SUPERHEATING CRITICAL FIELD FOR SUPERCONDUCTORS OF THE FIRST KIND

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The stability of the superconducting state of a bulky superconductor of the first kind located in an external magnetic field is investigated on basis of the Ginzburg-Landau theory of superconductivity. For this purpose, the second variation of the thermodynamic potential of the system is calculated as a functional of the parameter of superconducting ordering in the vicinity of the Meissner superconducting state. In an equilibrium state that is stable with respect to small fluctuations, the potential has a minimum and the second variation is positive. The critical value of the external magnetic field strength for which the second variation of the potential changes sign and the superconducting state becomes unstable is found.

In a critical magnetic field H_c, a bulky superconductor of the first kind experiences an equilibrium transition from the Meissner superconducting state to the normal state, which is thermodynamically more convenient at fields stronger than H_c. At the same time, in the Meissner state the main mass of the superconductor is in a zero field and is therefore stable against fluctuations. For not too strong fields, the stability remains in force also in the surface layer, where the field differs from zero. Thus, superheating of the metastable superconducting state becomes possible in fields stronger than H_c, up to the critical superheating field $H_{c2} > H_c$ in which the superconducting state first becomes unstable.

As already noted earlier in the calculation of the superheating field for super conductors of the second kind[1,2], the mathematical basis for the determination of the stability limit in superconductors is the Ginzburg-Landau theory of superconductivity, which is based on introducing the parameter of superconducting-ordering, which is characteristic of the general theory of second-order phase transitions[4]. The thermodynamic potential of this system is represented in the form of a functional of the ordering parameter. In the equilibrium state, the potential should have a minimum. From this we determine the equilibrium equations for the ordering parameter, which describes the superconducting states in the magnetic field, and an investigation of the second variation of the potential in the vicinity of this state makes it possible to determine the limits of its stability.

In the present paper we investigate, using a method similar to that used in[1], the stability of the superconducting state in a magnetic field for extremely soft superconductors of the first kind ($\varkappa \ll 1$, when $\varkappa$ is the Ginzburg-Landau theory parameter). For simplicity we consider temperatures close to critical. In this case, according to the Ginzburg-Landau theory[3], the thermodynamic potential of the superconductor in an external field H_0 can be represented in dimensionless variables in the following form:

$$\Omega = \Omega_G + \int d\mathbf{V}(\mathbf{H}-\mathbf{H}_0)^2,$$

$$\Omega_G = \int dV\left[\frac{(1-|\psi|^2)^2}{2} + \left|\frac{i}{\varkappa}\nabla\psi + \mathbf{A}\psi\right|^2\right], \tag{1}$$

where ψ is the ordering parameter ($|\psi| \leq 1$) and $\mathbf{A}$ is the vector potential of the magnetic field, $\mathbf{H}$ = curl $\mathbf{A}$, and is determined from Maxwell's equations

$$\operatorname{rot}\mathbf{H} = {}^1/_2\mathbf{j}, \quad \operatorname{div}\mathbf{H} = 0,$$

$$\mathbf{j} = -\frac{\delta\Omega_G}{\delta\mathbf{A}} = -\frac{i}{\varkappa}(\psi^*\nabla\psi - \psi\nabla\psi^*) - 2\mathbf{A}\psi^*\psi. \tag{2}$$

From the condition that the thermodynamic potential Ω (1) be a minimum and from (2) we get the Ginzburg-Landau equations[3] describing a superconductor in a magnetic field. For a superconductor filling the half-space $z > 0$ (the magnetic field $\mathbf{H}$ is directed along the x axis and the vector potential $\mathbf{A}$ along the y axis), these equations take the form ($\psi = \psi^* = f$)

$$\frac{1}{\varkappa^2}\frac{d^2f}{dz^2} + (1-f^2)f = A^2f, \quad \left.\frac{df}{dz}\right|_{z=0} = 0, \quad f(\infty) = 1,$$

$$\frac{d^2A}{dz^2} = f^2A, \quad \left.\frac{dA}{dz}\right|_{z=0} = -H_0, \quad A(\infty) = 0. \tag{3}$$

As seen from (3), the ordering parameter varies slowly (the characteristic distances are $\sim 1/\varkappa \gg 1$), whereas the magnetic field decreases rapidly at distances on the order of unity (on the order of the depth of penetration in ordinary units). Therefore we can neglect the term with the magnetic field A^2f in Eq. (8) for the function f throughout at distances $z > 1$. The corresponding solution of this equation, satisfying the boundary condition as $z \to \infty$, can be readily obtained by quadratures:

$$z > 1, \quad f = \operatorname{th}(\varkappa z/\sqrt{2} + \xi) \quad (f_0 = \operatorname{th}\xi). \tag{4}$$

At small distances ($z \lesssim 1$) near the surface of the sample, the ordering parameter f is approximately constant. It follows therefore from (3) that the magnetic field is determined by the relations

$$A = \frac{H_0}{f_0}e^{-f_0z}, \quad H = H_0e^{-f_0z}. \tag{5}$$

in the same region, in view of the inequality $\varkappa \ll 1$ we can put in Eq. (3) for the ordering parameter f

$$z \lesssim 1, \quad f = f_0 + \varphi, \quad \varphi \ll f_0, \quad \varphi'' = \varkappa^2f_0A^2, \quad \varphi(0) = \varphi'(0) = 0,$$

$$\varphi = \frac{1}{2}\left(\frac{\varkappa H_0}{f_0}\right)^2\left(z - \frac{1-\exp[-2f_0z]}{2f_0}\right). \tag{6}$$

By making the functions (4) and (6), as well as their derivatives, continuous at $z \sim 1$ we arrive finally at the following relation between the magnetic field H_0 and the value f_0 of the ordering parameter near the surface of the sample:

$$H_0 = 2^{1/4}\varkappa^{-1/2} f_0 \sqrt{1-f_0^2}. \tag{7}$$

As seen from (7), the solution (4), (5), and (6) of the Ginzburg-Landau equations (3), which describes the Meissner state of a bulky superconductor, exists formally up to the maximum field on the surface[1)] $H_{max} = 1/2^{3/4}\kappa^{1/2}$ or $H_{max} = H_c/2^{1/4}\kappa^{1/2} \gg H_c$ in ordinary units (accordingly $f_{min} = 1/\sqrt{2}$). The problem is to determine the stability limit of this solution, i.e., the critical field H_{c2} at which the second variation of the potential Ω (1) in the vicinity of this solution reverses sign. Calculation of the second variation $\delta^2\Omega$ yields after some simplifications the following results[2)]:

$$\delta^2\Omega = \frac{1}{2}\int dV\Big\{ f^2(\delta\psi+\delta\psi^*)^2 - 2(1-f^2)\,|\delta\psi|^2 + 2\left|\left(\frac{i}{\varkappa}\nabla + \mathbf{A}\right)\delta\psi\right|^2 + \delta\mathbf{A}\Big[f\frac{i}{\varkappa}\nabla(\delta\psi-\delta\psi^*) - (\delta\psi-\delta\psi^*)\frac{i}{\varkappa}\nabla f + 2f\mathbf{A}(\delta\psi+\delta\psi^*)\Big]\Big\}, \tag{8}$$

where the variations of the field $\delta\mathbf{A}$ and $\delta\mathbf{H}$ satisfy, in accordance with (2), the equations

$$\begin{gathered}\operatorname{rot}\delta\mathbf{H} = {}^1\!/_2\delta\mathbf{j}, \qquad \operatorname{rot}\delta\mathbf{A} = \delta\mathbf{H},\\ \delta\mathbf{j} = -[fi\varkappa^{-1}\nabla(\delta\psi-\delta\psi^*) - (\delta\psi-\delta\psi^*)\,i\varkappa^{-1}\nabla f] - 2f\mathbf{A}(\delta\psi+\delta\psi^*) - 2f^2\delta\mathbf{A}.\end{gathered} \tag{9}$$

From (9) it follows that $\delta\mathbf{A}$ can be represented in the form

$$\delta\mathbf{A} = -\frac{i}{2\varkappa}\nabla\left(\frac{\delta\psi-\delta\psi^*}{f}\right) + \delta\mathbf{A}', \tag{10}$$

$$\operatorname{rot}\operatorname{rot}\delta\mathbf{A}' = -f^2\delta\mathbf{A}' - f\mathbf{A}(\delta\psi+\delta\psi^*). \tag{11}$$

Further transformations of expression (8) for the second variation $\delta^2\Omega$ are based on the fact that the "dangerous" fluctuations $\delta\psi$ of the ordering parameter, as can be readily seen, are localized at distances on the order of $1/\kappa$ (on the order of the coherence length in ordinary units), whereas the field $\delta\mathbf{A}'$, according to (11), is concentrated near the surface of the sample at distances on the order of unity. Since $\mathbf{A}$ and $\delta\mathbf{A}' \sim 1/\sqrt{\kappa}$ (see (7)), it follows that it suffices to retain in the integral (8) only the terms quadratic in the field, and linear terms can be neglected. Substituting, with allowance for these remarks, expression (10) into formula (8) and simplifying with the aid of Eq. (3) for the function f, we obtain after combining similar terms:

$$\delta^2\Omega = \frac{1}{4}\int dV\left[\left(\frac{\nabla u}{\varkappa}\right)^2 + (3f^2-1)u^2 + A^2u^2 + 4fA\delta A'u\right], \qquad u = \delta\psi + \delta\psi^*. \tag{12}$$

In (12) it is necessary to substitute the already-obtained functions f(z) (4) and A(z) (5). Owing to the first term $(\nabla u/\kappa)^2$ in (12), the dependence of the fluctuations of $u(x, y, z)$ on x and y only increases the integrand in (12), and to determine the degree of stability it is sufficient to vary ψ in the class of functions that depend only on z. In this case, after renormalizing the potential to unit surface of the superconductor, $\delta^2\Omega$ (12) takes the form

$$\delta^2\Omega = \frac{1}{4}\int_0^\infty dz\left[\left(\frac{1}{\varkappa}\frac{du}{dz}\right)^2 + (3f^2-1)u^2 + A^2u^2 + 4fA\delta A'u\right], \tag{13}$$

and (11) is rewritten as

$$\frac{d^2}{dz^2}\delta A' - f^2\delta A' = fAu, \qquad \frac{d}{dz}\delta A'\Big|_{z=0} = 0, \qquad \delta A'(\infty) = 0. \tag{14}$$

The solution of the last equation of (14), in the same approximation as above (see equations (3) and (5)), yields

$$\delta A' = -\frac{1}{2}\int_0^\infty dz'\left[e^{-f_0(z+z')} + e^{-f_0|z-z'|}\right]A(z')u(z'). \tag{15}$$

Formulas (13), (15), (4), and (5) determine the second variation of the potential, $\delta^2\Omega$, in the form of a bilinear functional of the fluctuation of u(z), viz., $\delta^2\Omega = (u, Lu)$, where L is a self-adjoint operator under the condition that $du/dz|_{z=0} = 0$. The problem consists of determining the critical value f_0^{cr} of the parameter f_0 (4), and accordingly H_0^{cr} (7), at which $\delta^2\Omega$ reverses sign for a certain function u(z). Since the substitution $u \to Cu$ leads to the transformation $\delta^2\Omega \to C^2\delta^2\Omega$, it is convenient to put

$$\int_0^\infty dz\,u^2 = (u, u) = 1$$

and to consider the functional $\delta^2\Omega$ in the class of normalized functions. Then, as can be readily seen from (13), the functional $\delta^2\Omega$ is bounded from below and consequently there exists a minimum of $\delta^2\Omega$:

$$\min\delta^2\Omega = \lambda,$$

where λ is the eigenvalue of the boundary-value problem

$$Lu - \lambda u = 0, \qquad \frac{du}{dz}\Big|_{z=0} = 0, \qquad u(\infty) = 0.$$

The quantity λ is obviously a function of f_0 and for f_0^{cr} we have $\lambda(f_0^{cr}) = 0$.

Thus, the final calculation of f_0^{cr} and of the critical superheat field $H_{c2} = H_0(f_0^{cr})$ (7) reduces to a determination of the parameter f_0^{cr} as the eigenvalue of the following boundary-value problem (see formulas (13), (15), (4), (5), and (7)):

$$\frac{1}{\varkappa^2}\frac{d^2u}{dz^2} - \left[3\,\mathrm{th}^2\left(\frac{\varkappa z}{\sqrt{2}} + \xi\right) - 1\right]u = -g(z)u_0,$$

$$\frac{du}{dz}\Big|_{z=0} = 0, \quad u(\infty) = 0, \quad g(z) = -\frac{\sqrt{2}(1-f_0^2)}{\varkappa}(1 + 2f_0z)e^{-2f_0z}. \tag{16}$$

The function (z) in the right side of (16), in view of the slow variation, is taken at the point z = 0.

The solution of (16) is

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1)Authors of other papers on the critical superheating field for super conductors, (see, for example, [5] and the literature sited therein) start from the erroneous premise that in the general case the limits of the existence of solutions of the Meissner type coincide with the physical superheat limit of the Meissner state, i.e., with the limits of its stability. This leads, in particular, to an incorrect value of the critical superheating field for superconductors of the second kind with $\kappa \gg 1$.

2)According to (2), the magnetic field is a functional of the ordering parameter ψ. Therefore, in varying ψ it is necessary to calculate $\delta\mathbf{A}$, $\delta\mathbf{H}$, $\delta^2\mathbf{A}$, and $\delta^2\mathbf{H}$. However, the terms $\delta^2\mathbf{A}$ and $\delta^2\mathbf{H}$ vanish in the expression for $\delta^2\Omega$ after integration by parts when account is taken of the boundary conditions $\delta\mathbf{H}(0) = 0$ and $\delta\mathbf{H}(\infty) = 0$.

$$u(z) = C_1(z)v_1(z) + C_2(z)v_2(z),$$

$$C_1 = \frac{\varkappa^2 u_0}{D}\int_a^z (gv_2)\,dz', \quad C_2 = \frac{\varkappa^2 u_0}{D}\int_z^\infty (gv_1)\,dz', \tag{17}$$

where

$$\left(C_1\frac{dv_1}{dz} + C_2\frac{dv_2}{dz}\right)\Big|_{z=0} = 0, \quad D = v_1\frac{dv_2}{dz} - v_2\frac{dv_1}{dz} = \text{const},$$

$$v_1(z) = \text{const}/\text{ch}^2\left(\frac{\varkappa z}{\sqrt{2}} + \xi\right), \tag{18}$$

and $v_2(z)$ is any solution of the homogeneous equation (16), linearly independent of $v_1(z)$ and increasing as $z \to \infty$. Writing down the condition $u_0 = u(0)$, we get from (17):

$$1 = -\varkappa^2 v_1(0)\left(\frac{dv_1}{dz}\Big|_{z=0}\right)^{-1}\int_0^\infty g(z)\,dz.$$

From this, with allowance for formulas (16), (18), and (7), we get

$$f_0^{\text{cp}} = 2^{-1/2}, \quad H_{c2} = 2^{-3/4}\varkappa^{1/2} \quad (H_{c2} = 2^{-1/4}\varkappa^{-1/2}H_c)$$

Thus, in the limiting case of $\kappa \ll 1$, for superconductors of the first kind, the stability limit of the superconducting state in a magnetic field coincides with the formal limit of existence of this state H_{max} (see above). This situation differs from that superconductors of the second kind. As shown in[1,2], in the case when $\kappa \gg 1$ the stability limit is $H'_{c1} < H_c$, where H_c is the thermodynamic critical field, which coincides in this case with H_{max}. In the field H'_{c1} there arises an instability against formation of vortex nuclei that give rise to a transition to a mixed state[6]. The mechanism of the instability consists in this case of the fact that in the field H'_{c1} the energy barrier that is overcome during the production of the nuclei vanishes as a result of the vanishing of the elasticity of the vortex filaments, which plays a role similar to the surface energy in the case of a superconductor of the first kind. The result obtained above therefore makes it possible to state that in superconductors of the first kind with $\kappa \ll 1$ the instability in the field H_{c2} is not connected with any critical changes of the nuclei of the normal phase. Since the "dangerous" fluctuation $u = \delta\psi + \delta\psi^*$ depends only on z, there arises in the field H_{c2} a new stability with respect to the continuous transition to the normal state over the entire surface of the superconductor.

It is of interest to calculate the dependence of the critical superheating field on the parameter κ for arbitrary superconductors, and particularly to calculate the value of κ at which the superheating field coincides with H_c. These calculations could be readily extended formally to arbitrary temperatures[1], on the basis of the microscopic theory of superconnectivity[7]. However, in the general case these calculations can apparently be carried out only numerically.

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