AMPLITUDE OF A NONRELATIVISTIC SQUARE GRAPH

L. D. BLOKHINTSEV and E. TRUGLIK

Institute of Nuclear Physics, Moscow State University

Submitted July 6, 1967

Zh. Eksp. Teor. Fiz. 53, 2176–2185 (December, 1967)

The explicit form of the amplitude for all types of nonrelativistic square graphs with constant vertex functions is obtained. The analytic properties of the amplitudes are derived and the branches of real proper singularities on the physical sheet are singled out.

1. In the graph theory of direct nuclear reactions at low and intermediate energies[1-3] the direct reaction amplitude is written in the form of a nonrelativistic Feynman graph (or a sum of several Feynman graphs) with the corresponding vertex functions. The criterion for the choice of this or that graph is (together with the magnitude of the vertex functions) the distance of the singularities of the amplitude for a given graph as a function of the external kinematic invariants, from the physical region of the reaction under consideration. The graphs whose singularities lie close to the physical region are rapidly varying functions and their contribution must be taken into account explicitly. The graphs with far singularities depend weakly on their variables. One may therefore hope to account for these graphs by replacing their sum by a constant; in particular cases this constant can be small compared to the amplitudes of the graphs taken along explicitly.

It follows from these considerations that it is necessary to investigate the singularities of the nonrelativistic graphs. The properties (among these, also the analytic properties) of the nonrelativistic graphs have been considered earlier.[4-6] In[4,6] the explicit form of the amplitude for a nonrelativistic triangular graph with constant vertices was obtained, and the singularities of this amplitude on the physical sheet were determined. In the present paper we study the properties of the next, more complicated graph, viz., the nonrelativistic square graph with arbitrary masses of the virtual particles. In Sec. 2 we consider the general form of the amplitude of this graph. In Secs. 3 to 5 the explicit form is obtained, and the analytic properties of three different types of square graphs are investigated.[1)]

2. The amplitude (matrix element) $\mathcal{M}_{nl}$ for an arbitrary nonrelativistic Feynman graph with n internal lines and l independent closed loops has the form[2)]

$$\mathcal{M}_{nl}=\left[\frac{i}{(2\pi)^4}\right]^l\int\prod_{i=1}^{l}d\mathbf{q}_i\,d\varepsilon_i\frac{\sum\limits_{(\mu)}\prod\limits_{k=1}^{v}\mathcal{M}_k}{\prod\limits_{j=1}^{n}(\mathscr{E}_j-\mathbf{k}_j^2/2m_j+i\delta)},\quad\delta\to+0. \tag{1}$$

Here $\mathbf{k}_j$, $\mathscr{E}_j$, and m_j are the momentum, the kinetic energy, and the mass corresponding to the j-th internal line; $\mathbf{q}_i$ and ϵ_i are variables of integration, for which we may choose the l independent four-vectors ($\mathbf{k}_j$, $\mathscr{E}_j$); $v = n - l + 1$ is the number of vertices in the graph; $\mathcal{M}_k$ is the vertex function at the k th vertex which is equal to the matrix element for the process described by this vertex; and $\sum\limits_{(\mu)}$ denotes the summation over the projections of the spins of the virtual particles. The quantities $\mathcal{M}_{nl}$ (and $\mathcal{M}_k$) are connected with the S matrix in the following way:

$$S_{if}=\delta_{if}-i(2\pi)^4\mathcal{M}_{nl}\delta^{(4)}(P_i-P_f), \tag{2}$$

where P_i and P_f are the four-momenta of the system before and after the interaction, respectively.

The vertex functions $\mathcal{M}_k$ depend on the four-momenta of the virtual and external particles of the graph. In the following we shall be interested in the analytic properties of the amplitude $\mathcal{M}_{nl}$ connected with the structure of the graph itself and not with the vertex functions. We therefore regard the quantities $\mathcal{M}_k$ as constants. Using formula (19) of[4], we obtain in this case for the amplitude of a square graph ($n = 4$, $l = 1$)

$$\mathcal{M}_{41}=-\frac{1}{2^{5/2}\pi}\left(\prod_{i=1}^{4}m_i\right)\sum_{(\mu)}\left(\prod_{k=1}^{4}\mathcal{M}_k\right)\int\limits_0^1\left(\prod_{i=1}^{4}d\alpha_i\right)\delta\left(\sum_{k=1}^{4}\alpha_k-1\right)$$
$$\times\,\delta\left(\sum_{i=1}^{4}\omega_i m_i\alpha_i\right)\left(\sum_{1\leqslant i<k\leqslant 4}\alpha_i\alpha_k\omega_i\omega_k m_i m_k\eta_{ik}-i\delta\right)^{-3/2}. \tag{3}$$

Here $\omega_i = +1\,(-1)$ if the i-th internal line is directed in (against) the clock sense.[3)] The Galilei-invariant quantities $\eta_{ik} = \eta_{ki}$ were determined in[4] for an arbitrary non-relativistic single-loop graph. In the most important case (for applications) of a square graph, where there is one external line at each vertex, corresponding to a particle on the mass shell, the quantities η_{ik} are expressed in the following form:

$$\eta_{ii+1}=-\frac{M_{ii+1}}{m_i m_{i+1}}Q_{ii+1},\quad i=1,2,3,4;\,^{4)} \tag{4}$$

$$\eta_{ii+2}=\frac{M_{ii+2}}{\omega_i\omega_{i+2}m_i m_{i+2}}\left(\frac{\mathbf{p}_{ii+2}^2}{2M_{ii+2}}-E_{ii+2}-Q_{ii+2}\right),\quad i=1,2. \tag{5}$$

Here M_{ii+1}, $\mathbf{p}_{ii+1}$, and E_{ii+1} are the mass, the momentum, and the kinetic energy of the external particle at the vertex (i, i + 1); $Q_{ii+1} = m_a + m_b - m_c$, where m_a,

[1)]Some properties of the relativistic square graph have been considered in [7,8]; the explicit form of its amplitude cannot be obtained.

[2)]Everywhere in the following $\hbar = c = 1$.

[3)]As is known, the virtual lines in a nonrelativistic graph can be given a definite direction (cf., for example,[4]).

[4)]The quantity η_{41} corresponding to the vertex (4.1) is obtained from this definition by setting i = 4 in (4) and replacing the index 5 by 1.

m_b, and m_c are the masses of the particles (internal and external) which enter in the vertex (i, i + 1) describing the decay (or synthesis) $c \rightleftarrows a + b$;

$$A_{ii+2} = \sigma_{ii+1} A_{ii+1} + \sigma_{i+1i+2} A_{i+1i+2} \qquad (A = M, \mathbf{p}, E, Q),$$

and $\sigma_{ii+1} = +1$ (−1) if the external particle enters (leaves) the vertex (i, i + 1).

Making in (3) the transformation of variables

$$\alpha_k = x_k \left(m_k \sum_{l=1}^{4} \frac{x_l}{m_l} \right)^{-1}, \qquad k = 1, 2, 3, 4,$$

we obtain

$$\mathcal{M}_{41} = -\frac{1}{2^{5/2}\pi} \sum_{(\mu)} \prod_{k=1}^{4} \mathcal{M}_k F, \tag{6a}$$

$$F = \int_0^1 \left(\prod_{i=1}^{4} dx_i \right) \delta \left(\sum_{i=1}^{4} x_i - 1 \right) \delta \left(\sum_{k=1}^{4} \omega_k x_k \right) \left(\sum_{1 \leqslant i < k \leqslant 4} \omega_i \omega_k x_i x_k \eta_{ik} - i\delta \right)^{-3/2}. \tag{6b}$$

For fixed η_{ii+1} the quantity F is a function of the two (complex) variables η_{13} and η_{24}. In the following we shall investigate the behavior of $F(\eta_{13}, \eta_{24})$ for real values of η_{13} and η_{24}. We shall assume that all $\eta_{ii+1} < 0$; this condition means that in each vertex of the graph any of three particles represented by lines converging in this vertex is stable against decay into the two other particles [cf. (4)]. The quantity F with $\eta_{ii+1} \geq 0$ can be obtained by analytic continuation of the expressions given below in the corresponding variable η_{ii+1}.

There exist three topologically inequivalent types of nonrelativistic square graphs, which differ from one another by the relative directions of the internal lines[5)] (Figs. 1 a, b, and c). The external lines in the vertices (3, 4) and (4, 1) in Fig. 1 a and in the vertices (2, 3) and (4, 1) in Fig. 1 b may correspond to ingoing as well as to outgoing particles. Let us consider all three types of square graphs one by one.

3. For the graph of Fig. 1 a, formula (6b) has the form[6)]

$$F \equiv F_1 = \int_0^1 \left(\prod_{i=1}^{4} dx_i \right) \delta \left(\sum_{i=1}^{4} x_i - 1 \right) \delta(-x_1 + x_2 - x_3 - x_4)$$
$$\times (-x_1 x_2 \eta_{12} + x_1 x_3 \eta_{13} + x_1 x_4 \eta_{14} - x_2 x_3 \eta_{23} - x_2 x_4 \eta_{24} + x_3 x_4 \eta_{34} - i\delta)^{-3/2}. \tag{7}$$

Let us consider the region of values of η_{13} and η_{24} for which the last factor in (7) is different from zero over the entire region of integration. In this region $F_1(\eta_{13}, \eta_{24})$ is analytic and can be obtained from (7) in explicit form (cf., for example,[9]):[7)]

$$F_1 = \frac{1}{2C_1(p_1 - q_1)} \left\{ (s_1 + p_1 r_1) \int_0^{1/2} \frac{dx}{(x - p_1)[R_1(x)]^{1/2}} \right.$$
$$- (s_1 + q_1 r_1) \int_0^{1/2} \frac{dx}{(x - q_1)[R_1(x)]^{1/2}} - (t_1 + p_1 v_1) \int_0^{1/2} \frac{dx}{(x - p_1)[Q_1(x)]^{1/2}}$$
$$\left. + (t_1 + q_1 v_1) \int_0^{1/2} \frac{dx}{(x - q_1)[Q_1(x)]^{1/2}} \right\} \tag{8}$$

or

$$F_1 = \frac{1}{2C_1(p_1 - q_1)} \left\{ - \frac{s_1 + p_1 r_1}{[R_1(p_1)]^{1/2}} \ln \frac{f_1(-p_1) - 2[R_1(p_1)]^{1/2}}{f_1(-p_1) + 2[R_1(p_1)]^{1/2}} \right.$$
$$+ \frac{s_1 + q_1 r_1}{[R_1(q_1)]^{1/2}} \ln \frac{f_1(-q_1) - 2[R_1(q_1)]^{1/2}}{f_1(-q_1) + 2[R_1(q_1)]^{1/2}} + \frac{t_1 + p_1 v_1}{[Q_1(p_1)]^{1/2}} \ln \frac{\varphi_1(-p_1) - 2[Q_1(p_1)]^{1/2}}{\varphi_1(-p_1) + 2[Q_1(p_1)]^{1/2}}$$
$$\left. - \frac{t_1 + q_1 v_1}{[Q_1(q_1)]^{1/2}} \ln \frac{\varphi_1(-q_1) - 2[Q_1(q_1)]^{1/2}}{\varphi_1(-q_1) + 2[Q_1(q_1)]^{1/2}} \right. . \tag{9}$$

In order to continue F_1 from the region of analyticity we must know the singularities of F_1 and the rules for by-passing them. According to the general theory of the singularities of Feynman amplitudes,[10–12,4] each graph has, besides its proper singularities, the singularities of all "reduced" graphs, obtained from the original graph by removing part of the internal lines and contracting the vertices joined by these lines. For the graph of Fig. 1 a, the reduced graphs having singularities in η_{13} and η_{24} are the three triangular graphs with the internal lines (1, 2, 3), (1, 2, 4) and (2, 3, 4) and the loop with the lines (2, 4) [the triangular graph (1, 2, 3) and the loop (1, 3) are identically zero (cf., for example,[4])]. The loop (2, 4) has a root-type singularity (normal threshold) for $\eta_{24} = 0$, and the triangular graphs have the following anomalous logarithmic singularities:[4]

$$\eta_{13} = -[(-\eta_{12})^{1/2} + (-\eta_{23})^{1/2}] \equiv L_4^+, \tag{10}$$

$$\eta_{24} = -[(-\eta_{12})^{1/2} - (-\eta_{14})^{1/2}] \equiv L_3^- \qquad (\eta_{14} < \eta_{12}), \tag{11}$$

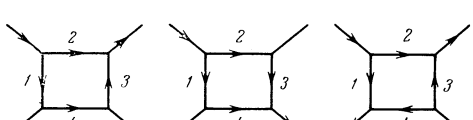
$$\eta_{24} = -[(-\eta_{23})^{1/2} - (-\eta_{34})^{1/2}]^2 \equiv L_1^- \qquad (\eta_{34} < \eta_{23}). \tag{12}$$

The inequalities $\eta_{14} < \eta_{12}$ and $\eta_{34} < \eta_{23}$ are necessary (and sufficient) conditions for the existence of the singularities $\eta_{24} = L_3^-$ and $\eta_{24} = L_1^-$, respectively. Thus F_1 has from one to three triangular singularities. In the neighborhood of the singularity $\eta_{ij} = L_k^\pm$ the quantity $F_1 \sim \ln(\eta_{ij} - L_k^\pm)$. This also holds for the amplitudes F_2 and F_3 considered below.)

Let us now consider the proper singularities of F_1. For these singularities to occur, it is necessary (but in general not sufficient) that the poles p_1 and q_1 in (8) coincide. The condition $p_1 = q_1$ leads to the equation

$$D(\eta_{13}, \eta_{24}) \equiv \eta_{13}^2 \eta_{24} + \eta_{24}^2 \eta_{13} - \eta_{13}\eta_{24}(\eta_{12} + \eta_{23} + \eta_{34} + \eta_{14})$$
$$+ \eta_{13}(\eta_{12} - \eta_{14})(\eta_{23} - \eta_{34}) + \eta_{24}(\eta_{12} - \eta_{23})(\eta_{14} - \eta_{34})$$
$$+ (\eta_{12} + \eta_{34} - \eta_{14} - \eta_{23})(\eta_{12}\eta_{34} - \eta_{14}\eta_{23}) = 0. \tag{13}$$

Equation (13) may also be obtained from the equation for the proper singularity of an arbitrary single-loop graph [formula (23) of[4]] for n = 4. The curve $D(\eta_{13}, \eta_{24}) = 0$, which consists in general of several branches, will be called the curve of the proper singularities and will be



a b c

FIG. 1. The different types of square graphs.

[5)]A nonrelativistic square graph whose internal lines are all directed in (or against) the sense of the clock vanishes identically (cf., for example,[4]).

[6)]The indices 1, 2, 3 of the function F correspond to the graphs of Figs. 1a, b, and c, respectively.

[7)]The notation used below is explained in the Appendix.

Table I

	$L^- < L_i^-$ $\eta_{14} < \eta_{12},\ \eta_{34} < \eta_{23}$	$\eta_{14} < \eta_{12},\ \eta_{34} > \eta_{23}$	$\eta_{14} > \eta_{12},\ \eta_{34} < \eta_{23}$
$L_2^+ < L_4^+ < L_2^-$	$\eta_{24}^+ [T_i^- < \eta_{13}, L_i^- < \eta_{24} < 0]$ $\eta_{24}^- [L_4^+ < \eta_{13} < T_i^-, T_4^+ < \eta_{24} < L_i^-]$	$\eta_{24}^- [L_4^+ < \eta_{13} < T_3^-, T_4^+ < \eta_{24} < L_3^-]$	$\eta_{24}^- [L_4^+ < \eta_{13} < T_1^-, T_4^+ < \eta_{24} < L_1^-]$
$L_2^- < L_4^+$	$\eta_{24}^+ [T_i^- < \eta_{13}, L_i^- < \eta_{24} < 0]$ $\eta_{24}^+ [T_k^- < \eta_{13} < L_4^+, L_i^- < \eta_{24} < T_4^+]$	$\eta_{24}^+ [T_3^- < \eta_{13} < L_4^+, L_3^- < \eta_{24} < T_4^+]$	$\eta_{24}^+ [T_1^- < \eta_{13} < L_4^+, L_1^- < \eta_{24} < T_4^+]$

Table II

	$\eta_{14} > \eta_{12}, \eta_{34} > \eta_{23}$	$\eta_{14} < \eta_{12},\ \eta_{34} > \eta_{23}$	$\eta_{14} > \eta_{12}, \eta_{34} < \eta_{23}$
$L_4^+ < L_2^+$	$\eta_{24}^+ [\eta_{13} < L_4^+, 0 < \eta_{24} < T_4^+]$	$\eta_{24}^+ [T_3^- < \eta_{13} < L_4^+, L_3^- < \eta_{24} < T_4^+]$	$\eta_{24}^+ [T_1^- < \eta_{13} < L_4^+, L_1^- < \eta_{24} < T_4^+]$

denoted by the symbol Γ. Solving (13), for example, for η_{24}, we obtain

$$\eta_{24} = \eta_{24}^{\pm} \equiv \frac{1}{2\eta_{13}} \{-\eta_{13}^2 + \eta_{13}(\eta_{12} + \eta_{23} + \eta_{34} + \eta_{14}) - (\eta_{12} - \eta_{14})(\eta_{23} - \eta_{34}) \pm [\Delta(123)\Delta(134)]^{1/2}\}. \quad (14)$$

From this and from the analogous solution for η_{13} it follows that the only asymptotes of the curve Γ parallel to the η_{13} and η_{24} axes are these axes themselves ($\eta_{24} = 0$, $\eta_{13} = 0$) and that the only tangents to the curve Γ parallel to the η_{13} and η_{24} axes are the straight lines

$$\eta_{24} = L_1^{\pm}, \quad \eta_{24} = L_3^{\pm}, \quad \eta_{13} = L_2^{\pm}, \quad \eta_{13} = L_4^{\pm}, \quad (15)$$

these are, respectively, solutions to the equations

$$\Delta(234) = 0, \quad \Delta(124) = 0, \quad \Delta(134) = 0, \quad \Delta(123) = 0. \quad (16)$$

The curve Γ is tangent to the straight lines (15) in the following points:[8)]

$$\eta_{13}^{\pm}(L_i^{\pm}) \equiv T_i^{\pm} \quad (i = 1, 3), \qquad \eta_{24}^{\pm}(L_k^{\pm}) \equiv T_k^{\pm} \quad (k = 2, 4). \quad (17)$$

The main problem is to single out the branches of the curve Γ which correspond to the singularities of the amplitude F_1 on the physical sheet. Near these branches it is necessary that in each of the two integrations which must actually be carried out in (7), the two singularities of the integrand pinch the contour of integration as they approach each other. The curves of the real proper singularities of F_1 on the physical sheet singled out by this criterium are given in Tables I and II. The region of variation of η_{13} and η_{24} for a given singular branch is indicated in square brackets.

The location of the curves of the singularities depends on the relations between the quantities $\eta_{i\,i+1}$. As an example, we show in Fig. 2 the location of the branches of the curve of proper singularities Γ (and of the triangular singularities) for the case where $L_2^+ < L_4^+ < L_2^-$, $L_1^- < L_3^-$, $\eta_{14} < \eta_{12}$, $\eta_{34} < \eta_{23}$. The heavy lines represent the parts which are singular on the physical sheet.

Near the singular part $\eta_{24} = \eta_{24}^{\pm}(\eta_{13})$ the amplitude F_1 (as well as the amplitudes F_2 and F_3 considered below) behaves like $F_1 \sim [\eta_{24} - \eta_{24}^{\pm}]^{-1/2}$. In bypassing the singularities of F_1 the physical rim is selected by adding small imaginary parts to η_{13} and η_{24} (cf., for example,[6]):

$$\eta_{13} \to \eta_{13} - i\delta, \quad \eta_{24} \to \eta_{24} + i\delta, \quad \delta \to +0. \quad (18)$$

The simplest example for the graph of Fig. 1a is the graph describing elastic Nd scattering (Fig. 1a, where the lines 1, 2, and 3 correspond to nucleons, and line 4 to the deuteron).

4. For the graph of Fig. 1b, formula (6b) takes the form

$$F \equiv F_2 = \int_0^1 \Big(\prod_{i=1}^4 dx_i\Big) \delta\Big(\sum_{i=1}^4 x_i - 1\Big) \delta(-x_1 + x_2 + x_3 - x_4) \times (-x_1x_2\eta_{12} - x_1x_3\eta_{13} + x_1x_4\eta_{14} + x_2x_3\eta_{23} - x_2x_4\eta_{24} - x_3x_4\eta_{34} - i\delta)^{-3/2}. \quad (19)$$

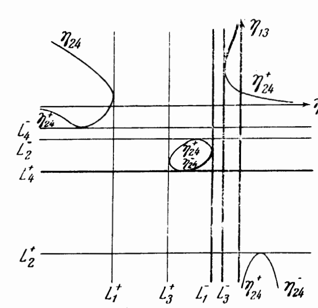


FIG. 2. Example for the location of the proper singularities of the graph of Fig. 1a.

The quantity F_2 is given by formulas which are obtained from (8) and (9) by replacing the index 1 by 2. F_2 has the normal thresholds $\eta_{13} = 0$, and $\eta_{24} = 0$ and the following triangular singularities:

$$\begin{aligned} \eta_{13} &= L_2^- \quad (\text{for } \eta_{14} < \eta_{34}), \\ \eta_{13} &= L_4^- \quad (\text{for } \eta_{23} < \eta_{12}), \\ \eta_{24} &= L_1^- \quad (\text{for } \eta_{23} < \eta_{34}), \\ \eta_{24} &= L_3^- \quad (\text{for } \eta_{14} < \eta_{12}). \end{aligned} \quad (20)$$

Depending on the relation between the quantities $\eta_{i\,i+1}$, the function F_2 therefore has from zero to four triangular singularities.

The physical branches of the curve of proper singularities for the amplitude F_2 are singled out by the same method as in the case of the amplitude F_1. The results are given in Tables III to V. Besides the cases given in

8)Formulas (13) to (17) (and the text referring to them) are common to all types of square graphs.

Table III

		$\eta_{14} < \eta_{34}$	$\eta_{14} > \eta_{34}$
$L_1^- < L_3^-, \quad L_4^- < L_2^-$	$\eta_{14} < \eta_{12}$	$\eta_{24}^+ [L_2^- < \eta_{13} < T_3^-, \ L_3^- < \eta_{24} < T_2^-]$	$\eta_{24}^+ [0 < \eta_{13} < T_3^-, \ L_3^- < \eta_{24}]$
	$\eta_{14} > \eta_{12}$	$\eta_{24}^+ [L_2^- < \eta_{13}, \ 0 < \eta_{24} < T_2^-]$	$\eta_{24}^+ [0 < \eta_{13}, \ 0 < \eta_{24}]$
$L_1^- > L_3^-, \quad L_4^- < L_2^-$	$\eta_{23} < \eta_{34}$	$\eta_{24}^+ [L_2^- < \eta_{13} < T_1^-, \ L_1^- < \eta_{24} < T_2^-]$	$\eta_{24}^+ [0 < \eta_{13} < T_1^-, \ L_1^- < \eta_{24}]$
	$\eta_{23} > \eta_{34}$	$\eta_{24}^+ [L_2^- < \eta_{13}, \ 0 < \eta_{24} < T_2^-]$	$\eta_{24}^+ [0 < \eta_{13}, \ 0 < \eta_{24}]$

Table IV

		$\eta_{23} < \eta_{12}$	$\eta_{23} > \eta_{12}$
$L_3^- < L_1^+$, $L_2^- < L_4^+$, $\eta_{14} = (\eta_{ii+1})_{min}$	$\eta_{23} < \eta_{34}$	$\eta_{24}^+ [L_2^- < \eta_{13} < T_3^-, L_3^- < \eta_{24} < T_2^-]$ $\eta_{24}^+ [\eta_{13} < T_1^-, L_1^- < \eta_{14} < 0]$ $\eta_{24}^+ [L_4^- < \eta_{13} < 0, \ \eta_{24} < T_4^-]$	$\eta_{24}^+ [L_2^- < \eta_{13} < T_3^-, L_3^- < \eta_{24} < T_2^-]$ $\eta_{24}^+ [\eta_{13} < T_1^-, L_1^- < \eta_{24} < 0]$
	$\eta_{23} > \eta_{34}$	$\eta_{24}^+ [L_2^- < \eta_{13} < T_3^-, L_3^- < \eta_{24} < T_2^-]$ $\eta_{24}^+ [L_4^- < \eta_{13} < 0, \ \eta_{24} < T_4^-]$	$\eta_{24}^+ [L_2^- < \eta_{13} < T_3^-, L_3^- < \eta_{24} < T_2^-]$

Table V

		$\eta_{14} < \eta_{34}$	$\eta_{14} > \eta_{34}$
$L_1^+ < L_3^+ < L_3^- < L_1^-$, $L_2^+ < L_4^+ < L_4^- < L_2^-$, $\eta_{12} = (\eta_{ii+1})_{max}$	$\eta_{23} < \eta_{34}$	$\eta_{24}^- [T_3^- < \eta_{13} < L_4^-, \ \ T_4^- < \eta_{24} < L_3^-]$ $\eta_{24}^+ [L_2^- < \eta_{13} < 0, \ \eta_{24} < T_2^-]$ $\eta_{24}^+ [\eta_{13} < T_3^-, \ L_3^- < \eta_{24} < 0]$	$\eta_{24}^- [T_3^- < \eta_{13} < L_4^-, \ T_4^- < \eta_{24} < L_3^-]$ $\eta_{24}^+ [\eta_{13} < T_3^-, \ L_3^- < \eta_{24} < 0]$
	$\eta_{23} > \eta_{34}$	$\eta_{24}^- [T_3^- < \eta_{13} < L_4^-, \ T_4^- < \eta_{24} < L_3^-]$ $\eta_{24}^+ [L_2^- < \eta_{13} < 0, \ \eta_{24} < T_2^-]$	$\eta_{24}^- [T_3^- < \eta_{13} < L_4^-, \ T_4^- < \eta_{24} < L_3^-]$

these tables, two more cases are possible, the tables for which are obtained from Tables III to V by making the following replacements:

1) $\eta_{14}(\eta_{12}) \rightleftarrows \eta_{23}(\eta_{34})$, $L_1^-(L_2^-) \rightleftarrows L_3^-(L_4^-)$, $T_1^-(T_2^-) \rightleftarrows T_3^-(T_4^-)$ in Table III; (21)

2) the same replacement (21) in Table IV;

3) $L_1^-(T_1^-) \rightleftarrows L_3^-(T_3^-)$ and $\eta_{14} \lessgtr \eta_{12} \rightleftarrows \eta_{23} \lessgtr \eta_{34}$ in Table V;

4) $L_2^-(T_2^-) \rightleftarrows L_4^-(T_4^-)$ and $\eta_{23} \lessgtr \eta_{12} \rightleftarrows \eta_{14} \lessgtr \eta_{34}$ in Table V;

5) the same replacement (21) in Table V.

Tables III to V and the tables obtained from these with the help of the replacements 1) to 5) contain all real proper singularities of the graph of Fig. 1b on the physical sheet.

The rule for bypassing the singularities of F_2 is

$$\eta_{13} \to \eta_{13} + i\delta, \quad \eta_{24} \to \eta_{24} + i\delta, \quad \delta \to +0. \tag{22}$$

An important example for the graph of Fig. 1b is the graph for the reactions (t, p) and (He^3, n). In[13] the usual wave function formalism was used to calculate the differential cross section for the reaction B^{10}(t, p)B^{12}, assuming a mechanism corresponding to the graph of Fig. 1b, where the internal lines 1 and 3 correspond to neutrons, the line 2 to a deuteron, and the line 4 to the nucleus B^{11}. The fair agreement with experiment obtained in this work can apparently be explained by the circumstance that in this graph the proper singularity in the variable η_{13} (connected with the invariant momentum transfer) lies close to the physical region (practically at the same distance as the less singular logarithmic singularity corresponding to the reduced triangular graph with the internal lines 1, 3, and 4). It is interesting to investigate the "square" mechanism for the reactions C^{12}(t, p)C^{14} and O^{16}(t, p)O^{18}. For these reactions the corresponding square graphs have no triangular singularities at all and therefore the proper square singularities can play an essential role.

5. For the graph of Fig. 1c, we have

$$F \equiv F_3 = \int_0^1 \left(\prod_{i=1}^4 dx_i \right) \delta\left(\sum_{i=1}^4 x_i - 1 \right) \delta(-x_1 + x_2 - x_3 + x_4) \cdot$$
$$(-x_1x_2\eta_{12} + x_1x_3\eta_{13} - x_1x_4\eta_{14} - x_2x_3\eta_{23} + x_2x_4\eta_{24} - x_3x_4\eta_{34} - i\delta)^{-3/2}. \tag{23}$$

Formulas for F_3 are obtained from (8) and (9) by replacing the index 1 by 3. F_3 has no normal thresholds in the variables η_{13} and η_{24}; for arbitrary relations between the η_{ii+1}, F_3 has four triangular singularities: $\eta_{13} = L_{2,4}^+$, $\eta_{24} = L_{1,3}^+$.

The amplitude F_3 always has the following proper singularities on the physical sheet:

$$\eta_{24}^-[T_i^+ < \eta_{13}, \eta_{24} < L_i^+], \quad L_i^+ = \min(L_1^+, L_3^+) \tag{24}$$

and

$$\eta_{24}^-[\eta_{13} < L_k^+, T_k^+ < \eta_{24}], \quad L_k^+ = \min(L_2^+, L_4^+). \tag{25}$$

Moreover, for $L_i^+ < L_k^+ < \min(L_i^-, L_k^-)$ the branch

$$\eta_{24}^+[L_k^+ < \eta_{13} < T_j^+, L_j^+ < \eta_{24} < T_k^+], \tag{26}$$

is singular, and for $L_i^+ < L_i^- < L_k^+$ the branch

$$\eta_{24}^-[T_j^+ < \eta_{13} < L_k^+, T_k^+ < \eta_{24} < L_j^+]. \tag{27}$$

is singular. In (26) and (27), i, k = 2, 4 (i ≠ k), and

$L_j^+ = \max(L_1^+, L_3^+)$. The singularities of F_3 are bypassed in the following way:

$$\eta_{13} \to \eta_{13} - i\delta, \quad \eta_{24} \to \eta_{24} - i\delta, \quad \delta \to +0. \tag{28}$$

The simplest example for the graph of Fig. 1c is the graph for elastic dd scattering, where all internal lines are nucleons and all external lines are deuterons. For this graph $\eta_{12} = \eta_{23} = \eta_{34} = \eta_{14} = \eta$ and F_3 has the form

$$F_3 = 8[\eta_{13}\eta_{24}(\eta_{13} + \eta_{24} - 4\eta)]^{-1/2} \operatorname{arctg}\left\{\left[\frac{-\eta_{13}\eta_{24}}{4\eta(\eta_{13} + \eta_{24} - 4\eta)}\right]^{1/2}\right\}. \tag{29}$$

The curve of proper singularities for this graph degenerates into the straight line $\eta_{13} + \eta_{24} = 4\eta$. Moreover, the straight lines $\eta_{13} = 4\eta$ and $\eta_{24} = 4\eta$ are also singular (logarithmic singularities). We note that the vertex functions of this graph describing the break-up (or synthesis) of the deuteron are constants if we use the zero range approximation for the wave function of the deuteron. Therefore the explicit form of the amplitude for this graph is given by (6a) in good approximation, where F is determined by (29), and the vertex $\mathcal{M}_{d \rightleftarrows n+p}$ for the break-up or synthesis of the deuteron has the form[14]

$$\mathcal{M}_{d \rightleftarrows n+p} = -\frac{(8\pi\varkappa)^{1/2}}{m} C^{1\mu_d}_{1/2\mu_n 1/2\mu_p}, \tag{30}$$

where m is the nucleon mass, $\kappa = (m\epsilon)^{1/2}$ (ϵ is the binding energy of the deuteron), $C^{c\gamma}_{a\alpha b\beta}$ is a Clebsch-Gordan coefficient, and μ_d, μ_n, μ_p are the projections of the spins of the corresponding particles.

The authors thank É. I. Dolinskiĭ for discussions.

APPENDIX

The notation used in this work is the following:

$$d_{ijk} = \eta_{ij} + \eta_{ik} - \eta_{jk}; \tag{A.1}$$

$$\Delta(ijk) = \eta_{ij}^2 + \eta_{ik}^2 + \eta_{jk}^2 - 2\eta_{ij}\eta_{ik} - 2\eta_{ij}\eta_{jk} - 2\eta_{ik}\eta_{jk}. \tag{A.2}$$

The condition $\Delta(ijk) = 0$ is the equation for the singularities of the triangular graph obtained from the original square graph by omitting the internal lines l ($l \neq$ i, j, k) and contracting the vertices joined by these lines. Furthermore,

$$p_i = \frac{1}{2C_i}[-B_i + (B_i^2 - 4AC_i)^{1/2}], \quad q_i = \frac{1}{2C_i}[-B_i - (B_i^2 - 4AC_i)^{1/2}] \tag{A.3}$$

are the roots of the equation $C_i x^2 + B_i x + A = 0$;

$$A = -{}^1\!/_4\Delta(234), \quad B_1 = d_{413}d_{423} - 2\eta_{34}d_{412},$$
$$C_1 = -\Delta(134), \quad B_2 = d_{324}(\eta_{12} - \eta_{13} - \eta_{24} + \eta_{34}) - 2\eta_{23}d_{413}, \tag{A.4}$$

$$C_2 = -(-\eta_{12} + \eta_{13} + \eta_{24} - \eta_{34})^2 + 4\eta_{14}\eta_{23};$$
$$s_1 = -t_2 = -d_{324}, \quad r_1 = 2d_{314}, \quad t_1 = d_{423}, \quad v_1 = -2d_{413}, \tag{A.5}$$

$$s_2 = -d_{234}, \quad v_2 = r_2 = 2(-\eta_{12} + \eta_{13} + \eta_{24} - \eta_{34});$$
$$R_1(x) = R_3(x) = -\eta_{13}x^2 + \frac{x}{2}d_{312} - \frac{\eta_{23}}{4},$$
$$Q_1(x) = R_2(x) = -\eta_{14}x^2 + \frac{x}{2}d_{412} - \frac{\eta_{24}}{4}, \tag{A.6}$$

$$Q_2(x) = -\eta_{14}x^2 + \frac{x}{2}d_{413} - \frac{\eta_{34}}{4};$$

$$f_1(x) = f_3(x) = 2x[(-\eta_{12})^{1/2} + (-\eta_{23})^{1/2}] + (-\eta_{23})^{1/2},$$
$$\varphi_1(x) = f_2(x) = 2x[(-\eta_{12})^{1/2} + (-\eta_{24})^{1/2}] + (-\eta_{24})^{1/2},$$
$$\varphi_2(x) = 2x[(-\eta_{13})^{1/2} + (-\eta_{34})^{1/2}] + (-\eta_{34})^{1/2}. \tag{A.7}$$

The quantities B_3, C_3, s_3, r_3, t_3, v_3, $R_3(x)$, $Q_3(x)$, $f_3(x)$, $\varphi_3(x)$ are obtained from the expressions for B_2, C_2, ..., $\varphi_2(x)$ by making the replacements $\eta_{24} \rightleftarrows \eta_{23}$, $\eta_{13} \rightleftarrows \eta_{14}$. Finally,

$$L_1^\pm = -[(-\eta_{23})^{1/2} \pm (-\eta_{34})^{1/2}]^2, \quad L_2^\pm = -[(-\eta_{14})^{1/2} \pm (-\eta_{34})^{1/2}]^2, \tag{A.8}$$

$$T_3^\pm = \pm(\eta_{12}\eta_{14})^{1/2} + [(-\eta_{12})^{1/2}\eta_{34} \pm (-\eta_{14})^{1/2}\eta_{23}]/[(-\eta_{12})^{1/2} \pm (-\eta_{14})^{1/2}],$$
$$T_4^\pm = \pm(\eta_{12}\eta_{23})^{1/2} + [(-\eta_{12})^{1/2}\eta_{34} \pm (-\eta_{23})^{1/2}\eta_{14}]/[(-\eta_{12})^{1/2} \pm (-\eta_{23})^{1/2}].$$

$$L_3^\pm = -[(-\eta_{12})^{1/2} \pm (-\eta_{14})^{1/2}]^2, \quad L_4^\pm = -[(-\eta_{12})^{1/2} \pm (-\eta_{23})^{1/2}]^2;$$
$$T_1^\pm = \pm(\eta_{23}\eta_{34})^{1/2} + [(-\eta_{23})^{1/2}\eta_{14} \pm (-\eta_{34})^{1/2}\eta_{12}] / [(-\eta_{23})^{1/2} \pm (-\eta_{34})^{1/2}],$$
$$T_2^\pm = \pm(\eta_{14}\eta_{34})^{1/2} + [(-\eta_{14})^{1/2}\eta_{23} \pm (-\eta_{34})^{1/2}\eta_{12}] / [(-\eta_{14})^{1/2} \pm (-\eta_{34})^{1/2}], \tag{A.9}$$

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Translated by R. Lipperheide
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