PARAMETRIC EXCITATION OF PLASMA WAVES BY A HIGH-FREQUENCY ELECTRIC FIELD IN AN INHOMOGENEOUS PLASMA

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The behavior of an inhomogeneous plasma in an RF electric field is investigated. It is found that parametric excitation of the electron plasma wave and the ion-acoustic wave is possible if the wave frequency is approximately one-half the frequency of the electric field. The excitation conditions and the growth rates for the plasma waves are determined.

IN recent years a number of authors have investigated the behavior of a plasma in a high-frequency (RF) electric field.[1-7] Most of these authors have been concerned with the properties of a homogeneous plasma in a uniform RF electric field.[1-5] The density of a plasma of this kind is time-independent in the quiescent state. The case of an inhomogeneous magnetoactive plasma in a uniform electric field has been studied in[6,7]. However, the relative orientation of the RF electric field, the fixed magnetic field and the density gradient was such that the presence of the electric field in the quiescent state did not lead to a variation of plasma density with time: the electric field was parallel to the magnetic field and perpendicular to the density gradient.

It is clear, however, that one can find cases in which the presence of the RF electric field will produce a periodic variation of the density in time. In particular, in an isotropic inhomogeneous plasma, with which we shall be concerned below, if the RF electric field has a component along the density gradient the plasma density in the quiescent state will be a periodic function of time with period given by that of the electric field. Since the plasma is an oscillatory system with characteristic frequencies given by the electron plasma frequency $\omega_{Le} = (4\pi N_e e^2/m_e)^{1/2}$ and the ion-acoustic frequency $\omega_s = (\omega_{Li}/1 + \omega_{Le}^2/k^2 v_{Te}^2)^{1/2}$ ($\omega_{Li} = (4\pi N_i e_i^2/m_i)^{1/2}$ is the ion-plasma frequency, k is the wave number while $v_{Te} = (T_e/m_e)^{1/2}$ is the electron thermal velocity), which depend on density, the periodic variation of density with time caused by the RF electric field offers the possibility of parametric excitation[8] of plasma waves in the inhomogeneous plasma; this effect has been observed experimentally.[9,10]

Krenz and Kino[9] have shown experimentally that electron plasma oscillations can be excited in a plasma by means of an electric field that varies at frequency $2\omega_{Le}$. On the basis of a hydrodynamic analysis, these authors have considered parametric resonance at the electron plasma frequency.

Somewhat later, related experimental results were published by Kato et al.[10] In this experimental work, in addition to observing parametric excitation of electron plasma waves, the authors also observed parametric excitation of ion-plasma waves under conditions in which the plasma density was such that the ion-plasma frequency was half the frequency of the excitation electric field.

In the present work we develop the theory of parametric excitation of both RF and low-frequency plasma waves on the basis of the kinetic equations for the electrons and ions.

In an electric field $\mathbf{E}_0(\mathbf{r}, t)$ the kinetic equation for particles of species α is

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v}\frac{\partial f_\alpha}{\partial \mathbf{r}} + \frac{e_\alpha}{m_\alpha}\mathbf{E}_0(\mathbf{r}, t)\frac{\partial f_\alpha}{\partial \mathbf{v}} = 0, \tag{1}$$

where f_α is the distribution function for particles of species α in the quiescent state. The distribution function in the quiescent state will be determined by perturbation theory in terms of the field $\mathbf{E}_0(\mathbf{r}, t)$ (the conditions for application of this analysis are discussed in the Appendix). Thus, we shall assume that

$$f_\alpha = N_{0\alpha}(\mathbf{r}) f_{0\alpha} + f_{1\alpha},$$

where

$$f_{0\alpha} = \frac{1}{(2\pi)^{3/2} v_{T\alpha}^3}\exp\left(-\frac{v^2}{2v_{T\alpha}^2}\right),$$

where $f_{1\alpha}$ is determined from the linearized kinetic equation. Limiting ourselves to a first approximation in $\mathbf{E}_0(\mathbf{r}, t)$ we have

$$f_\alpha(\mathbf{v}, \mathbf{r}, t) = N_{0\alpha}(\mathbf{r}) f_{0\alpha}(v) - \frac{e_\alpha}{m_\alpha}\int_{-\infty}^{0} d\tau\, \mathbf{E}_0(\mathbf{r} + \mathbf{v}\tau, t + \tau) N_{0\alpha}(\mathbf{r} + \mathbf{v}\tau)\frac{\partial f_{0\alpha}}{\partial \mathbf{v}}. \tag{2}$$

The linearized (in terms of the perturbation) kinetic equation assumes the form

$$\frac{\partial f_\alpha'}{\partial t} + \mathbf{v}\frac{\partial f_\alpha'}{\partial \mathbf{r}} + \frac{e_\alpha}{m_\alpha}\mathbf{E}_0(\mathbf{r}, t)\frac{\partial f_\alpha'}{\partial \mathbf{v}} + \frac{e_\alpha}{m_\alpha}\mathbf{E}\frac{\partial f_\alpha}{\partial \mathbf{v}} = 0, \tag{3}$$

where f_α' is the perturbation in the quiescent distribution function while $\mathbf{E}$ is the field of the perturbation, which satisfies the Poisson equation

$$\operatorname{div}\mathbf{E} = \sum_\alpha 4\pi e_\alpha \int d\mathbf{v}\, f_\alpha'. \tag{4}$$

Assuming that the inhomogeneities in $\mathbf{E}_0(\mathbf{r}, t)$ and $N_{0\alpha}(\mathbf{r})$ are small over distances of the order of the perturbation wavelength, in the zeroth approximation in geometric optics we shall assume that f_α' and $\mathbf{E}$ are proportional to $e^{i\mathbf{k}\cdot\mathbf{r}}$, writing the kinetic equation in the form

$$\frac{\partial f_\alpha'}{\partial t}+i\mathbf{kv}f_\alpha'+\frac{e_\alpha}{m_\alpha}\mathbf{E}_0(\mathbf{r},t)\frac{\partial f_\alpha'}{\partial \mathbf{v}}$$
$$-i\frac{\mathbf{k}}{k^2}\sum_\beta\frac{4\pi e_\alpha e_\beta}{m_\alpha}\frac{\partial f_\alpha(\mathbf{v},\mathbf{r},t)}{\partial \mathbf{v}}\int d\mathbf{v}\, f_\beta'=0. \quad (5)$$

In what follows it will be convenient to introduce the new function

$$\Phi_\alpha(\mathbf{v},\mathbf{r},t)=f_\alpha'\left(\mathbf{v}+\frac{e_\alpha}{m_\alpha}\int_{-\infty}^{t}d\tau\,\mathbf{E}_0(\mathbf{r},\tau),\mathbf{r},t\right)$$
$$\times\exp\left\{i\frac{e_\alpha}{m_\alpha}\mathbf{k}\int_{-\infty}^{t}d\tau'\int_{-\infty}^{\tau'}d\tau\,\mathbf{E}_0(\mathbf{r},\tau)\right\}, \quad (6)$$

which satisfies the equation

$$\frac{\partial\Phi_\alpha}{\partial t}+i\mathbf{kv}\Phi_\alpha-i\frac{\mathbf{k}}{k^2}\frac{\partial}{\partial\mathbf{v}}f_\alpha\left(\mathbf{v}+\frac{e_\alpha}{m_\alpha}\int_{-\infty}^{t}d\tau\,\mathbf{E}_0(\mathbf{r},\tau),\mathbf{r},t\right)$$
$$\times\sum_\beta\frac{4\pi e_\alpha e_\beta}{m_\alpha}\int d\mathbf{v}\,\Phi_\beta\exp\left\{-ik\left(\frac{e_\beta}{m_\beta}-\frac{e_\alpha}{m_\alpha}\right)\int_{-\infty}^{t}d\tau'\int_{-\infty}^{\tau'}d\tau\,\mathbf{E}_0(\mathbf{r},\tau)\right\}=0. \quad (7)$$

For the case of a standing wave, in which

$$\mathbf{E}_0(\mathbf{r},t)=\mathbf{E}_0(\mathbf{r})\sin\omega_0 t, \quad (8)$$

we can make use of the relation

$$e^{ia\sin\omega_0 t}=\sum_{n=-\infty}^{\infty}J_n(a)e^{in\omega_0 t},$$

where $J_n(a)$ is the Bessel function of the first kind; we then reduce (7) to the form

$$\frac{\partial\Phi_\alpha}{\partial t}+i\mathbf{kv}\Phi_\alpha-i\frac{\mathbf{k}}{k^2}\frac{\partial}{\partial\mathbf{v}}f_\alpha\left(\mathbf{v}-\frac{e_\alpha\mathbf{E}_0(\mathbf{r})}{m_\alpha\omega_0}\cos\omega_0 t,\mathbf{r},t\right)$$
$$\times\sum_\beta\sum_{n=-\infty}^{\infty}J_n(a_\beta)e^{in\omega_0 t}\frac{4\pi e_\alpha e_\beta}{m_\alpha}\int d\mathbf{v}\,\Phi_\beta=0. \quad (9)$$

Here we have introduced the notation

$$a_\beta\equiv\left(\frac{e_\beta}{m_\beta}-\frac{e_\alpha}{m_\alpha}\right)\frac{\mathbf{kE}_0(\mathbf{r})}{\omega_0^2}.$$

We assume that the velocities associated with the oscillatory motion of the particles in the alternating electric field $\mathbf{v}_{E\alpha}=e_\alpha\mathbf{E}_0/m_\alpha\omega_0$ are small compared with $v_{T\alpha}$ the thermal velocities of the particles; we also assume that the distances traversed by the particles in one period of the field $\mathbf{E}_0$ are small, that is to say, $v_{T\alpha}/\omega_0$ is small compared with the scale of the inhomogeneities in $\mathbf{E}_0(\mathbf{r})$ and $N_{0\alpha}(\mathbf{r})$. Thus we find that

$$f_\alpha\left(\mathbf{v}-\frac{e_\alpha\mathbf{E}_0(\mathbf{r})}{m_\alpha\omega_0}\cos\omega_0 t,\mathbf{r},t\right)\approx N_{0\alpha}(\mathbf{r})f_{0\alpha}(v)$$
$$-\sin\omega_0 t\left(\mathbf{v}\frac{\partial}{\partial\mathbf{r}}\right)\left(\mathbf{A}^\alpha(\mathbf{r})N_{0\alpha}(\mathbf{r})\frac{\partial f_{0\alpha}(v)}{\partial\mathbf{v}}\right), \quad (10)$$

where $\mathbf{A}^\alpha(\mathbf{r})$ is the amplitude of the oscillatory motion of particle of species α:

$$\mathbf{A}^\alpha(\mathbf{r})=\frac{\mathbf{v}_{E\alpha}}{\omega_0}=\frac{e_\alpha\mathbf{E}_0(\mathbf{r})}{m_\alpha\omega_0^2}.$$

Making use of (10) we can now write the kinetic equations for the electrons and ions in terms of the Fourier components:

$$(1+\delta\varepsilon_e(\omega,\mathbf{k}))n_\omega^e+\frac{e_i}{e}\delta\varepsilon_e(\omega,\mathbf{k})\sum_{l=-\infty}^{\infty}J_l(a_i)n^i_{\omega+l\omega_0}$$
$$+i\psi_e(\omega,\mathbf{k})(n^e_{\omega+\omega_0}-n^e_{\omega-\omega_0})$$
$$+i\frac{e_i}{e}\psi_e(\omega,\mathbf{k})\sum_{l=-\infty}^{\infty}J_l(a_i)(n^i_{\omega+(l+1)\omega_0}-n^i_{\omega+(l-1)\omega_0})=0, \quad (11)$$

$$(1+\delta\varepsilon_i(\omega,\mathbf{k}))n_\omega^i+\frac{e}{e_i}\delta\varepsilon_i(\omega,\mathbf{k})\sum_{l=-\infty}^{\infty}J_l(a_i)n^e_{\omega-l\omega_0}$$
$$+i\psi_i(\omega,\mathbf{k})(n^i_{\omega+\omega_0}-n^i_{\omega-\omega_0})$$
$$+i\frac{e}{e_i}\psi_i(\omega,\mathbf{k})\sum_{l=-\infty}^{\infty}J_l(a_i)(n^e_{\omega-(l-1)\omega_0}-n^e_{\omega-(l+1)\omega_0})=0. \quad (12)$$

In these relations we have made use of the following notation:

$$n_\omega^\alpha=\int d\mathbf{v}\,\Phi_\omega^\alpha, \quad (13)$$

$$\delta\varepsilon_\alpha(\omega,\mathbf{k})=\frac{\omega_{L\alpha}^2}{k^2}\int\frac{d\mathbf{v}}{\omega-\mathbf{kv}}\mathbf{k}\frac{\partial f_{0\alpha}}{\partial\mathbf{v}}, \quad (14)$$

$$\psi_\alpha(\omega,\mathbf{k})=\frac{2\pi e_\alpha^2}{m_\alpha k^2}\int\frac{d\mathbf{v}}{\omega-\mathbf{kv}}\left(\mathbf{k}\frac{\partial}{\partial\mathbf{v}}\right)\left(\mathbf{v}\frac{\partial}{\partial\mathbf{r}}\right)$$
$$\times\left(N_{0\alpha}(\mathbf{r})\mathbf{A}^\alpha(\mathbf{r})\frac{\partial f_{0\alpha}(v)}{\partial\mathbf{v}}\right). \quad (15)$$

In view of the fact that $\psi_i(\omega,\mathbf{k})/\psi_e(\omega,\mathbf{k})$ is much smaller than unity in all cases of interest here, in what follows we neglect the effect of the inhomogeneity on the ion component of the plasma, that is to say, we assume that $\psi_i(\omega,\mathbf{k})=0$. We can then eliminate the quantity n^i from (11) and (12) and obtain an equation for n^e. In the limit $\psi_i(\omega,\mathbf{k})=0$, using (12) we find

$$n_\omega^i=-\frac{e}{e_i}\frac{\delta\varepsilon_i(\omega,\mathbf{k})}{1+\delta\varepsilon_i(\omega,\mathbf{k})}\sum_{l=-\infty}^{\infty}J_l(a_i)n^e_{\omega-l\omega_0}. \quad (16)$$

Substitution of this expression in (11) yields

$$(1+\delta\varepsilon_e(\omega,\mathbf{k}))n_\omega^e-\delta\varepsilon_e(\omega,\mathbf{k})\sum_{l=-\infty}^{\infty}J_l(a_i)\frac{\delta\varepsilon_i(\omega+l\omega_0,\mathbf{k})}{1+\delta\varepsilon_i(\omega+l\omega_0,\mathbf{k})}$$
$$\times\sum_{m=-\infty}^{\infty}J_m(a_i)n^e_{\omega+(l-m)\omega_0}+i\psi_e(\omega,\mathbf{k})(n^e_{\omega+\omega_0}-n^e_{\omega-\omega_0})$$
$$-i\psi_e(\omega,\mathbf{k})\sum_{l=-\infty}^{\infty}J_l(a_i)\left\{\frac{\delta\varepsilon_i(\omega+(l+1)\omega_0,\mathbf{k})}{1+\delta\varepsilon_i(\omega+(l+1)\omega_0,\mathbf{k})}\right.$$
$$\times\sum_{m=-\infty}^{\infty}J_m(a_i)n^e_{\omega+(l-m+1)\omega_0}-\frac{\delta\varepsilon_i(\omega+(i-1)\omega_0,\mathbf{k})}{1+\delta\varepsilon_i(\omega+(l-1)\omega_0,\mathbf{k})}$$
$$\left.\times\sum_{m=-\infty}^{\infty}J_m(a_i)n^e_{\omega+(l-m-1)\omega_0}\right\}=0. \quad (17)$$

We now investigate this equation separately for the high-frequency and low-frequency limits.

In the high-frequency region (RF), we have

$$\omega,\omega_0,|\omega-\omega_0|\gg kv_{Te},$$

in which case the ion motion can be neglected and we can write $\delta\epsilon_i(\omega,\mathbf{k})=0$. Under these conditions (17) assumes the simple form

$$\{1+\delta\varepsilon_e(\omega,\mathbf{k})\}n_\omega^e=-i\psi_e(\omega,\mathbf{k})(n^e_{\omega+\omega_0}-n^e_{\omega-\omega_0}). \quad (18)$$

In the low-frequency region, where

$$kv_{Te}\gg\omega,\omega_0,|\omega-\omega_0|\gg kv_{Ti},$$

the ion motion cannot be neglected. In this case we limit our analysis to the case of waves that propagate across the electric field $\mathbf{E}_0$. For these waves $\mathbf{k}\cdot\mathbf{E}_0=0$ and (17) reduces to the form

$$\frac{1+\delta\varepsilon_e(\omega,\mathbf{k})+\delta\varepsilon_i(\omega,\mathbf{k})}{1+\delta\varepsilon_i(\omega,\mathbf{k})}n_\omega^e$$
$$=-i\psi_e(\omega,\mathbf{k})\left\{\frac{n^e_{\omega+\omega_0}}{1+\delta\varepsilon_i(\omega+\omega_0,\mathbf{k})}-\frac{n^e_{\omega-\omega_0}}{1+\delta\varepsilon_i(\omega-\omega_0,\mathbf{k})}\right\}. \quad (19)$$

Although (18) follows formally from (19) in the limit $\delta\epsilon_i(\omega, \mathbf{k}) = 0$, it should be understood that (19) describes a low-frequency wave that propagates across the electric field $\mathbf{E}_0$ whereas (18) describes a high-frequency wave with arbitrary orientation of the wave vector with respect to $\mathbf{E}_0$.

We first investigate (19). Introducing the notation

$$\varepsilon(\omega,\mathbf{k}) = 1+\delta\varepsilon_e(\omega,\mathbf{k})+\delta\varepsilon_i(\omega,\mathbf{k}),$$
$$\varepsilon_i(\omega,\mathbf{k}) = 1+\delta\varepsilon_i(\omega,\mathbf{k}),$$

we can write (19) in the form

$$\frac{\varepsilon(\omega,\mathbf{k})}{\varepsilon_i(\omega,\mathbf{k})}n_\omega^e = -i\psi_e(\omega,\mathbf{k})\left\{\frac{n^e_{\omega+\omega_0}}{\varepsilon_i(\omega+\omega_0,\mathbf{k})}-\frac{n^e_{\omega-\omega_0}}{\varepsilon_i(\omega-\omega_0,\mathbf{k})}\right\}. \quad (20)$$

From this relation we can obtain a dispersion equation that contains continuous fractions:

$$\frac{\varepsilon(\omega,\mathbf{k})}{\varepsilon_i(\omega,\mathbf{k})} = i\psi_e(\omega,\mathbf{k})$$
$$\times\left\{\left[-\frac{\varepsilon(\omega-\omega_0,\mathbf{k})\varepsilon_i(\omega,\mathbf{k})}{i\psi_e(\omega-\omega_0,\mathbf{k})}+\frac{\varepsilon_i(\omega,\mathbf{k})}{-\dfrac{\varepsilon(\omega-2\omega_0,\mathbf{k})\varepsilon_i(\omega-\omega_0,\mathbf{k})}{i\psi_e(\omega-2\omega_0,\mathbf{k})}+\ldots}\right]^{-1}\right.$$
$$\left.-\left[\frac{\varepsilon(\omega+\omega_0,\mathbf{k})\varepsilon_i(\omega,\mathbf{k})}{i\psi_e(\omega+\omega_0,\mathbf{k})}+\frac{\varepsilon_i(\omega,\mathbf{k})}{\dfrac{\varepsilon(\omega+2\omega_0,\mathbf{k})\varepsilon_i(\omega+\omega_0,\mathbf{k})}{i\psi_e(\omega+2\omega_0,\mathbf{k})}+\ldots}\right]^{-1}\right\}. \quad (21)$$

Since $\psi_e(\omega, \mathbf{k})$ is proportional to a small quantity, the ratio of the electron excursion to the scale of the inhomogeneity, the fraction series can be truncated and to a first approximation the dispersion equation becomes

$$\varepsilon(\omega,\mathbf{k}) = \psi_e(\omega,\mathbf{k})\left\{\frac{\psi_e(\omega-\omega_0,\mathbf{k})}{\varepsilon(\omega-\omega_0,\mathbf{k})}+\frac{\psi_e(\omega+\omega_0,\mathbf{k})}{\varepsilon(\omega+\omega_0,\mathbf{k})}\right\}. \quad (22)$$

It is evident that from (18) we can also obtain a dispersion equation of the form (22) where, however, by $\epsilon(\omega, \mathbf{k})$ we are to understand $1 + \delta\epsilon_e(\omega, \mathbf{k})$. This is clear even without the derivation since, as we have already noted, (18) follows formally from (19) as $\delta\epsilon_i(\omega, \mathbf{k})$ approaches zero. Thus, both in the low-frequency and high-frequency cases it is necessary to investigate (22).

The functions $\epsilon(\omega, \mathbf{k})$ and $\psi_e(\omega, \mathbf{k})$ are complex. However, in those regions of frequency and wavelength in which weakly damped longitudinal waves can propagate in a uniform plasma, the imaginary parts of these functions are small compared with their real parts.

Let $\omega^0(\mathbf{k})$ be a root of the equation $\mathrm{Re}\,\epsilon(\omega, \mathbf{k}) = 0$ which corresponds to weakly damped waves and let $\omega_0 = 2\omega^0 + \eta$ where $\eta \ll \omega^0$. We shall seek a solution of (22) that does not differ greatly from $\omega^0(\mathbf{k})$. In accordance with this assumption we write $\omega = \omega^0 + \Delta$ where $\Delta \ll \omega^0$. Under these conditions, the second term in the curly brackets in (22) can be neglected compared with the first term. Furthermore, since $\psi_e(\omega, \mathbf{k})$ contains a small parameter, the imaginary part of this function (which is much smaller than the real part) can be neglected. Expanding $\epsilon(\omega, \mathbf{k})$ and $\epsilon(\omega - \omega_0, \mathbf{k})$ in powers of η and Δ and keeping the first non-vanishing terms, we obtain the following equation for Δ:

$$\Delta^2-\Delta\left(\eta-2i\frac{\mathrm{Im}\,\varepsilon(\omega^0,\mathbf{k})}{\partial\,\mathrm{Re}\,\varepsilon(\omega^0,\mathbf{k})/\partial\omega^0}\right)+\frac{\psi_e(\omega^0,\mathbf{k})\psi_e(-\omega^0,\mathbf{k})}{(\partial\,\mathrm{Re}\,\varepsilon(\omega^0,\mathbf{k})/\partial\omega^0)^2}$$
$$-\left(\frac{\mathrm{Im}\,\varepsilon(\omega^0,\mathbf{k})}{\partial\,\mathrm{Re}\,\varepsilon(\omega^0,\mathbf{k})/\partial\omega^0}\right)^2-i\eta\frac{\mathrm{Im}\,\varepsilon(\omega^0,\mathbf{k})}{\partial\,\mathrm{Re}\,\varepsilon(\omega^0,\mathbf{k})/\partial\omega^0}=0. \quad (23)$$

It is evident that $\mathrm{Re}\,\psi_e(\omega, \mathbf{k})$ is an even function of frequency. Hence, the solution of (23) will assume the form

$$\Delta=\frac{1}{2}\eta-i\frac{\mathrm{Im}\,\varepsilon(\omega^0,\mathbf{k})}{\partial\,\mathrm{Re}\,\varepsilon(\omega^0,\mathbf{k})/\partial\omega^0}$$
$$\pm i\left[\left(\frac{\psi_e(\omega^0,\mathbf{k})}{\partial\,\mathrm{Re}\,\varepsilon(\omega^0,\mathbf{k})/\partial\omega^0}\right)^2-\frac{1}{4}\eta^2\right]^{1/2}. \quad (24)$$

The second term on the right side of (24) yields the usual linear damping rate (cf. for example[11]). Excitation can only be due to the third term. Using (24) we obtain the following condition for excitation of plasma waves:

$$(\psi_e(\omega^0,\mathbf{k}))^2-(\mathrm{Im}\,\varepsilon(\omega^0,\mathbf{k}))^2>\frac{1}{4}\eta^2\left(\frac{\partial\,\mathrm{Re}\,\varepsilon(\omega^0,\mathbf{k})}{\partial\omega^0}\right)^2. \quad (25)$$

It is evident that the maximum growth rate is obtained for exact parametric resonance, in which case $\eta = 0$.

In order to evaluate the requirements on the inequality in (25) we should estimate the function $\psi_e(\omega, \mathbf{k})$. Let us assume the vector $\mathbf{k}$ is along the z axis and that $\mathbf{E}_0$ lies in the xOz plane. We then find that

$$\psi_e(\omega,\mathbf{k})=\frac{2\pi e^2}{T_ek^2}\left[J_+\left(\frac{\omega}{kv_{Te}}\right)-1\right]\frac{\partial A_x^eN_{0e}}{\partial x}$$
$$-\frac{2\pi e^2}{T_ek^2}\left\{\left(2-\frac{\omega^2}{k^2v_{Te}^2}\right)\left[J_+\left(\frac{\omega}{kv_{Te}}\right)-1\right]+1\right\}\frac{\partial A_z^eN_{0e}}{\partial z}, \quad (26)$$

where

$$J_+(x)=xe^{-x^2/2}\int_{i\infty}^{x}d\tau\, e^{\tau^2/2}. \quad (27)$$

Asymptotic expansions of the function $J_+(x)$ for small and large values of the argument can be found, for example, in[11]. Making use of the asymptotic values of $\epsilon(\omega, \mathbf{k})$ and $\psi_e(\omega, \mathbf{k})$ we can write (25) in explicit form. For the RF case ($\omega \gg kv_{Te}$) this inequality becomes

$$\frac{1}{4}\left(\frac{\mathrm{div}\,\mathbf{A}^eN_{0e}}{N_{0e}}\right)^2-\frac{\pi}{2}\frac{\omega_{Le}^6}{k^6v_{Te}^6}\exp\left\{-\frac{\omega_{Le}^2}{k^2v_{Te}^2}-3\right\}>\frac{\eta^2}{\omega_{Le}^2}. \quad (28)$$

For the low-frequency case ($kv_{Te} \gg \omega \gg kv_{Ti}$) from (25) we have

$$\frac{1}{4}\left(\frac{\mathrm{div}\,\mathbf{A}^eN_{0e}}{N_{0e}}\right)^2-\frac{\pi}{2}\frac{\omega_{Le}^2}{\omega_{Le}^2+k^2v_{Te}^2}\left[\left(\frac{m_ee_i}{m_i|e|}\right)^{1/2}\right.$$
$$\left.+\left(\frac{e_iT_e}{|e|T_i}\right)^{3/2}\exp\left\{-\frac{e_iT_e\omega_{Le}^2}{2|e|T_i(\omega_{Le}^2+k^2v_{Te}^2)}\right\}\right]^2$$
$$>\frac{\eta^2}{\omega_s^2}\left(1+\frac{k^2v_{Tg}^2}{\omega_{Le}^2}\right)^2 \quad (29)$$

It is evident from (28) and (29) that the conditions for parametric excitation of plasma waves can be satisfied quite easily.

In conclusion we determine the maximum growth rates for plasma waves assuming that Landau damping can be neglected for $\eta = 0$. For the RF case we have

$$\Delta_0\equiv\left|\frac{\psi_e(\omega^0,\mathbf{k})}{\partial\,\mathrm{Re}\,\varepsilon(\omega^0,\mathbf{k})/\partial\omega^0}\right|=\frac{\omega_{Le}}{4}\left|\frac{\mathrm{div}\,N_{0e}\mathbf{A}^e}{N_{0e}}\right|$$
$$-\frac{3}{8}\omega_{Le}\frac{k^2v_{Te}^2}{\omega_{Le}^2}\frac{1}{N_{0e}}\left|\frac{\partial N_{0e}A_x^e}{\partial x}\right|-\frac{21}{8}\omega_{Le}\frac{k^2v_{Te}^2}{\omega_{Le}^2}\frac{1}{N_{0e}}\left|\frac{\partial N_{0e}A_z^e}{\partial z}\right|. \quad (30)$$

In the particular case in which all vector quantities are in the same direction in a cold plasma we can recover the results of Krenz and Kino[9] from (30).

For the low-frequency case, the maximum growth rate (neglecting Landau damping) is

$$\Delta_0=\frac{\omega_s}{4}\frac{\omega_{Le}^2}{\omega_{Le}^2+k^2v_{Te}^2}\left|\frac{\mathrm{div}\,N_{0e}\mathbf{A}^e}{N_{0e}}\right|. \quad (31)$$

We emphasize once again that the excited low-frequency waves propagate across the perturbation electric field.

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APPENDIX

We consider the validity of the approximation for which (2) is obtained; we write the latter in the form

$$f_\alpha = N_{0\alpha} f_{0\alpha} + f_{1\alpha}.$$

For the case of standing waves $\mathbf{E}_0(\mathbf{r}, t) = \mathbf{E}_0(\mathbf{r}) \sin \omega_0 t$ we can write

$$f_{1\alpha} = \frac{1}{(2\pi)^3} \frac{e_\alpha}{T_\alpha} \operatorname{Re} \int d\mathbf{k} (\mathbf{E}_0 N_{0\alpha})_\mathbf{k} e^{i\mathbf{kr} - i\omega_0 t} \frac{\mathbf{v} f_{0\alpha}}{\omega_0 - \mathbf{kv} + i0},$$

where

$$(\mathbf{E}_0 N_{0\alpha})_\mathbf{k} = \int d\mathbf{r} \mathbf{E}_0(\mathbf{r}) N_{0\alpha}(\mathbf{r}) e^{-i\mathbf{kr}}.$$

Under these conditions we obtain the following expression for the density perturbation:

$$n_{1\alpha} = \frac{1}{(2\pi)^3} \frac{e_\alpha}{T_\alpha} \operatorname{Re} \int d\mathbf{k}\, e^{i\mathbf{kr} - i\omega_0 t} (\mathbf{E}_0 N_{0\alpha})_\mathbf{k} \frac{\mathbf{k}}{k^2} \left[J_+\left(\frac{\omega}{k v_{Te}} \right) - 1 \right].$$

Using the asymptotic expansions for the function $J_+(x)$ we find that $n_{1\alpha}$ is of order

$$n_{1\alpha} \sim \operatorname{div}(N_{0\alpha}(\mathbf{r}) \mathbf{A}^e(\mathbf{r})).$$

Thus, the requirement that the density perturbation be small is equivalent to the requirement that the ratio of the amplitude of the particle excursion in the field be small compared with the scale size of the field inhomogeneity and the density inhomogeneity (or that the growth rate be small compared with the frequency).

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