THERMODYNAMICS OF ELECTRON DIAMAGNETISM FOR THE CASE OF AN INHOMOGENEOUS MAGNETIC MOMENT

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An electron gas with a nonuniform magnetic moment is considered. The Gibbs free energy (formula (2.6)) and magnetic moment (formula (2.8)) are derived in the one-dimensional case; the energy levels (formula (2.19)) and wave functions (formulas (2.13) and (2.18)) are found in the two-dimensional case.

1. INTRODUCTION

$\mathbf{I}$T has been customarily assumed that electronic diamagnetism is a weak magnetism and leads to a very small magnetic susceptibility χ. Shoenberg[1] was the first to call attention to the fact that electronic diamagnetism can lead to values of χ on the order of unity and larger. Of course, a diamagnetic increment $\delta\theta$, oscillating with the magnetic field, to any thermodynamic potential θ is always small: inasmuch as the magnetic moment has a relativistic origin, the "magnetic energy" $B^2/8\pi$ (B—magnetic induction) acquires a factor $(v/c)^2$ (the square of the velocity v of the charges is involved because the sign of the velocity v should be immaterial), and inasmuch as the oscillations are connected with passage through the Fermi energy of the last "magnetic branch" (Fig. 1), an additional factor $(\hbar\Omega/\epsilon_0)^{1/2}$ appears in the quasiclassical case (this form of the factor corresponds to the quasiclassical quantization rules (see[2]); Ω—cyclotron frequency, ϵ_0—Fermi energy).

It must also be taken into account that the Landau quantization becomes manifest only in the case when the electron has time to execute a large number of revolutions during the time τ between collisions, i.e., if $\Omega\tau/2\pi \gg 1$ (then the natural width of the level h/τ is much smaller than the distance $\hbar\Omega$ between levels). In the opposite case only the electrons moving between the collisions within a time on the order of $2\pi/\Omega$ are significant; their number is exponentially small, on the order of $\exp(-2\pi r\alpha/l)$, $\alpha \sim 1$, l—mean free path, r—Larmor radius; an exact calculation yields $\alpha = \pi$.

The temperature "smears" the Fermi boundary (the concept of the "last branch" loses its exact meaning) and leads to an "effective" time τ_{eff}, wherein $\hbar/\tau_{eff} = T$. As a result $\delta\theta \sim \exp(-2\pi^2 T/\hbar\Omega)$. The amplitude
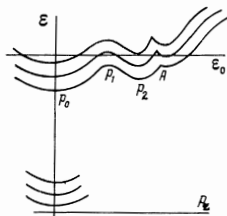

FIG. 1.

$\delta\theta$ of the quantum oscillations has finally the following order of magnitude[1]:

$$\delta\theta \sim \frac{B^2}{8\pi}\left(\frac{v}{c}\right)^2\left(\frac{\hbar\Omega}{\epsilon_0}\right)^{1/2}\exp\left(-\frac{2\pi^2}{\Omega\tau}\right)\exp\left(-\frac{2\pi^2 T}{\hbar\Omega}\right)\ll\frac{B^2}{8\pi}. \quad (1.1)$$

From simple physical considerations it is easy to find the period of the quantum oscillations. To this end, we call attention to the manner in which a magnetic field changes the form of the spectrum. Owing to the presence of a conserving continuous parameter p_z (z—direction of magnetic field, p—quasimomentum of the conduction electron), the spectrum in the magnetic field remains continuous, only the boundaries of the spectrum shift (thus, in the absence of a spin, we have for the free electron $\epsilon_{min} = \hbar\Omega/2$). However, with increasing magnetic field, the multiplicity of degeneracy at a given energy changes abruptly, since the branches of the spectrum "vanish" (going over into the region of energies larger than the given one), leading to the known singularity in the state density. This produces also a singularity in the thermodynamic potential, when the number of the branches n changes by unity, i.e., when[2]

$$1 = \delta n = \delta\left(\frac{cS}{ehB}\right) = \frac{cS}{eh}\delta(B^{-1}).$$

Hence

$$\delta(B^{-1}) = eh/cS, \quad (1.1a)$$

where $S = S(\epsilon, p_z)$ is the arc of the section of the surface $\epsilon(p) = \epsilon$ by the plane $p_z = \text{const}$.

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[1] We note that the relativistic factor $(v/c)^2$ is of purely quantum origin, since $v \sim h/ma$, where h—Planck's constant, m—of the order of the effective mass of the charge, and a—average distance between charges; the factor $B^2(v/c)$ can be represented also in the more customary: $n_0\hbar\Omega \cdot \hbar\Omega/\epsilon_0$, where $\hbar\Omega = \mu B$ (μ—Bohr magneton for the conduction electron), of the order of the diamagnetic energy of the charge in the magnetic field, n_0—charge density, $n_0\hbar\Omega$—energy of the "fully oriented" diamagnetic moments, and $\hbar\Omega/\epsilon_0$—relative number of oriented moments in the field B.

[2] The fact that the number of branches is cS/ehB follows from the correspondence principle: $\delta\epsilon = \hbar\Omega = \hbar eH/m^*c$, $m^* = (2\pi)^{-1}\partial S/\partial\epsilon$ (a proof of the correspondence principle for an arbitrary dispersion law is given in[3]). We note incidentally that in the calculations it is not necessary to take into account the change of the Fermi energy ϵ_0, since it is of the order of $\delta\epsilon$, and the effects of interest to us are of the order of $(\delta\epsilon)^{1/2}$.

Inasmuch as for a degenerate Fermi gas we are interested in the passage of the branches through the limiting energy ϵ_0, it is necessary to take S at $\epsilon = \epsilon_0$ and at p_z corresponding either to the passage of the branch or its "part" through ϵ_0 ($p_z = p_0, p_1, p_2, \ldots$ on Fig. 1), or else the passage of a point of the type A through ϵ_0. The points p_i correspond to $\epsilon_n'(p_i) = 0$. But

$$\left(\frac{\partial \epsilon}{\partial p_z}\right)_n = \frac{\partial(\epsilon, n)}{\partial(p_z, n)} = \frac{\partial(\epsilon, S)}{\partial(p_z, S)} = \frac{\partial(\epsilon, S)}{\partial(\epsilon, p_z)} \bigg/ \frac{\partial(p_z, S)}{\partial(\epsilon, p_z)}$$

$$= -\left(\frac{\partial S}{\partial p_z}\right)_\epsilon \bigg/ \left(\frac{\partial S}{\partial \epsilon}\right)_{p_z}$$

Therefore p_i corresponds to the points at which either $S'(p_z) = 0$, i.e., $S(p_0)$ has an extremum, or else (formally) $\partial S/\partial \epsilon = \infty$ (cross sections with self-intersection or bordering between open and closed sections).

Thus, the period of the oscillations in B^{-1} is determined by formula (1.1), where S corresponds either to the extremal or to the singular cross section at $\epsilon = \epsilon_0$.

The order of magnitude of the oscillation period δB relative to the magnetic induction[3] B (see Fig. 1) is

$$\delta(\epsilon_0/\hbar\Omega) \sim 1, \quad \delta B/B \sim \hbar\Omega/\epsilon_0. \quad (1.1b)$$

As a result, the amplitude of the oscillations of the susceptibility $\chi = \partial M/\partial B = -\partial^2\delta\theta/\partial B^2$ (M—magnetic moment) turns out to be of the order of

$$\chi \sim \frac{\delta\theta}{(\delta B)^2} \sim \left(\frac{v}{c}\right)^2 \Delta^{-3} \exp\left(-\frac{2\pi^2}{\Omega\tau} - \frac{2\pi^2 T}{\hbar\Omega}\right),$$

$$\Delta \sim \left(\frac{\hbar\Omega}{\epsilon_0}\right)^{1/2} \quad (1.2)$$

and at sufficiently low temperatures and for pure samples, when

$$\Omega \gtrsim 2\pi^2/\tau, \; 2\pi^2 T/\hbar, \quad (1.3)$$

it can reach arbitrarily large values and thus correspond to strong magnetism. In particular, for $|\chi| \gtrsim 1$ it is necessary to have

$$\epsilon_0/\hbar\Omega \gtrsim (c/v)^{4/3} \sim 10^3, \quad (1.4)$$

and the total number of oscillations in the interval $|\chi| \gtrsim 1$ at a given temperature T is of the order of $(\epsilon_0/2\pi^2 T)(v/c)^{4/3}$.

Thus, a distinguishing feature of electron diamagnetism is that when $T = 0$ and $\tau = \infty$ it becomes the stronger, the weaker the magnetic field: $M \sim B^{-1/2}$, $\chi \sim B^{-3/2}$. (The estimate (1.2) can, of course, be derived from the formulas of[5,6], if account is taken of the fact that

$$e^2/h = e^2 a/ha \sim e^2/a p_0 \sim \epsilon_0/p_0 \sim v_0,$$

where p_0 is the Fermi momentum and a is the average distance between the conduction electrons). We note incidentally that the relative values of the diamagnetic increment of the electronic specific heat and the electronic compressibility are of the order of Δ.

The non-oscillating increment $\delta\theta_L$ to θ (due to the Landau diamagnetism) is due to all the "branches" on Fig. 1 and is therefore appreciably larger than (1):

$$\delta\theta_L \sim n\hbar\Omega\hbar\Omega/\epsilon_0 \sim B^2(v/c)^2.$$

However, it varies smoothly and its contribution to the total values of χ and M is small at low temperatures (1.3).

Since the thermodynamically stable states are only those corresponding to $\partial H/\partial B > 0$ (see, e.g.,[7]), i.e., $4\pi\chi < 1$, the increase of χ in a bounded sample can lead in the case of $4\pi\chi_{max} > 1$ to a stratification into magnetic phases with different values of B, i.e., to the appearance of diamagnetic domains (this was noted by Condon[8]; the form of the wall between the domains was found by Privorotskiĭ[9]).

This is physically connected with the fact that the magnetic field changes the density of the states and consequently also the internal energy of the electron gas, and when $4\pi\delta > 1$ the "realignment" of the density of states, connected with the change of B occurring during the stratification into phases, becomes "convenient."

The homogeneous state of the magnet can become unstable, however, even earlier, when the stratification into domains is thermodynamically inconvenient or when more than one conduction-electron band exist in the diamagnet, so that $\chi = \sum_\alpha \chi_\alpha$ (the summation is over different bands and over the extremal and singular cross sections). In this case it is possible that although a homogeneous state is convenient for this value of χ in a single-component system, for certain bands and cross sections, in the absence of interaction between them, an instability against stratification has already set in:

$$4\pi\chi_0 = 4\pi\sum_\beta \chi_\beta > 1$$

(β are those of the α for which $\chi_\beta > 0$). Here, as shown in[10], the appearance of spatial periodicity of the magnetic moment may turn out to be convenient. (It is noted in[11] that a periodic structure appears when $4\pi\chi > 1$, but the state considered there is thermodynamically absolutely unstable.)

Inasmuch as not only χ but also $M/H \propto H^{-1/2}$ have singularities when $H \to 0$, $T/H \lesssim$ const and $\tau H \gtrsim$ const (i.e., for example, when $T = 0$, $\tau = \infty$, and $H \to 0$), it may turn out that $|M| \gtrsim |H|$. To this end it is necessary, in accordance with (1.1), to have

$$2\pi^2\hbar/\tau, \; 2\pi^2 T \lesssim \hbar\Omega < \epsilon_0(v/c)^4,$$

i.e.,

$$T \lesssim \frac{1}{2\pi^2}\epsilon_0\left(\frac{v}{c}\right)^4, \quad l > 2\pi^2 a\left(\frac{c}{v}\right)^4,$$

or $T \lesssim 10^{-4} - 10^{-5}$ deg and $l \gtrsim 10 - 10^3$ cm. Since experiments at such temperatures and mean free paths are not realistic at present, we shall assume throughout that

$$T \gg \frac{1}{2\pi^2}\epsilon_0\left(\frac{v}{c}\right)^4, \quad |M| \ll |H|. \quad (1.4a)$$

We emphasize that all the effects of interest to us take place in sufficiently weak magnetic fields, and the difficulties of the observation are connected with the

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[3]We have been referring throughout to B. It would be important to ascertain whether the thermodynamic potential is an oscillating function of B^{-1} or H^{-1}. The form of a thermodynamic potential in a magnetic field is determined by the quantization rules, which are connected with the vector potential A, with curl A = B (since div B = 0). It is therefore natural to expect $M = M(B^{-1})$. This was first noted by Shoenberg[1] and then demonstrated by Pippard[4]. The physical cause of this fact will be discussed later.

need for obtaining low temperatures and high stability and homogeneity of the magnetic field (accurate to an oscillation period $\delta H \sim H \hbar \Omega / \epsilon_0$) and an exceedingly weak mosaic structure of the crystal.

The appearance of an inhomogeneous moment leads to the occurrence of an electrostatic potential φ. According to the conditions for thermodynamic equilibrium, the chemical potential ζ is constant throughout the system, and in order to ensure constancy of the total electron density (which follows from the electron neutrality condition, which is satisfied with accuracy of the order of $(a/r)^2$—see, e.g.[12]), a $\varphi(r)$ dependence is necessary. Since $e\varphi(r) \ll \hbar \Omega$ (as seen from the result), the value of φ can be disregarded in the quantization, so that if $n_0 = N_0 + \delta n_0$ ($\delta n_0(r) = - \partial \delta \Omega r / \partial \zeta$) is the oscillating part of n_0 with ζ constant, $\Omega(r)$ is the density of the potential Ω, and N_0 is the monotonic part of the electron density which does not depend on the magnetic field in the main approximation), we have

$$N_0(\zeta) = N_0(\zeta + e\varphi) + \delta n_0\{B(r)\},$$

$$e\varphi(r) = -\frac{1}{\nu(\epsilon_0)} \delta n_0\{B(r)\} \sim \hbar \Omega \left(\frac{\hbar \Omega}{\epsilon_0}\right)^{1/2} \exp\left(-\frac{2\pi^2 T}{\hbar \Omega} - \frac{2\pi^2}{\Omega \tau}\right), \quad (1.5)$$

where $\nu = dN_0/d\epsilon_0$.

The quantity ζ is determined from the equality of the total number of electrons in the magnetic field and in the absence of the field. Knowledge of ζ and $\varphi(r)$ makes it possible to determine the density of the electrons of each band at the point r.

Using the obtained formula and the previously given estimate for the increment of the thermodynamic potential, it is easy to verify that in the approximation of interest to us the potential φ has no effect on either the quantization in the magnetic field or on the magnetic susceptibility. Therefore all the subsequent calculations will be made without taking φ into account; by substituting the final results in (1.5) it is easy to obtain the value of $\varphi(r)$.

All the foregoing shows that the development of a consistent theory of stratification into diamagnetic domains, as well as of the spatial periodicity of the magnetic moment, calls for a determination of the thermodynamic characteristics of the electrons in the inhomogeneous field B, and the inhomogeneous increment can be regarded as small if M/B is small. The solution of this problem is the subject of the present article (the corresponding formula was presented without proof in[10]).

It is first necessary to show that the magnetic moment, as already mentioned, is determined only by the magnetic induction B.

The problem of interest to us, that of the behavior of the charged quasiparticles (conduction electrons in the magnetic field) is a typical field-theory problem concerning the behavior of a system of free charges in vacuum (in this case—vacuum for quasiparticles). Therefore, strictly speaking, it is necessary to consider only the microscopic magnetic field produced at a given point by all the charges moving along orbits having a radius of the order of the Larmor radius r. If the distance a between the charges is small compared with r, as is usually the case in metals (where $a \sim 10^{-8}$ and r is of the order of 10^{-3} cm when $B \sim 10^4$ Oe), the macroscopic magnetic field at a given point is determined by the
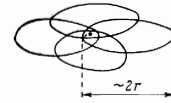

FIG. 2.

number of electrons $4\pi(r/a)^2$, and is therefore "self-averaged" (Fig. 2).

Such an averaged self-consistent field is, by definition, the magnetic induction B; it is only the induction which is felt by each of the electrons, and it is only the induction which determines the magnetic moment M. It is understandable that the connection between M and B is nonlocal: M at a given point is expressed in terms of the values of B at all points located at a distance $2r_{max}$ from the given point.

We now proceed to solve our problem.

2. THERMODYNAMIC POTENTIAL OF ELECTRON GAS IN AN INHOMOGENEOUS FIELD

A. According to the estimates of the preceding section, the magnetic moment, which is determined by the first derivative of the thermodynamic potential with respect to the induction, is of the following order of magnitude:

$$M \sim H(v/c)^2 \Delta^{-1} \sim \chi H \hbar \Omega / \epsilon_0.$$

Accordingly, the relative correction, due to the magnetic moment, to the distance between the diamagnetic levels is of the order of $\chi \hbar \Omega / \epsilon_0$. Such a correction cannot influence the derivation of formulas based only on the large quantity $(\hbar \Omega / \epsilon_0)^{-1}$, and leads qualitatively to the appearance of the quantity

$$\frac{\epsilon_0}{\hbar \Omega (1 + a\chi \hbar \Omega / \epsilon_0)} \approx \frac{\epsilon_0}{\hbar \Omega} - a\chi, \quad a \sim 1,$$

in the argument of the periodic function, in place of $\epsilon_0 / \hbar \Omega$, i.e., to an essential nonlinearity in M. The next higher approximation adds to the argument a value on the order of

$$\chi \hbar \Omega / \epsilon_0 \sim (v/c)^2 (\hbar \Omega / \epsilon_0)^{1/2},$$

i.e., a quantity which is certainly small. This shows that, first, it is possible to use the approximation linear in M for the argument of the periodic function and, second, it is sufficient to find only the main approximation, with respect to M, for the increment in the quantization rules, meaning that it is possible to use the correspondence principle

$$\delta \varepsilon = \hbar \Omega^*, \quad (2.1)$$

where Ω^* is the classical frequency of revolution of the electron in the field $B(r)$.

Let us first consider the dependence of B, on only one coordinate, when $B_x = B_y = 0$ and $B_z = B(y)$. In order to have $M = M_z(y)$, $H = H_z(y)$, and $\varphi = \varphi(y)$, we shall assume that the z axis coincides with one of the principal crystallographic axes.

The classical equation of motion, with allowance for the force acting on the spin in an inhomogeneous magnetic field, is of the form*

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*$[vB] \equiv v \times B$.

$$\dot{\mathbf{p}} = -e\nabla\varphi + ec^{-1}[\mathbf{v}\mathbf{B}] + \mu_0\nabla(\sigma\mathbf{B}),$$

$$\sigma = (0, 0, \pm 1), \quad \mathbf{v} = \dot{\mathbf{r}} = \partial\varepsilon/\partial\mathbf{p}.$$

In this equation μ_0 is the spin magnetic moment of the electron, the role of which reduces to a "renormalization" of the potential φ: $\varphi \to \varphi \mp \mu_0 B$. Inasmuch as this quantum addition (see formula (1.5) for φ) is not significant in the approximation of interest to us, it can be disregarded in the calculation of the distance between the diamagnetic levels. This means that it is possible to disregard the influence of the spin splitting of the levels on the Landau quantization.

For simplicity we confine ourselves to the case when the inhomogeneous addition to the induction (which is of the order of M) is small compared with the temperature. Then, in order to take into account the spin quantization, it is sufficient to add $\pm\mu_0 H$ to the diamagnetic levels $\varepsilon_{np_z P_X}$ obtained below. In this case the equations of motion are equivalent to the equations of one-dimensional motion with Hamiltonian

$$\varepsilon^*(y) = \varepsilon\{p_x(y), p_y; p_z\}, \quad p_x(y) = P_x + \frac{e}{c}\int^y B(y')dy', \quad (2.2)$$

where p_z and P_X play the role of parameters. The period of such a motion is

$$\Theta = \oint\frac{dy}{v_y} = \frac{\partial S^*}{\partial\varepsilon} \quad S^* = \oint dy\,dp_y. \quad (2.3)$$

Here $S^*(P_X, p_z) = c\tilde{S}/eH$ is the area under the curve

$$\varepsilon\left(P_x + \frac{e}{c}\int B(y')dy', p_y; p_z\right) = \varepsilon, \quad (2.4)$$

which lies in the plane (y, p_y). Since only M depends on y, and since we are interested only in the linear approximation in M, we have

$$S^* = \frac{2c}{e}\int_{p_x^{min}}^{p_x^{max}} p_y(p_x)dp_x\left|B\left(\frac{p_x - P_x}{eH/c}\right)\right). \quad (2.4a)$$

It follows from (2.1) and (2.3) that $\delta\varepsilon = 2\pi\hbar/\Theta$, i.e.,

$$\frac{\partial S^*}{\partial\varepsilon}\delta\varepsilon = h, \quad \delta S^* = h,$$

and

$$S^*(\varepsilon, p_z, P_x) = nh. \quad (2.5)$$

(A similar quantization rule was first obtained by Kosevich[13].)

Formulas (2.4) and (2.5) can be obtained also by perturbation theory, starting from the classical Hamilton-Jacobi equations (they are more convenient than Newton's equations, owing to the variation of the period of revolution), as well as from the fact that the mean value of the perturbing increment to the energy coincides with the classical time-average (see[14], Sec. 48).

It is significant that (2.1) means also that (2.5) can be used to determine $\delta\varepsilon$. The increments to the right side of (2.5) are not only small compared with nh (which is obvious, since $S^*/h \gg 1$, under the assumption that the quasiclassical approach holds), but also vary slowly.

B. In order to calculate the thermodynamic potential, it remains, in principle, to determine the density of the states. We shall again use the correspondence principle in the quasiclassical approach. (This is sufficient, since the density of states varies slowly in the approximations of interest to us.) Since in the classical approach the

density of states is (for a given spin-projection direction) $dydp_ydP_Xdp_Z/h^3$, and $dydp_y = dS^* \to \delta S^* = h$, in the general case of the presence of several bands α, the increment Ω_m added by the quantization to the thermodynamic potential Ω is equal to

$$\Omega_m = -T\frac{2}{h^3}\sum_{\substack{\alpha \\ \sigma=\pm 1}}\int dP_x\,dp_z\sum_n\ln\left\{\exp\left[\frac{\zeta - \varepsilon_{np_z P_x}^\alpha - \sigma\mu_0 H}{T}\right] + 1\right\}.$$

This formula differs from the Lifshitz-Kosevich (LK) formula[5] in a homogeneous field only in the presence of an additional integration with respect to P_X. Therefore we obtain under the integral with respect to P_X the well known expression for Ω_{LK} in a homogeneous field:

$$\Omega_m = \sum_\alpha\int_{-\infty}^\infty dP_x\Omega_{LK}^{(\alpha)}\left\{\frac{2c}{eh}\int_{p_x^{min}}^{p_x^{max}} p_y(p_x)dp_x\left|B\left(\frac{p_x - P_x}{eH/c}\right)\right\}\right.$$

$$\qquad\qquad (2.6)$$

$$= \sum_\alpha\int_{-\infty}^\infty dP_x\Omega_{LK}^{(\alpha)}\left\{\frac{cS_\alpha}{ehH} - \frac{2c}{ehH^2}\int_{p_x^{min}}^{p_x^{max}} p_y(p_x)M\left(\frac{p_x - P_x}{eH/c}\right)\right\}dp_x,$$

where Ω_{LK} is the potential Ω_m in the homogeneous case and S is the area under the curve $\varepsilon(p_x, p_y) = \varepsilon$. (The divergence in (2.16) at B = const is connected with the fact that Ω_m pertains to an infinite volume.) The density Ω_m, which is equal to $\text{Tr}\{\hat{\Omega}_m\delta(y - \hat{y})\}$, is, of course, finite; the substitution $p_X \to eHy/c$ yields

$$\Omega_m = \int\Omega_{m'}dy. \quad (2.7)$$

We obtain from (2.6) the moment $M(y) = -\delta\Omega_m/\delta B$:

$$M(y) = \sum_\alpha\left\langle M_{LK}^\alpha\left\{\left\langle B^{-1}\left(y + \frac{cp_x - cp_x'}{eH}\right)\right\rangle_\alpha\right\}\right\rangle_\alpha, \quad (2.8)$$

where

$$\langle h\rangle_\alpha = \frac{2}{S_{ext}^\alpha}\int_{p_{x\alpha}^{min}}^{p_{x\alpha}^{max}} p_y^\alpha(p_x)h(p_x)dp_x, \quad (2.9)$$

and $M_{LK}^\alpha(B^{-1})$ is the moment of the given band and of the given extremal or singular section in a homogeneous field, as obtained by Lifshitz and Kosevich[5]. (Formulas (2.8) and (2.9) were cited without proof in[10].)

It remains to go over from the variable ζ—the chemical potential—to the number of particles, but this results in only a small increment to (2.8), which is natural, since the small increments to all the thermodynamic potentials in terms of the corresponding variables coincide, as is well known (see, e.g.,[15]).

C. Equation (2.8) determines M(y); a thermodynamically stable solution of this equation was given for different cases in[16]. If the sum over α contains only one term, it is possible to write in the homogeneous case an explicit formula for M (see also[1]); in the presence of several terms, M(H) is generally speaking (if the different periods of the oscillations are not commensurate) a non-periodic function, and there is no explicit formula for it.

We shall show how to find M(H) in the presence of a single period. Let (see Fig. 3)

$$4\pi M = f(H + 4\pi M), \quad B - H = f(B),$$

where f has a period H_0. It is clear that it is sufficient to consider the solution in the intervals AB and AC. Assume, for concreteness, that AC corresponds to the
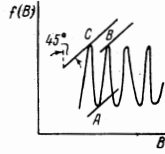
FIG. 3.

interval from H_1 to H_2. We formally continue $B(H)$ with period $H_2 - H_1 = h$:

$$B = \sum_{k=-\infty}^{\infty} b_k e^{2\pi i k H/h}, \qquad b_{-k} = b_k^*.$$

Then

$$b_k = \frac{1}{h} \int_{H_1}^{H_2} B(H) e^{-2\pi i k H/h} dH = -\frac{1}{2\pi i k} \int_{H_1}^{H_2} e^{-2\pi i k H/h} B'(H) dH$$

$$= \frac{1}{2\pi i k} \int_{B_1}^{B_2} \exp\left\{-\frac{2\pi i k (B - f(B))}{h}\right\} dB.$$

D. In the two-dimensional case the solution of the problem is much more complicated, and we shall therefore obtain only the form of the wave functions and the energy levels. Let $B = B_z(x, y)$, so that the vector potential can be chosen in the form

$$A = A_x(x, y) = -\int^y B(x, y') dy'. \qquad (2.10)$$

Further, as before, let

$$B = B_0 + B_1, \quad A = -B_0 y + A_1, \quad |B_1| \ll |B_0|, \quad |A_1| \ll |A_0|, (2.11)$$

with B_0 independent of the coordinates, and B_1 varying over x and over y at distances on the order of r. Then

$$\varepsilon = \varepsilon_0(p_x, p_y) + \frac{e}{c} v_x A_1, \quad p_x = P_x + \frac{eB_0}{c} y. \qquad (2.12)$$

In the main approximation (at $A_1 = 0$), the energy levels, as is well known, are infinitely degenerate in P_x: $\varepsilon = \epsilon^{(0)}(n, p_z)$. Therefore the "regular" zeroth-approximation wave forms are

$$\psi = \int c(P_x) \psi^{(0)}_{n P_x p_z} dP_x, \qquad (2.13)$$

where $\psi^{(0)}_{n P_x p_z}$ are the zeroth-approximation eigenfunctions corresponding to the quantum numbers P_x, p_z, and n.

The coefficients $c(P_x)$ and the correction ϵ' to $\epsilon^{(0)}(n, p_z)$ satisfy the following equation (see[14], Sec. 39)

$$\int V_{nn; P_x P'_x} c(P'_x) dP'_x = \varepsilon' c(P_x), \quad V = \frac{e}{c} v_x A_1. \qquad (2.14)$$

In the quasiclassical case in the main approximation, as can be readily verified, the matrix elements $V_{nn; P_x P'_x}$ are equal to

$$V_{nn; P_x P'_x} = \frac{1}{\Theta} \int_0^{\Theta} dt \int_{-\infty}^{\infty} \exp\left\{\frac{i}{\hbar}(P_x - P'_x)x\right\}$$

$$\times \frac{e}{c} v_x(t) A_1\left(x, \frac{p_x(t) - P_x}{eH/c}\right) dx, \qquad (2.15)$$

where $v_x(t)$, $p_x(t)$, and Θ are determined from the solution of the classical equation $\dot{\mathbf{p}} = ec^{-1} \mathbf{v} \times \mathbf{H}$ followed by substitution of $\epsilon = \epsilon^{(0)}(n, p_z)$. Using this equation, we obtain after simple transformations

$$V_{nn; P_x P'_x} = \frac{c}{eTH^2} \oint p_y(p_x) B_1\left(\frac{P_x - P'_x}{\hbar}, \frac{p_x - P_x}{eH/c}\right) dp_x; \qquad (2.16)$$

$$\varepsilon(p_y(p_x), p_x; p_z) = \varepsilon^{(0)}(n, p_z), \qquad \tilde{B}_1(k, y) = \int_{-\infty}^{\infty} e^{ikx} B_1(x, y) dx. \qquad (2.17)$$

The function $\tilde{B}_1$ varies rapidly with respect to its first argument (it changes appreciably over a distance $\delta P_x \sim \hbar/r$) and slowly with respect to the second argument (at distances $\delta P_x \sim eHr/c \sim p_0 \sim \hbar/a \gg \hbar/r$, where a is the interatomic distance). If P_x were constant in the second argument of B_1 in (2.16), then (2.14) would have the solution $c(P_x) = a \exp(ix_0 P_x/\hbar)$, and we shall therefore seek $c(P_x)$ in the form

$$c(P_x) = a \exp\left\{\frac{i}{\hbar} \int^{P_x} x_0(P'_x) dP'_x\right\}, \qquad (2.18)$$

where $x_0(P_x)$ is a slowly varying function. Substituting (2.18) in (2.14) we get

$$\varepsilon' = \frac{\tilde{S}}{H \partial \tilde{S}/\partial \varepsilon} \bar{B}_1(x_0, y_0), \quad y_0 = -\frac{cP_x}{eH}, \qquad (2.19)$$

$$\bar{B}_1 = \left\langle B_1\left(x_0, y_0 + \frac{cP_x}{eH}\right)\right\rangle = \frac{1}{S} \oint p_y(p_x) B_1\left(x_0, \frac{cp_x - cP_x}{eH}\right) dp_x. \qquad (2.20)$$

Thus, ϵ' is a continuous quantum number, and the solutions $x_0 = x_0(P_x)$ (the number of which, generally speaking, is discrete) of Eq. (2.19) determine, in accordance with (2.13) and (2.18), the wave functions at specified n, p_z, and ϵ'.

As expected, if B_1 does not depend on x, then formula (2.19) yields the previously-obtained correction to the main approximation for the energy as a function of n, p_z, and P_x (the latter being conserved). To calculate the density of states $\nu(E)$ we can use its well known connection with the Green's function (see, e.g.,[17]).

The obtained formulas pertain only to an ideal electron gas and do not take into account, for example, the Fermi-liquid interaction (incidentally, the latter does not influence the period of the oscillations and the order of magnitude of their amplitude—see[18]). However, the concrete form of $M(r)$ is immaterial, and the character of this function can be obtained from very general considerations.

As shown at the beginning of this section, in the general case all the quantities are periodic functions, the argument of which is a linear function of the magnetic moment. This means that the density of any thermodynamic potential $\theta_m(r)$ of a magnet should, with account of the translation symmetry, have the form (as before, B is directed along a crystallographic axis)

$$\theta_m(r) = \sum_\alpha f_\alpha\left\{4\pi \int K_\alpha(r - r') M(r') dr'\right\}, \qquad (2.21)$$

the central symmetry of the crystal ensuring the evenness of $K_\alpha(r)$; the factor 4π has been introduced for convenience; the sum is taken over different bands and cross sections. Moreover, the period of these functions is determined only by the quantization, which is given, as shown above, only by the correspondence principle, i.e., which does not depend on the interaction. This means that in the general case we obtain formula (2.8); only the formula for the "homogeneous" $M^\alpha_{LK}(B^{-1})$ is missing, and all that remains of it is the order of magnitude of the oscillation amplitude and the period of the oscillations.

Let us write, using (2.21) and recognizing that the thermodynamic potential of the entire sample is equal to

$$\tilde{\theta} = \int \theta(r) dr, \qquad (2.21a)$$

an expression for the magnetic susceptibility. Putting $M = M_0 + M_1(r)$ (M_0—homogeneous moment, M_1-infinitesimally small increment), we get the increment $\tilde{\theta}_1$ of second order in $4\pi M_1 = B_1$ to the potential $\tilde{\theta}$:

$$\tilde{\theta}_1 = -\frac{1}{2} \sum_\alpha f_\alpha'' \left\{ 4\pi M_0 \int K_\alpha(\rho) d\rho \right\} \int B_1(r') B_1(r'') dr' dr''$$
$$\times \int K_\alpha(\rho') K_\alpha(\rho' + r' - r'') d\rho'.$$

Using the definition $\chi = -\delta^2 \tilde{\theta} / \delta B_1^2$, we get

$$\chi(r) = -\sum_\alpha f_\alpha'' \left\{ 4\pi M_0 \int K_\alpha(\rho) d\rho \right\} \int K_\alpha(\rho') K_\alpha(r + \rho') d\rho', \quad (2.22)$$

$$\tilde{\theta}_1 = -\frac{1}{2} \int \chi(r - r') B_1(r') B_1(r'') dr' dr''. \qquad (2.23)$$

Going over to Fourier components

$$B_1(r) = \int \tilde{B}_1(k) e^{ikr} dr,$$

we get

$$\tilde{\chi}(k) = \sum \tilde{\chi}_\alpha(k),$$

$$\tilde{\chi}_\alpha(k) = \chi_\alpha |\tilde{K}_\alpha(k)|^2 / |\tilde{K}_\alpha(0)|^2, \quad \tilde{K}_\alpha(k) = \int K_\alpha(r) \cos(kr) dr; \qquad (2.24)$$

$$\tilde{\theta}_1 = -\frac{(2\pi)^3}{2} \int \tilde{\chi}(k) |\tilde{B}_1(k)|^2 dk, \qquad (2.25)$$

where $\chi_\alpha = \chi_\alpha(0)$ is the susceptibility in the homogeneous field B.

In the case investigated by us (see formula (2.6) we have $K(r) \geq 0$. The same takes place also in the general case. Consequently, according to (2.24),

$$|\tilde{\chi}_\alpha(k)| \leq |\chi_\alpha|, \quad \text{sign } \tilde{\chi}_\alpha(k) = \text{sign } \chi_\alpha, \qquad (2.26)$$

and if $\chi_\alpha > 0$, then

$$\tilde{\chi}_\alpha(k) \leq \chi_\alpha. \qquad (2.27)$$

From this, in particular, follows the statement used in[16]: if $4\pi\tilde{\chi}(k) = 1$ but $4\pi\chi = 4\pi\Sigma_\alpha \chi_\alpha < 1$, with $\chi_{\alpha_1} > 0$ and $\chi_{\alpha_2} < 0$, then $4\pi\Sigma_{\alpha_1}\chi_{\alpha_1} > 1$. In fact, taking (26) into account, we get

$$4\pi \sum \tilde{\chi}_{\alpha_1} \geq 4\pi \sum \tilde{\chi}_{\alpha_1}(k) = 1 - 4\pi \sum \tilde{\chi}_{\alpha_2}(k) = 1 + 4\pi \sum |\tilde{\chi}_{\alpha_2}(k)| > 1.$$

3. FUNDAMENTAL EQUATION OF THE PROBLEM AND THE POSSIBILITY OF ITS SOLUTION

The fundamental equation of the problem is the equation for the self-consistent magnetic moment $M(r)$, which follows from the requirement that the thermodynamic potential be a minimum (formulas (2.21), (2.21a)), namely $\delta\theta/\delta M = 0$. It must be recognized here that the formulas obtained in the preceding section pertained only to the "intrinsic" magnetic energy θ_m, which is connected with the quantization of the levels in the magnetic field, and that the total potential θ is equal to (see formula (1.2a) of[16])

$$\theta(r) = \theta_m(r) + 2\pi M^2(r). \qquad (3.1)$$

As a result, the equation for $M(r)$ takes the form

$$M(r) = \sum_\alpha \int K_\alpha(r'') f_\alpha' \left\{ 4\pi \int K_\alpha(r') M(r - r' + r'') dr' \right\} dr'', \quad (3.2)$$

with $f_\alpha' \sim \chi g_\alpha$, where $g_\alpha \sim 1$.

Equation (3.2) can be solved in general form near a second-order phase transition point and near the criti-

cal point, where the inhomogeneous part of the magnetic moment is small compared with the period δB of the oscillations with respect to B. A solution was obtained in this case in[10,16] (see also the solution for the domain wall in the homogeneous case near the critical point in[9]). A solution can be similarly obtained also for $|\chi| \gg 1$, if the inhomogeneous part of the moment is small compared with the homogeneous one, i.e., with the quantity $\chi \delta B \sim \chi H \hbar \Omega / \epsilon_0$, when it is necessary to put in the main approximation the homogeneous part M_0 of the moment in the left side of (3.2). In the simplest case, if the sum of (3.2) has only one term, then

$$f' \left\{ 4\pi \int K(r') M(r - r') dr' \right\} = M_0 / \int K(r) dr. \qquad (3.3)$$

Solving (3.3) with respect to the argument of the function f' (this equation can have for $|\chi| \gg 1$ a large number of solutions μ_n, $f'(\mu_n) = M_0/\int K(r)dr$, numbered by the index n), we obtain for $M_1 = M - \mu_n [4\pi\int K(r)dr]^{-1}$ the equation

$$\int K(r - r') M_1(r') dr' = 0. \qquad (3.4)$$

From (3.4) we obtain the main approximation for M_1:

$$M_1 = \text{Re} \sum_l A_l \exp(i\varkappa_l r), \quad \int K(r) \cos(\varkappa_l r) dr = 0. \qquad (3.5)$$

The next approximation is obtained in the same manner as in[16].

In the region where the inhomogeneous part of the moment is large (of the order of $\chi \delta B$), the function oscillates rapidly, and the main approximation is obtained from the saddle points r_j, so that (for simplicity we assume as before that (3.2) has only one term)

$$M = \sum_i A_i K(r - r_i). \qquad (3.6)$$

Only a few terms under the f' sign in (3.2) differ from zero, since $K(r)$, naturally, vanishes outside the "attainability limits" of the classical orbit.

Substitution of (3.6) in (3.2) makes it possible to reduce the problem to an algebraic one. We recall that K should have an extremum at the points r_j, so that

$$\sum_i A_i \int \varkappa(r' - r_i) \frac{\partial}{\partial r_j} \varkappa(r_j - r') dr' = 0. \qquad (3.7)$$

When $4\pi\chi \sim 1$ it is impossible to obtain a solution of (3.2) in closed form far from the second-order phase transition points and the critical point. It can only be stated that the amplitude of the spatially-periodic part of the magnetic moment (where the periodic structure arises, see[1,16]) is of the order of $\delta B \sim H(v/c)^2(\hbar\Omega/\epsilon_0)^{-1/2}$, and that the difference of the values of the magnetic moment in the domains, which is obtained directly from the form of the $H = H(B)$ curve for homogeneous B, is of the same order.

[1] D. Shoenberg, Phil. Trans. Roy. Soc. (London) A255, 85 (1962).

[2] M. Ya. Azbel', Zh. Eksp. Teor. Fiz. 39, 878 (1960) [Sov. Phys.-JETP 12, 608 (1961)].

[3] M. Ya. Azbel', ibid. 46, 929 (1964) [19, 634 (1964)].

[4] A. B. Pipard, Proc. Roy. Soc. (London) A272, 192 (1963).

[5] I. M. Lifshitz and A. M. Kosevich, Zh. Eksp. Teor. Fiz. 29, 730 (1955) [Sov. Phys.-JETP 2, 636 (1956)].

[6] Yu. A. Bychkov, ibid. 39, 1401 (1960) [12, 971 (1961)].

[7] L. D. Landau and E. M. Lifshitz, Élektrodinamika sploshnykh sred (Electrodynamics of Continuous Media), Gostekhizdat, 1957, Sec. 30 (Addison-Wesley, 1960).

[8] J. H. Condon, Phys. Rev. 145, 526 (1965); Abstracts, LT-10, Moscow, 1966, p. 310.

[9] I. A. Privorotskiǐ, ZhETF Pis. Red. 5, 280 (1967) [JETP Lett. 5, 228 (1967)].

[10] M. Ya. Azbel', ibid. 5, 282 (1967) [5, 230 (1967)].

[11] M. P. Greene, A. Coughton, and J. Quinn, Abstracts, LT-10, Moscow, 1966, p. 309. J. J. Quinn, J. Phys. Chem. Solids, 24, 933 (1963). Phys. Rev. Lett. 16, 731 (1966).

[12] M. Ya. Azbel' and V. G. Peschanskiǐ, Zh. Eksp. Teor. Fiz. 49, 572 (1965) [Sov. Phys.-JETP 22, 399 (1965)].

[13] A. M. Kosevich, ibid. 35, 738 (1958) [8, 512 (1959)].

[14] L. D. Landau and E. M. Lifshitz, Kvantovaya mek-hanika (Quantum Mechanics), Fizmatgiz 1963 [Addison-Wesley, 1958)].

[15] L. D. Landau and E. M. Lifshitz, Statisticheskaya fizika (Statistical Physics), Nauka, 1964, Secs. 15 and 24 [Addison-Wesley, 1958].

[16] M. Ya. Azbel', Zh. Eksp. Teor. Fiz. 53, 1751 (1967) [Sov. Phys.-JETP 26, 1003 (1968)].

[17] A. A. Abrikosov, L. P. Gor'kov, and I. E. Dzyalo-shinskiǐ, Metody kvantovoǐ teorii polya v statisticheskoǐ fizike (Quantum Field Theoretical Methods in Statistical Physics), Fizmatgiz, 1962, Sec. 7 [Pergamon, 1965].

[18] I. M. Lifshitz and M. I. Kaganov, Usp. Fiz. Nauk 78, 411 (1962) [Sov. Phys.-Uspekhi 5, 878 (1963)]. Yu. A. Bychkov and L. P. Gor'kov, Zh. Eksp. Teor. Fiz. 41, 1592 (1962) [Sov. Phys.-JETP 14, 1132 (1963)].

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