HYDROMAGNETIC STABILITY OF A PLASMA IN A QUASI-UNIFORM MAGNETIC FIELD

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Submitted June 5, 1967

Zh. Eksp. Teor. Fiz. 53, 2063–2069 (1967)

Conditions are derived for the hydromagnetic stability of an arbitrary equilibrium plasma configuration confined by a quasi-uniform magnetic field.

1. EQUILIBRIUM, VARIATIONAL PRINCIPLE, AND COORDINATE SYSTEM

THE equations of equilibrium of a plasma in a magnetic field **B** are

$$\nabla p = [\mathbf{jB}], \qquad \mathbf{j} = \operatorname{rot}\mathbf{B}, \qquad \operatorname{div}\mathbf{B} = 0. \tag{1.1*}$$

We shall examine confined plasma configurations whose magnetic surfaces form a system of imbedded toroidal surfaces surrounding the magnetic axis. From (1.1), it follows that the vectors **j** and **B** lie on the magnetic surfaces that coincide with the surfaces of constant pressure p.

For the intrinsic stability of an ideally conducting plasma, it is necessary and sufficient that the potential energy[1]

$$\delta w = \frac{1}{2}\int\left\{(\operatorname{rot}[\xi\mathbf{B}])^2 + \gamma p(\operatorname{div}\xi)^2 + (\xi\nabla p)\operatorname{div}\xi + [\mathbf{j}\xi]\operatorname{rot}[\xi\mathbf{B}]\right\}d\tau \tag{1.2}$$

be positive for arbitrary displacements ξ that satisfy the condition that $\xi_\perp$ be zero on the plasma boundary Σ.

To obtain the conditions of stability, it is convenient to use a system of curvilinear coordinates x^1, x^2, x^3, related to the magnetic surfaces in such a way that coordinates x^1 and x^2 change along the magnetic surfaces, while x^3 changes in the perpendicular direction. For the coordinate x^3 it is convenient to select the running volume V enclosed by the system of magnetic surfaces, reckoned from the magnetic axis V = 0, which is a closed space curve s. In this system, the vectors **j** and **B** will each have two nonzero contravariant components $j^i = \{j^1, j^2, 0\}$ and $B^i = \{B^1, B^2, 0\}$, while ∇p has only one covariant component $\partial p/\partial x^3 = \sqrt{g}(j^1B^2 - j^2B^1)$.

As was shown in[2,3], it is possible to introduce a "natural" surface coordinate system, in which $x^1 = \theta$ and $x^2 = \zeta$ are the cyclic coordinates with unity periodicity, the determinant of the metric tensor is g = 1, and the contravariant components of **j** and **B** are respectively equal to

$$j^i = \{\dot I, \dot J, 0\}, \qquad B^i = \{\dot\chi, \dot\Phi, 0\}. \tag{1.3}$$

Here differentiation with respect to V is denoted by a dot, J(V) and I(V) are the longitudinal and azimuthal currents, while $\Phi(V)$ and $\chi(V)$ are the longitudinal and azimuthal magnetic fluxes inside the magnetic surface bounding the volume V.

The length element which defines the metric of this coordinate system can be written as

$$d\mathbf{r} = \mathbf{e}_1dx^1 + \mathbf{e}_2dx^2 + \mathbf{e}_3dx^3 = [\nabla x^2\nabla x^3]dx^1 + [\nabla x^3\nabla x^1]dx^2 + [\nabla x^1\nabla x^2]dx^3. \tag{1.4}$$

Here $g_{ik} = \mathbf{e}_i\cdot\mathbf{e}_k$, $a^i = \mathbf{a}\cdot\nabla x^i$, $a_i = \mathbf{e}_i\cdot\mathbf{a}$, $\partial/\partial x^i = \mathbf{e}_i\cdot\nabla$, and the vectors $\mathbf{e}_i$ can be represented as

$$\mathbf{e}_1 = \dot p^{-1}(\dot\Phi\mathbf{j} - \dot J\mathbf{B}), \qquad \mathbf{e}_2 = \dot p^{-1}(\dot I\mathbf{B} - \dot\chi\mathbf{j}),$$
$$\mathbf{e}_3 = k_1\mathbf{e}_1 + k_2\mathbf{e}_2 + \nabla V/|\nabla V|^2. \tag{1.5}$$

From the equation **j** = curl **B** it follows that the functions k_1 and k_2 satisfy the following "magnetic differential equations":

$$\mathbf{B}\nabla k_1 = -\ddot\chi - \frac{\mathbf{e}_2}{|\nabla V|^2}\left(\mathbf{j} - \frac{2}{|\nabla V|^2}[\nabla V, (\nabla V\nabla)\mathbf{B}]\right),$$
$$\mathbf{B}\nabla k_2 = -\ddot\Phi + \frac{\mathbf{e}_1}{|\nabla V|^2}\left(\mathbf{j} - \frac{2}{|\nabla V|^2}[\nabla V, (\nabla V\nabla)\mathbf{B}]\right). \tag{1.6}$$

2. CONDITIONS FOR HYDROMAGNETIC STABILITY

If **f** and **F** denote the vectors

$$\mathbf{f} = \operatorname{rot}[\xi\mathbf{B}] + \mathbf{B}\operatorname{div}\xi + \mathbf{D}\xi^3, \qquad \mathbf{F} = [\mathbf{je}], \tag{2.1}$$

where **D** and **e** are defined by their contravariant components $D^i = \{\partial B^1/\partial x^3, \partial B^2/\partial x^3, 0\}$, $e^i = \{0, 0, 1\}$ in the given surface coordinate system x^1, x^2, x^3, then we can show that the following identity holds:

$$\operatorname{div}(\xi\mathbf{F})\xi^3\mathbf{B} = (\xi\mathbf{F})f^3 + (\mathbf{fF})\xi^3. \tag{2.2}$$

Equality (2.2) permits us to transform the expression (1.2) into

$$\delta w = \frac{1}{2}\int\{(\operatorname{rot}[\xi\mathbf{B}] + [\mathbf{je}]\xi^3)^2 + \gamma p(\operatorname{div}\xi)^2 + [\mathbf{je}](\mathbf{D} - [\mathbf{je}])(\xi^3)^2\}d\tau. \tag{2.3}$$

We observe that the expression (2.3) differs from the corresponding expression obtained in[3] in that the vector $\mathbf{n} = \nabla V/|\nabla V|^2$ is replaced by $\mathbf{e} = \sqrt{g}[\nabla x^1\nabla x^2]$.

Denoting the derivative with respect to $x^3 = V$ by a dot, we introduce the surface functions

$$\dot p = \dot I\dot\Phi - \dot J\dot\chi, \qquad \Omega = \dot I\ddot\Phi - \dot J\ddot\chi, \qquad S = \dot\chi\ddot\Phi - \dot\Phi\ddot\chi. \tag{2.4}$$

The function

$$S = \dot\Phi^2\frac{d}{dV}\frac{d\chi}{d\Phi}$$

describes the shear of the magnetic lines of force on the neighboring magnetic surfaces, while the function Ω, which equals $\dot p\ddot\Phi/\dot\Phi$ for S = 0, characterizes "minimum $\bar{\mathrm{B}}$."

In the natural coordinates $x^1 = \theta$, $x^2 = \zeta$, $x^3 = V$, formula (2.3) becomes

$$\delta w = \frac{1}{2}\int\{(\operatorname{rot}[\xi\mathbf{B}] + [\mathbf{je}]\xi^3)^2 + \gamma p(\operatorname{div}\xi)^2 - (\Omega + [\mathbf{je}]^2)(\xi^3)^2\}d\tau. \tag{2.5}$$

A. The resulting expression (2.5) for the potential energy δw permits us immediately to obtain a sufficient condition for the stability of the plasma

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*[jb] ≡ j × B.

$$\Omega + [\mathbf{je}]^2 \leqslant 0. \tag{2.6}$$

Let us introduce the notion of a quasi-uniform magnetic field, assuming small quantities of order ϵ containing the derivative of the magnetic field with respect to V, i.e., we set $\mathbf{B} = \mathbf{B}(x^1, x^2, \epsilon V)$. Thus the quantities j, $\dot{p}$, and S will be of order ϵ, while Ω will be of order ϵ^2. Since the vector $\mathbf{e} = \mathbf{e}_3$ when g = 1, and it follows from (1.6) that the coefficients k_1 and k_2 are small quantities of order ϵ, we have, to terms of order ϵ,

$$\mathbf{e} \approx \nabla V / |\nabla V|^2. \tag{2.7}$$

Consequently, under the condition of quasi-uniformity of the magnetic field, the sufficient criterion for stability (2.6) can be written in invariant form

$$\Omega + \mathbf{j}^2 / |\nabla V|^2 \leqslant 0. \tag{2.8}$$

B. The condition of quasi-uniformity also permits us to obtain a necessary and sufficient criterion for plasma stability.

Let us denote the contravariant components of the displacement ξ by $\xi^i = \{\xi_\theta, \xi_\zeta, \xi_V\}$ and introduce the combinations $\mu = \dot{\Phi}\xi_\theta - \dot{\chi}\xi_\zeta$ and $\eta = \dot{J}\xi_\theta - \dot{I}\xi_\zeta$. Then

$$\mathbf{B} = \mathbf{e}_1\dot{\chi} + \mathbf{e}_2\dot{\Phi}, \qquad \mathbf{j} = \mathbf{e}_1\dot{I} + \mathbf{e}_2\dot{J},$$
$$\xi = \mathbf{e}_1\xi_\theta + \mathbf{e}_2\xi_\zeta + \mathbf{e}_3\xi_V = \dot{p}^{-1}(\mu\mathbf{j} - \eta\mathbf{B}) + \xi_V\mathbf{e}_3. \tag{2.9}$$

The expression for curl$[\boldsymbol{\xi} \times \mathbf{B}]$ can be written as

$$\operatorname{rot}[\xi\mathbf{B}] = \left(\frac{\partial\mu}{\partial\zeta} - \frac{\partial}{\partial V}\xi_V\dot{\chi}\right)\mathbf{e}_1 - \left(\frac{\partial\mu}{\partial\theta} + \frac{\partial}{\partial V}\xi_V\dot{\Phi}\right)\mathbf{e}_2 + (\mathbf{B}\nabla\xi_V)\mathbf{e}_3. \tag{2.10}$$

Representing ξ_V and μ by expansions in the nonuniformity parameter ϵ:

$$\xi_V = \xi_V{}^0 + \varepsilon\xi_V{}^1 + \ldots, \qquad \mu = \mu^0 + \varepsilon\mu^1 + \ldots \tag{2.11}$$

and also expanding $\dot{\Phi}$ and $\dot{\chi}$ in ϵV, we obtain

$$\operatorname{rot}[\xi\mathbf{B}] = \left\{\frac{\partial\mu^0}{\partial\zeta} - \dot{\chi}\xi_V{}^0 + \varepsilon\left(\frac{\partial\mu^1}{\partial\zeta} - \dot{\chi}\xi_V{}^1 - \ddot{\chi}\frac{\partial}{\partial V}V\xi_V{}^0\right)\right\}\mathbf{e}_1$$
$$- \left\{\frac{\partial\mu^0}{\partial\theta} + \dot{\Phi}\xi_V{}^0 + \varepsilon\left(\frac{\partial\mu^1}{\partial\theta} + \dot{\Phi}\xi_V{}^1 + \ddot{\Phi}\frac{\partial}{\partial V}V\xi_V{}^0\right)\right\}\mathbf{e}_2$$
$$+ \{\mathbf{B}\nabla\xi_V{}^0 + \varepsilon(\mathbf{B}\nabla\xi_V{}^1 + V\mathbf{D}\nabla\xi_V{}^0)\}\,\mathbf{e}_3. \tag{2.12}$$

We shall seek a sufficient condition of stability by the method of successive approximations in the parameter ϵ. In the zeroth approximation, the quantity $\delta w > 0$ if curl $[\boldsymbol{\xi} \times \mathbf{B}] \neq 0$. The requirement curl$[\boldsymbol{\xi} \times \mathbf{B}] = 0$ reduces to the equations

$$\dot{\chi}\xi_V{}^0 = \partial\mu^0/\partial\zeta, \qquad \dot{\Phi}\xi_V{}^0 = -\partial\mu^0/\partial\theta, \qquad \mathbf{B}\nabla\xi_V{}^0 = 0. \tag{2.13}$$

These equations can be satisfied only if, in the zeroth approximation (in which S = 0), all the lines of force are closed: $n\dot{\Phi} = m\dot{\chi}$, and moreover, $\xi_V^0 = \xi_V^0(u, V)$, where $u = m\theta - n\zeta$, so that $\mathbf{B}\cdot\nabla u = 0$[3].

Further, let us write the vectors **B**, **j**, and curl$[\boldsymbol{\xi} \times \mathbf{B}]$ as

$$\mathbf{B} = \alpha_1[\mathbf{eB}] + \beta_1[\mathbf{ej}] + \gamma_1\mathbf{e}, \qquad \mathbf{j} = \alpha_2[\mathbf{eB}] + \beta_2[\mathbf{ej}] + \gamma_2\mathbf{e},$$
$$\operatorname{rot}[\xi\mathbf{B}] = a\mathbf{B} + b\mathbf{j} + c\mathbf{e}, \tag{2.14}$$

where the coefficients α_i, β_i, and γ_i are equal to

$$\alpha_1 = -\frac{[\mathbf{eB}][\mathbf{ej}]}{\dot{p}\mathbf{e}^2}, \quad \beta_1 = \frac{[\mathbf{eB}]^2}{\dot{p}\mathbf{e}^2}, \quad \gamma_1 = \frac{\mathbf{eB}}{\mathbf{e}^2},$$
$$\alpha_2 = -\frac{[\mathbf{je}]^2}{\dot{p}\mathbf{e}^2}, \quad \beta_2 = \frac{[\mathbf{eB}][\mathbf{ej}]}{\dot{p}\mathbf{e}^2}, \quad \gamma_2 = \frac{\mathbf{ej}}{\mathbf{e}^2}, \tag{2.15}$$

and the coefficients a, b, and c are defined by the formulas (2.12). In these notations,

$$(\operatorname{rot}[\xi\mathbf{B}] + [\mathbf{je}]\,\xi_V)^2 = \left\{\left(b\alpha_2 - b\alpha_1\frac{\beta_2}{\beta_1} + \xi_V\frac{\alpha_1}{\beta_1}\right)[\mathbf{eB}]\right.$$
$$\left. + \frac{1}{\mathbf{e}^2}\left(a + b\frac{\beta_2}{\beta_1} - \frac{\xi_V}{\beta_1}\right)[\mathbf{e}[\mathbf{Be}]] + (c + \gamma_1 a + \gamma_2 b)\,\mathbf{e}\right\}^2. \tag{2.16}$$

Since all three vectors in the braces are mutually orthogonal, expression (2.16) can only decrease if we discard the last two vectors. Thus, to first approximation in ϵ, we obtain

$$(\operatorname{rot}[\xi\mathbf{B}] + [\mathbf{je}]\xi_V{}^0)^2 \geqslant \frac{1}{[\mathbf{Be}]^2}\left(\mathbf{B}\nabla\mu^1 + S\frac{\partial}{\partial V}V\xi_V{}^0 + [\mathbf{je}][\mathbf{Be}]\xi_V{}^0\right)^2. \tag{2.17}$$

In addition, discarding the positive term $\gamma p(\operatorname{div}\boldsymbol{\xi})^2$ in (2.5), we have

$$\delta w \geqslant \frac{1}{2}\int\left\{\frac{1}{[\mathbf{Be}]^2}\left(\mathbf{B}\nabla\mu^1 + S\frac{\partial}{\partial V}V\xi_V{}^0 + [\mathbf{je}][\mathbf{Be}]\,\xi_V{}^0\right)^2\right.$$
$$\left. - (\Omega + [\mathbf{je}]^2)(\xi_V{}^0)^2\right\}d\tau. \tag{2.18}$$

Let angular brackets denote the average along the closed line of force **B**:

$$\langle f\rangle = \oint f\frac{dl}{B}\Big/\oint\frac{dl}{B}. \tag{2.19}$$

Integrating along the line of force in (2.18) and using Schwarz inequality $\langle a^2\rangle\langle b^2\rangle \geq \langle ab\rangle^2$, where

$$a = \left(\mathbf{B}\nabla\mu^1 + S\frac{\partial}{\partial V}V\xi_V{}^0 + [\mathbf{je}][\mathbf{Be}]\,\xi_V{}^0\right)\Big/\,|[\mathbf{Be}]|,$$
$$b = |[\mathbf{Be}]|,$$

we get

$$\delta w \geqslant \frac{1}{2}\int\left\{\langle[\mathbf{Be}]^2\rangle^{-1}\left\langle S\frac{\partial}{\partial V}V\xi_V{}^0 + [\mathbf{je}][\mathbf{Be}]\,\xi_V{}^0\right\rangle^2\right.$$
$$\left. - \langle\Omega + [\mathbf{je}]^2\rangle(\xi_V{}^0)^2\right\}d\tau. \tag{2.20}$$

Moreover, since the boundary condition $\xi_V^0|_\Sigma = 0$ yields

$$\int\frac{\partial}{\partial V}(V\xi_V{}^0)\xi_V{}^0\,dV = \int(\xi_V{}^0)^2\,dV + \frac{1}{2}\int V\frac{\partial}{\partial V}(\xi_V{}^0)^2\,dV$$
$$= \frac{1}{2}\int(\xi_V{}^0)^2\,dV, \tag{2.21}$$

application of the Schwarz inequality

$$\int(\xi_V{}^0)^2\,dV\int\left(\frac{\partial}{\partial V}V\xi_V{}^0\right)^2 dV \geqslant \left\{\int\xi_V{}^0\frac{\partial}{\partial V}(V\xi_V{}^0)\,dV\right\}^2$$
$$= \frac{1}{4}\left\{\int(\xi_V{}^0)^2\,dV\right\}^2 \tag{2.22}$$

leads to the following sufficient criterion of stability:

$$\langle S/2 + [\mathbf{je}][\mathbf{Be}]\rangle^2 - \langle[\mathbf{Be}]^2\rangle\langle\Omega + [\mathbf{je}]^2\rangle \geqslant 0. \tag{2.23}$$

Under the condition of quasi-uniformity of the magnetic field, criterion (2.23), according to (2.7), assumes the form

$$\left\langle\frac{S}{2} + \frac{\mathbf{jB}}{|\nabla V|^2}\right\rangle^2 - \left\langle\frac{\mathbf{B}^2}{|\nabla V|^2}\right\rangle\left\langle\Omega + \frac{\mathbf{j}^2}{|\nabla V|^2}\right\rangle \geqslant 0. \tag{2.24}$$

It can be shown[4] that the necessary and sufficient criteria for stability with respect to local disturbances[3,5,6] can be transformed into the same form. Thus, inequality (2.24) is a necessary and sufficient condition for the stability of a plasma in a quasi-uniform magnetic field.

3. STABILITY OF THE PLASMA IN THE NEIGHBORHOOD OF THE MAGNETIC AXIS

It is convenient to write the stability criterion (2.24) in the form

$$\frac{S^2}{4}+\left(S\left\langle\frac{\mathbf{jB}}{|\nabla V|^2}\right\rangle-\Omega\left\langle\frac{\mathbf{B}}{|\nabla V|^2}\right\rangle\right)$$
$$-\left(\left\langle\frac{\mathbf{j}^2}{|\nabla V|^2}\right\rangle\left\langle\frac{\mathbf{B}^2}{|\nabla V|^2}\right\rangle-\left\langle\frac{\mathbf{jB}}{|\nabla V|^2}\right\rangle^2\right)\geqslant 0. \qquad (3.1)$$

Here the first term defines the stabilizing action of shear, the first term in the parentheses describes the stabilization by minimum-$\overline{B}$, while the term in the second parentheses is positive by the Schwarz inequality. We note that averaging along the line of force may be replaced approximately by an averaging over the volume of an infinitesimally thin layer between neighboring magnetic surfaces

$$\langle f\rangle \approx \frac{d}{dV}\int f\,d\tau.$$

In the neighborhood of the magnetic axis $V \to 0$, and all quantities appearing in (3.1) may be expanded in powers of V. In this connection, the terms appearing in the first parentheses are of order $1/V^2$ and it suffices to calculate them in a rough approximation, whereas the second parentheses contain mutually compensating terms of order $1/V^4$, and care must be exerted in calculating them in order to preserve all quantities of order $1/V^2$.

Using the approximate relations

$$\Phi \approx \dot{\Phi} V \approx B_s\sigma, \qquad J \approx \dot{J} V \approx j_s\sigma, \qquad (3.2)$$

where σ is the cross-section area, and B_S and j_S are taken on the magnetic axis s, we obtain

$$S\left\langle\frac{\mathbf{jB}}{|\nabla V|^2}\right\rangle-\Omega\left\langle\frac{\mathbf{B}^2}{|\nabla V|^2}\right\rangle\approx\left\langle\frac{1}{|\nabla V|^2}\right\rangle B_s^2\dot{p}\frac{\ddot{\Phi}}{\dot{\Phi}}. \qquad (3.3)$$

To calculate the second parentheses in (3.1), we take into account the fact that $\mathbf{j}_S(s) = \mathrm{const}\cdot\mathbf{B}_S(s)$ and represent $\mathbf{j}$ and $\mathbf{B}$ as

$$\mathbf{j}=\mathbf{j}_s+\mathbf{j}_1, \qquad \mathbf{B}=\mathbf{B}_s+\mathbf{B}_1, \qquad (3.4)$$

where $\mathbf{j}_1$ and $\mathbf{B}_1$ are small quantities of order V.

Up to terms of order $1/V^2$, we have

$$\left\langle\frac{\mathbf{j}^2}{|\nabla V|^2}\right\rangle\left\langle\frac{\mathbf{B}^2}{|\nabla V|^2}\right\rangle-\left\langle\frac{\mathbf{jB}}{|\nabla V|^2}\right\rangle^2$$
$$\approx\left\langle\frac{1}{|\nabla V|^2}\right\rangle\left\langle\frac{\mathbf{j}_s^2\mathbf{B}_1^2+\mathbf{B}_s^2\mathbf{j}_1^2-2(\mathbf{j}_s\mathbf{B}_s)(\mathbf{j}_1\mathbf{B}_1)}{|\nabla V|^2}\right\rangle \qquad (3.5)$$
$$-\left\langle\frac{\mathbf{B}_s\mathbf{j}_1}{|\nabla V|^2}\right\rangle^2-\left\langle\frac{\mathbf{j}_s\mathbf{B}_1}{|\nabla V|^2}\right\rangle^2+2\left\langle\frac{\mathbf{B}_s\mathbf{j}_1}{|\nabla V|^2}\right\rangle\left\langle\frac{\mathbf{j}_s\mathbf{B}_1}{|\nabla V|^2}\right\rangle.$$

It is possible to show that the last three terms on the right side are of order unity and they can be neglected. To transform the remaining terms, we use the equilibrium equation $|\nabla p|^2 = \mathbf{j}^2\mathbf{B}^2 - (\mathbf{j}\cdot\mathbf{B})^2$ expanded in $\mathbf{B}_1$ and $\mathbf{j}_1$:

$$|\nabla p|^2\approx \mathbf{j}_s^2\mathbf{B}_1^2+\mathbf{B}_s^2\mathbf{j}_1^2-2(\mathbf{j}_s\mathbf{B}_s)(\mathbf{j}_1\mathbf{B}_1)-(\mathbf{j}_s\mathbf{B}_1)^2$$
$$-(\mathbf{B}_s\mathbf{j}_1)^2+2(\mathbf{j}_s\mathbf{B}_1)(\mathbf{B}_s\mathbf{j}_1). \qquad (3.6)$$

As a result, we find

$$\left\langle\frac{\mathbf{j}^2}{|\nabla V|^2}\right\rangle\left\langle\frac{\mathbf{B}^2}{|\nabla V|^2}\right\rangle-\left\langle\frac{\mathbf{jB}}{|\nabla V|^2}\right\rangle^2\approx\left\langle\frac{1}{|\nabla V|^2}\right\rangle \qquad (3.7)$$
$$\cdot\left\{\dot{p}^2+\left\langle\frac{(\mathbf{j}_s\mathbf{B}_1-\mathbf{B}_s\mathbf{j}_1)^2}{|\nabla V|^2}\right\rangle\right\}.$$

Here the expression in the parentheses can also be written as

$$(\mathbf{j}_s\mathbf{B}_1-\mathbf{B}_s\mathbf{j}_1)^2\approx\left(j_s\frac{\partial B_s}{\partial\rho}-B_s\frac{\partial j_s}{\partial\rho}\right)^2\rho^2=\left(B_s^2\rho\frac{\partial}{\partial\rho}\frac{j_s}{B_s}\right)^2, \qquad (3.8)$$

where ρ is the distance from the magnetic axis s.

If the longitudinal flux Φ is taken to be the argument of all the surface functions, then the stability criterion in the neighborhood of the magnetic axis s can be written as

$$\frac{1}{4}\left(\frac{\chi''}{V'}\right)^2+\left\langle\frac{1}{|\nabla\Phi|^2}\right\rangle\left\{p'\left(B_s^2\frac{V''}{V'}-p'\right)\right.$$
$$\left.-B_s^4\left\langle\left(\rho\frac{\partial}{\partial\rho}\frac{j_s}{B_s}\right)^2\Big/|\nabla\Phi|^2\right\rangle\right\}\geqslant 0, \qquad (3.9)$$

where the primes denote differentiation with respect to Φ.

Let us consider separately the stability condition for configurations without longitudinal current $j_S \approx 0$, and for axisymmetric configurations, which cannot have equilibrium without a longitudinal current.

A. In the absence of a longitudinal current $j_S = 0$, the stability condition (3.9) reduces to the minimum-$\overline{B}$ requirement:

$$-p'\leqslant -B_s^2V''/V'. \qquad (3.10)$$

Representing the plasma pressure p as

$$p=p_0(1-\Phi/\Phi_\Sigma), \qquad (3.11)$$

we obtain a condition on the ratio of the plasma pressure to the magnetic pressure $\beta = 2p/B_S^2$:

$$\beta\leqslant -2\Phi_\Sigma V''/V'. \qquad (3.12)$$

B. For axisymmetric configurations[7] we have

$$B_s=I_A(\psi)/r, \qquad j_s=rp'(\psi)-I_A(\psi)I_A'(\psi)/r, \qquad (3.13)$$
$$\psi=-\chi/2\pi,$$

where r is the distance from the axis of symmetry, $R - r = \rho\cos\omega$, R is the radius of the magnetic axis. Since ψ is proportional to V, then according to (3.9), we get

$$\frac{1}{4}\left(\frac{\chi''}{V'}\right)^2+\left\langle\frac{p'}{|\nabla\Phi|^2}\right\rangle$$
$$\times\left\{B_s^2\frac{V''}{V'}-p'\left(1+\frac{16\pi^2B_s^2}{\chi'^2}\left\langle\frac{(r-R)^2}{|\nabla\Phi|^2}\right\rangle\right)\right\}\geqslant 0. \qquad (3.14)$$

The stability condition (3.14) is valid for all arbitrary axial sections of magnetic surfaces.

For magnetic surfaces with circular cross sections, where $\Phi \approx B_S\pi\rho^2$, $V \approx 2\pi^2R\rho^2$, the stability condition (3.14) can be written as[8]

$$\frac{1}{4}\left(\frac{\chi''}{V'}\right)^2+\left\langle\frac{p'}{|\nabla\Phi|^2}\right\rangle\left\{B_s^2\frac{V''}{V'}-p'\left(1+\frac{2}{\chi'^2}\right)\right\}\geqslant 0. \qquad (3.15)$$

For cylindrical geometry this condition becomes Suydam's criterion[9]. For toroidal geometry, it reduces to a bound on the longitudinal current: $Rj_S/B_S \leq 2$.

The author is greatly indebted to Academician M. A. Leontovich for discussions on this work.

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Translated by C. K. Chu
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