{z'}^{z}\frac{\delta(v_z)}{v_z}dz'=-J\frac{M}{e}\Big/\frac{d\varphi}{dz'}. \tag{8}$$

For the ion density, we find

$$n_i(z)=\int_0^{\sqrt{-2e\varphi(z)/M}}F_i\,dv_z=J\int_0^z dz'\left\{\frac{2e}{M}[\varphi(z')-\varphi(z)]\right\}^{-1/2}. \tag{9}$$

The axial distribution of the potential and the density in the stationary state is determined from the equation of quasineutrality ($n_e = n_i = n$), which, with account of (4) and (8), has the form

$$\exp\left(\frac{e\varphi}{T_e}\right)=\frac{J}{n_0}\int_0 dz'\left\{\frac{2e}{M}[\varphi(z')-\varphi(z)]\right\}^{-1/2}. \tag{10}$$

The solution of this equation is given in the work of Langmuir and Tonks,[3] such that

$$\frac{e\varphi_0}{T_e}=-0.85,\quad J=\frac{n_0}{L}\sqrt{\frac{T_e}{M}},\quad n|_{z=\pm L/2}=0{,}43n_0. \tag{11}$$

It is assumed that in the stationary state the radial density gradient differs from zero ($\kappa \ne 0$).

5. For excitations of the form

$$A'=A_1(z)\exp(im\varphi-i\omega t) \tag{12}$$

we can obtain from Eq. (3) the following set of equations for the excited ion distribution function:

$$\begin{aligned}
&v_z\frac{\partial F_i'}{\partial z}-\frac{e}{M}\frac{d\varphi}{dz}\frac{\partial F_i'}{\partial v_z}=i\omega F_i'-Y-\frac{1}{r}\frac{\partial rW}{\partial r}+\frac{e}{M}\frac{\partial F_i}{\partial v_z}\frac{\partial\varphi'}{\partial z},\\
&v_z\frac{\partial Y}{\partial z}-\frac{e}{M}\frac{d\varphi}{dz}\frac{\partial Y}{\partial v_z}=i\omega Y+\frac{m^2}{r^2}\frac{e}{M}F_i\varphi'-\frac{im}{r}\omega_{iH}W,\\
&v_z\frac{\partial W}{\partial z}-\frac{e}{M}\frac{d\varphi}{dz}\frac{\partial W}{\partial v_z}=i\omega W-\frac{e}{M}F_i\frac{\partial\varphi'}{\partial r}-\frac{i}{m}\omega_{iH}rY,
\end{aligned} \tag{13}$$

where

$$F_i'=\int f_i'\,dv_\varphi\,dv_r;\quad W=\int v_rf_i'\,dv_\varphi\,dv_r;\quad Y=\frac{im}{r}\int v_\varphi f_i'\,dv_\varphi\,dv_r. \tag{14}$$

In the limit $\omega_{iH} \gg \omega$, v_z/L, the system (13) reduces to the equation

$$v_z\frac{\partial F_i'}{\partial z}-\frac{e}{M}\frac{d\varphi}{dz}\frac{\partial F_i'}{\partial v_z}=i\omega F_i'+\frac{ie}{Mr}\frac{m}{\omega_{iH}}\frac{\partial F_i}{\partial r}\varphi'+\frac{e}{M}\frac{\partial F_i}{\partial v_z}\frac{\partial\varphi'}{\partial z}. \tag{15}$$

The characteristics of this equation are

$$\frac{Mv_z^2}{2}=e\varphi(z')-e\varphi(z),$$

$$\frac{dF_i'}{dz}=\left[i\omega F_i'+\frac{ie}{Mr}\frac{m}{\omega_{iH}}\frac{\partial F_i}{\partial r}\varphi'+\frac{e}{M}\frac{\partial F_i}{\partial v_z}\frac{\partial\varphi'}{\partial z}\right]\Big/v_z(z,z'). \tag{16}$$

Integrating along the characteristics according to the calculation scheme of [2], we can obtain the following expression for the excited ion density:

$$\begin{aligned}
n_{1i}(z)=&\int_0^{\sqrt{-2e\varphi(z)/M}}F_{1i}\,dv_z=\int_0^z\frac{i\omega n_{1i}(\xi)+\dfrac{ie}{M}\dfrac{m}{r}\dfrac{\varkappa}{\omega_{iH}}n(\xi)\varphi_1(\xi)}{\{2e[\varphi(\xi)-\varphi(z)]/M\}^{1/2}}d\xi\\
&-\frac{e}{M}\int_0^z d\xi\frac{d\varphi_1(\xi)}{d\xi}\int_0^\xi\frac{\partial F_i(\xi,z')}{dz'}\frac{dz'}{\{2e[\varphi(z)'-\varphi(z)]/M\}^{1/2}}.
\end{aligned} \tag{17}$$

The nonlocal character of the expression for the excited ion density is associated with processes of removal of the excitations in the ambipolar electric field to the ends of the column, as a result of which the excited ion density at each point z is determined by all the excitations appearing in the interval $0 < z' < z$.

Under the assumption of quasineutrality, $n_e' = n_i'$, one can obtain the following equation for the excited potential $\varphi_1(z)$ by use of (4):

$$n(z)\varphi_1(z)=i\lambda\int_0^z\frac{n(\xi)\varphi_1(\xi)}{v_z(z,\xi)}d\xi-\frac{e}{M}\int_0^z d\xi\frac{d\varphi_1}{d\xi}\int_0^\xi\frac{\partial F_i}{\partial z'}\frac{dz'}{v_z(z,z')}, \tag{18}$$

where

$$\lambda=\omega+\frac{T_e}{M}\frac{m}{r}\frac{\varkappa}{\omega_{iH}},$$

and the solution of the problem reduces to the determination of the eigenfunctions φ_{1n} and the eigenvalues λ_n of this linear homogeneous singular integro-differential equation with a non-symmetric and non-hermitian kernel. The solution of such a problem is difficult to say the least and we shall use an approximate method for derivation of the dispersion relation.

In the following, an analysis is given for the case of parabolic profiles of the density and the potential:

$$\varphi=-A\tilde z^2;\quad n=n_0(1-\alpha\tilde z^2);\quad \tilde z=2z/L, \tag{19}$$

where, according to (11),

$$A=0.85T_e/e,\quad \alpha=0.57.$$

Stationary profiles of the potential and density (19) are given in Fig. 1.

In this approximation, Eq. (17) for the excited ion density has the form

$$\begin{aligned}
n_{1i}(z)=&\int_0^{\tilde z}\frac{ixn_{1i}(\xi)+ieT_e^{-1}\delta n(\xi)\varphi_1(\xi)}{\sqrt{\tilde z^2-\xi^2}}d\xi\\
&+n_0\frac{e}{T_e}\frac{\mu}{\tilde z^2}\int_0^z\sqrt{\tilde z^2-\xi^2}\frac{d^2\varphi_1}{d\xi^2}d\xi,
\end{aligned} \tag{20}$$

where

$$x=\omega\left(\frac{2eA}{M}\right)^{-1/2}\frac{L}{2},\quad \delta=\frac{T_e}{M}\frac{m}{r}\frac{\varkappa}{\omega_{iH}}\left(\frac{2eA}{M}\right)^{-1/2}\frac{L}{2},$$

$$\mu=\frac{J}{n_0}\frac{T_e}{M}\left(\frac{2eA}{M}\right)^{-3/2}\frac{L}{2}\approx 0{,}22. \tag{21}$$

In the derivation of (20), we have assumed that

$$d\varphi_1/dz|_{z\to 0}\to 0. \tag{22}$$

This requirement is due to the fact that the stationary ambipolar field for the parabolic density profile tends

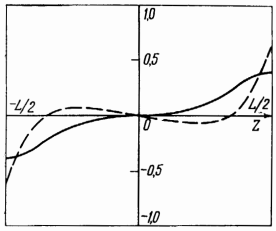

FIG. 2. Profile of the first mode $\varphi_1(z)/\varphi_1$ (solid curve) and profile of the third mode (dashed).

to zero as $z \to 0$, and satisfaction of (22) is necessary for the validity of the linearization of Eq. (3), which has been carried out in the vicinity of $z = 0$.

The dispersion equation is determined from the condition of neutrality of the excitations:

$$\int_0^1 n_e' \, d\tilde{z} = \int_0^1 n_i' \, d\tilde{z}, \tag{23}$$

where n_e' and n_i' are determined by (4) and (20).

6. In order to find an expression from Eq. (20) for the excited ion density, we give the profile of the excited potential $\varphi_1(z)$. The symmetric

$$\varphi_1(z) = \varphi_1(-z) \tag{24}$$

and antisymmetric

$$\varphi_1(z) = -\varphi_1(-z) \tag{25}$$

distributions of the excited potential relative to the plane $z = 0$ are possible.

The symmetric profiles have an even number of nodes ($p = 2, 4, \ldots$) in the interval $-1 \le \tilde{z} \le 1$, and the point $z = 0$ is necessarily an extremum point for them. The antisymmetric profiles have an odd number of nodes ($p = 1, 3, \ldots$,) in this interval (one node is necessarily at the point $z = 0$). Thus the first, fundamental mode ($p = 1$) has a single node at the point $z = 0$. If we specify the profile of the potential $\varphi_1(z)$ in the form of polynomials, then, as is seen from Eq. (22), the profile of the antisymmetric potential begins only with terms proportional to z^3.

In what follows, we shall consider the excitation of the first two antisymmetric modes ($p = 1, 3$) and the first two symmetric modes ($p = 2, 4$). The specific choice of the profiles of the excited potential is a difficult one. Therefore, we shall proceed in the following fashion.

A. The first, fundamental mode: $p = 1$. We choose the profile of this mode in the form

$$\varphi_1(z) = \varphi_1\tilde{z}^3(1 + A_1\tilde{z}^2). \tag{26}$$

If it is assumed that flow takes place at the conducting end, then $d\varphi_1/dz|_{\tilde{z}=\pm1} = 0$, whence $A_1 = -0.6$. The profile of the first mode is shown in Fig. 2.

By substituting the profile (26) in Eq. (20), we find the expression for the excited ion density:

$$n_{1i}(z) = n_0 \frac{e\varphi_1}{T_e} \tilde{z} \left(\frac{0,44}{1 - ix} + \frac{0.67i\delta - 0.35}{1 - 0,67ix} \tilde{z}^2 - \frac{0.62i\delta}{1 - 0,53ix} \tilde{z}^4 + \frac{0.16i\delta}{1 - 0,46ix} \tilde{z}^6 \right). \tag{27}$$

The dispersion equation (23) for the fundamental mode has the form

$$0.1 = \frac{0.22}{1 - ix} + \frac{0.16i\delta - 0.09}{1 - 0.67ix} - \frac{0.11i\delta}{1 - 0.53ix} + \frac{0.02i\delta}{1 - 0.46ix}. \tag{28}$$

On the stability boundary ($\mathrm{Im}\, x = 0$), it splits into two equations relative to x and δ (the imaginary and real parts of the dispersion equation are equated to zero). From these two equations, it can easily be established that

$$x_{\lim} \approx 1, \quad \delta_{\lim} \approx -1,3 \tag{29}$$

on the boundary of stability of the first mode.

Simple analysis[2)] shows that the first mode is excited when $|\delta| < 1.3$. This condition can be written in the form

$$\omega_{iH} > 0.6 \frac{s|\varkappa|}{a} L \sqrt{\frac{T_e}{M}} \tag{30}$$

by taking (21) into account. In the derivation of (30), it was considered that $m = 2s$.

For hydrogen, for example, under the experimental conditions of Vlasov ($L = 40$ cm, $T_e = 10$ eV) the critical magnetic field for $s = 1$: $H_{cr} \approx 500$ Oe; for this case, it was assumed that the decay constant of the secondary plasma is equal to the radius of the chamber ($|\kappa|^{-1} = a \approx 4$ cm). For $\kappa < 0$, waves are excited with $m > 0$ (helical waves). Here the frequency of oscillation, according to (21), (29), is

$$f = \frac{\omega}{2\pi} \approx \frac{0.4}{L} \sqrt{\frac{T_e}{M}}. \tag{31}$$

For hydrogen, under the experimental conditions of Vlasov, $f \approx 30$ kHz, which agrees well with the frequency of the fundamental harmonic excited in the experiment.

In what follows, we shall assume that for the remaining modes ($p = 2, 3, 4$) the conditions

$$\delta_{\lim} = -2.6; \ -3.9; \ -5.2; \quad x_{\lim} = 2; \ 3; \ 4, \tag{32}$$

hold on the boundary of stability, in accord with (1), (2), and (29). The conditions (32) allow us to determine the profiles of the remaining modes.

B. Second mode (symmetric). We seek the profile of this mode in the form

$$\varphi_1(z) = \varphi_1(1 + A_2\tilde{z}^2 + B_2\tilde{z}^4). \tag{33}$$

The constants A_2 and B_2 are determined from the conditions that on the boundary of stability

$$x_{\lim} = 2; \quad \delta_{\lim} = -2.6. \tag{34}$$

The expression for the excited ion density, according to (20) and (33), has the form

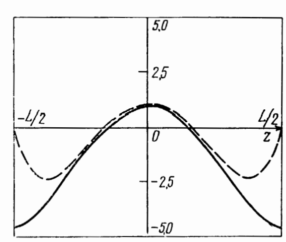

FIG. 3. Profile of the second mode (solid curve) and profile of the fourth mode (dashed).

[2)] Infinitesimally small deviations from the stability limit are considered.

$$n_{1i} = n_0 \frac{e\varphi_1}{T_e}\frac{\pi}{2}\Big[\frac{i\delta + 0.22A_2}{1-0{,}57ix} + \frac{i\delta(A_2-0.57)+0.66B_2}{2(1-0.79ix)}\tilde{z}^2 + \frac{0.38i\delta(B_2-0.57A_2)}{1-0.59ix}\tilde{z}^4 - \frac{0.18i\delta B_2}{1-0.49ix}\tilde{z}^6\Big]. \quad (35)$$

The dispersion equation (20) on the boundary of stability reduces to a system of two equations relative to the coefficients A_2 and B_2, whence

$$A_2 = -10.54; \quad B_2 = 4.88. \quad (36)$$

The profile of the second mode determined in this fashion (Fig. 3) actually has two modes in the interval $-1 \le \tilde{z} \le 1$.

C. Third mode (antisymmetric). The profile of the third mode was chosen by us in the form

$$\varphi_1(z) = -\varphi_1\tilde{z}^3(1 + A_3\tilde{z}^2 + B_3\tilde{z}^4). \quad (37)$$

The constants A_3 and B_3 are determined from the condition that on the boundary of stability (Im x = 0) we will have

$$x_{\text{lim}} = 3; \quad \delta_{\text{lim}} = -3{,}9. \quad (38)$$

By forming the dispersion equation analogously to the case of the previous modes, and splitting it on the stability boundary into two equations relative to A_3 and B_3, we find, by means of (38), that

$$A_3 = -2.37; \quad B_3 = 0.73. \quad (39)$$

(The profile of the third mode (Fig. 2) has three modes in the interval $-1 \le \tilde{z} \le 1$.)

D. Fourth mode (symmetric). The profile of the fourth mode is expressed in the form

$$\varphi_1(z) = \varphi_1(1 + A_4\tilde{z}^2 + B_4\tilde{z}^4 + C_4\tilde{z}^6). \quad (40)$$

The constants A_4, B_4, and C_4 are determined from the condition that on the stability limit we have

$$x_{\text{lim}} = 4; \quad \delta_{\text{lim}} = -5{,}2 \quad (41)$$

and the nodes of the mode p = 4 are located at equal distances. These conditions allow us to choose the values of the coefficients A_4, B_4, and C_4 uniquely:

$$A_4 = -9.86; \quad B_4 = 4.06; \quad C_4 = 4.80. \quad (42)$$

The profile thus determined (Fig. 3) actually has four nodes in the interval $-1 \le \tilde{z} \le 1$.

Without doubt, such a method of determination of the form of the higher modes is not entirely rigorous. The aim of the present calculations was to trace the character of the change in the profile of the excited modes upon satisfaction of (32).

7. Thus, it is shown that under conditions when ions from a discharge flow through the ends in a positive longitudinal ambipolar electric field, a drift oscillation is possible of the spatial modes to the longitudinal ambipolar sound at the frequencies

$$\omega = \frac{2p}{L}\sqrt{\frac{2e\varphi_0}{M}} \approx \frac{2{,}6p}{L}\sqrt{\frac{T_e}{M}} \quad (43)$$

under conditions when

$$\omega_{iH} > 0{,}6\frac{s}{p}\frac{|\varkappa|}{a}L\sqrt{\frac{T_e}{M}}. \quad (44)$$

In the case when flow of ions takes place only at one end, the mechanism of excitation of the ambipolar sound remains unchanged, but it is necessary to replace L/2 in the formulas (43) and (44) by L—the length of the discharge region (the flow in this case takes place at both ends of the tube).

The mechanism of excitation that has been considered can appear in a Penning low pressure discharge, where the removal of ions in the ambipolar field takes place at the cathode.

In conclusion, I express my deep gratitude to B. B. Kadomtsev for his suggesting the problem and for valued critical comments, and also to M. A. Vlasov and A. V. Timofeev for interest in the work.

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Note added in proof (September 12, 1967). When the length of the free path of the ions is small ($\lambda_i < L/2$), the conditions for the excitation of longitudinal ambipolar sound appear at the very end of the free path, where there is a free removal of the ions at the ends in the longitudinal ambipolar field. Here, in order that the ions drift transversely to the column, satisfaction of the condition $\omega_{iH}\tau_i > 1$ is necessary. It must be expected that here the maximum of the amplitude of the ambipolar sound is reached near the ends. A similar situation arises in the case of a transverse ambipolar field.

Translated by R. T. Beyer
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