INSTABILITY OF A PLASMA WITH AN ISOTROPIC DISTRIBUTION FUNCTION

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It is shown that a plasma can be unstable if the velocity distribution function for the ions or the electrons is nonMaxwellian, even if it is isotropic. The plasma, under these conditions, can be unstable against perturbations with wavelengths of the order of or smaller than the mean Larmor radius for the corresponding particle.

IT is well known that an anisotropic velocity distribution function for charged particles in a plasma can lead to an instability even in the absence of macroscopic motion or spatial inhomogenieties. One might expect that a plasma in which the electron and ions distribution functions are isotropic would be stable. However, it has recently been shown by one of the present authors (L. V. Korablev) that a plasma in a magnetic field can be unstable if, in addition to the main mass of electrons and ions, there is a small number of fast ions which might be, for example, the products of a thermonuclear reaction; the distribution function for these ions is isotropic and characterized by a small energy spread about a mean value ϵ_0. A plasma of this kind can support oscillations at a frequency close to harmonics of the ion-cyclotron frequency with wavelengths of the order of the Larmor radius of the fast ions.[1] In this work, an investigation was made of the stability for electrostatic oscillations, the conclusions being valid for conditions such that the velocity of the fast ions is smaller than the phase velocity of the magnetoacoustic wave.

In Section 1 of the present work we consider the stability of a plasma whose density is higher than that treated in[1]; specifically, we assume the condition $H^2 \ll 8\pi n\epsilon_0$. Under these conditions the results obtained in[1] no longer apply and the question of stability at frequencies close to the ion frequencies remains open. We shall show here that fast ions can excite magnetoacoustic waves as a result of the Cerenkov effect. In Sec. 2 we indicate the possibility of an instability if the electron distribution function is isotropic but non-Maxwellian.

1. In the frequency region far below the ion-cyclotron frequency the magnetoacoustic oscillations of a low-density plasma ($8\pi p/H^2 \ll 1$) are described by the dispersion relation[2]

$$\omega = kv_A - i\frac{\sqrt{\pi}}{4}\frac{k_\perp^2}{k_z}\frac{m}{M}v_{Te} + i\pi\omega_{Hi}^2\frac{n'}{n}\int\frac{dv_z}{v_z}\frac{\partial g(v_z)}{\partial v_z}\delta(\omega - k_z v_z);$$

$$g(v_z) = \pi\int_0^\infty f_0(v_z^2 + v_\perp^2)J_1^2\left(\frac{k_\perp v_\perp}{\omega_{Hi}}\right)v_\perp^2\,dv_\perp^2,$$

$$\int f_0\,d\mathbf{v} = 1. \tag{1}$$

Here, k_z and $k_\perp$ and v_z and $v_\perp$ are the components of the wave vector and the velocity for the ions along and perpendicular to the magnetic field

$$v_{Te} = \left(\frac{2T_e}{m}\right)^{1/2}, \quad v_A = \frac{H}{\sqrt{4\pi nM}}, \quad \omega_{Hi} = \frac{eH}{Mc},$$

while J_1 is the Bessel function. The second term on the right side of Eq. (1) takes account of the damping due to resonance ($v_z = kv_A/k_z$) electrons and it is obtained under the assumption that $kv_A/k_z \ll v_{Te}$. The third term is due to the small number of fast ions with mean energy $\epsilon_0 = MV_0^2/2$.

In the simplest case, in which the distribution function for the fast ions is of the form $\delta(\epsilon - \epsilon_0)$, the function $g(v_t)$ can be computed easily and γ the growth rate can be written in the form

$$\gamma = -\frac{\sqrt{\pi}}{4}\frac{k_\perp^2}{k_z}\frac{m}{M}v_{Te} - \pi\frac{n'}{n}\frac{\omega_{Hi}^2}{k_z V_0}xJ_0(x)J_1(x),$$

$$x = \frac{k_\perp}{\omega_{Hi}}\left(V_0^2 - \frac{k^2}{k_z^2}v_A^2\right)^{1/2}. \tag{2}$$

The last term in the expression for γ can be positive for $x > 2.4$, that is to say $k_\perp > 2.4\,\omega_{Hi}/V_0$ and the instability arises if

$$\frac{n'}{n} \gtrsim \frac{m}{M}\frac{v_{Te}}{V_0}. \tag{3}$$

We recall that in the derivation of Eq. (1) it is assumed that $v_{Te} > V_0$. If this is not the case the electron damping will be exponentially small and waves characterized by $k_\perp > 2.4\omega_{Hi}/V_0$ are always unstable. The growth rate is a maximum when $k_\perp \sim 3\omega_{Hi}/V_0$ and the order of magnitude of the growth rate is given by

$$\gamma \approx \frac{n'}{n}\omega_{Hi}\frac{V_0}{v_A}. \tag{4}$$

The energy spread of the fast particles about the mean value ϵ_0 causes a smoothing of the function $g(v_z)$ in Eq. (1). It is qualitatively evident that the function $g(v_z)$ becomes monotonic and the instability disappears when $\Delta\epsilon/\epsilon_0 \gtrsim \omega_{Hi}^2/k_\perp^2 V_0^2$. In other words, waves characterized by $k_\perp \sim \omega_{Hi}/V_0$ are unstable for very broad distributions. A numerical calculation for a distribution function taken in the form

$$f_0 \sim \exp(-\varepsilon/T)(1 + \alpha\varepsilon),$$

shows that the instability develops if $\alpha > 3/2$. The condition for validity of Eq. (1) $\omega = kv_A \ll \omega_{Hi}$ and the instability condition $k_\perp V_0 > 2.4\omega_{Hi}$ are compatible if $\epsilon_0 \gg (2.4)^2 H^2/8\pi n$.

The instability leads to a broadening of the energy distribution of the fast particles in the direction of lower energy. This follows from energy conservation. A detailed analysis of the quasilinear relaxation process is obviously complicated. For this reason we limit ourselves to a qualitative discussion and several estimates.

The distribution of fast particles which is initially almost monoenergetic will be smeared out during the growth of the instability from an initial level w_0 to the final value $\sim n'\epsilon_0$, that is to say, in a time $\omega_{Hi}^{-1}(nv_A/n'V_0) \times \ln(n'\epsilon_0/w_0)$. If fast particles are not produced during this time then a stable distribution $f_0(v_\perp; v_z)$ is established as determined by the integral equation $\gamma = 0$ where γ is given by Eq. (1). However, if there is a weak source n'/τ of fast particles, for example, a thermonuclear reaction (in practice we always have $n'\omega_{Hi}V_0/nv_A \gg 1/\tau$) then a weak instability remains. Under these conditions approximately half the energy of the fast particles will go to the excitation of waves which then damp on the electrons with a damping rate $\gamma_e \approx -\omega_{Hi}mv_{Te}/Mv_A$. Hence the steady-state energy density of the waves is given approximately by

$$w = \sum_k \frac{H_k^2}{8\pi} \approx \frac{n'\varepsilon_0}{\tau\gamma_e} \approx \frac{n'\varepsilon_0}{\tau\omega_{Hi}} \frac{Mv_A}{mv_{Te}} \tag{5}$$

or can be even smaller if there are other (nonlinear) absorption mechanisms.

The macroscopic flux of charged particles of species α across the magnetic field caused by this instability can be interpreted as drift under the effect of a frictional force **F** arising in the emission of waves by the particles. Hence, if we use the discussion leading to Eq. (5), the following estimate obtains:

$$|nv_\perp{}^\alpha| = \left|\frac{c}{e_\alpha H^2}[\mathbf{FH}]\right| = \left|\frac{c}{e_\alpha H^2}\sum_k [\mathbf{kH}]\gamma_k \frac{w_k}{\omega_k}\right| \lesssim \frac{c}{e_\alpha H v_A}\frac{n'}{\tau}\varepsilon_0. \tag{6*}$$

The loss of energy due to the additional flux of particles to the wall is $V_0/\omega_{Hi}a$ times (a is the plasma radius) smaller than the energy of the charged particles formed as a result of the thermonuclear reaction.

2. We now consider an unstable plasma with an isotropic electron distribution function. In this case the plasma can support ion-acoustic waves. In the frequency region $\omega_{Hi} \ll \omega \ll \omega_{He}$ and for phase velocities $\omega/k \gg v_{Ti}$ the dispersion relation for electrostatic oscillations of a uniform plasma can be written in the form

$$1 + 2\frac{\omega_{pe}^2}{k^2v_0^2} = \frac{\omega_{pi}^2}{\omega^2} - \frac{\omega_{pe}^2}{k^2}\int \frac{\omega dv_z}{\omega - k_z v_z}\frac{1}{v_z}\frac{\partial g(v_z)}{\partial v_z}, \tag{7}$$

$$g(v_z) = \pi \int_0^\infty f_0^e(v_z^2 + v_\perp^2) J_0^2\left(\frac{k_\perp v_\perp}{\omega_{He}}\right) dv_\perp^2$$

(conventional notation is used).

*[FH] ≡ F × H.

When $f_0^e = (2\pi v_0)^{-1}\delta(v_z^2 + v_\perp^2 - v_0^2)$, integrating over $v_\perp$ we find

$$1 + 2\frac{\omega_{pe}^2}{k^2v_0^2} = \frac{\omega_{pi}^2}{\omega^2} - \frac{\omega_{pe}^2}{k^2v_0}\int \frac{\omega dv_z}{\omega - k_z v_z}\frac{1}{v_z}\frac{\partial}{\partial v_z} \times\left\{J_0^2\left[\frac{k_\perp}{\omega_{He}}(v_0^2 - v_z^2)^{1/2}\right]\right\}. \tag{8}$$

The imaginary part of the integral over v_z, that arises as a result of the residue, becomes an alternating function of $k_\perp$ and ω/k_z for $k_\perp v_0/\omega_{He} > 2.4$ while the waves that satisfy this inequality are unstable if the ion damping is small, as is the case when $mv_0^2/2 \gg T_i$. Under these conditions, using Eq. (8) and the simplifying assumptions $\omega/k_z \ll v_0$ and $\gamma \ll \omega$ we find

$$\omega^2 = \omega_{pi}^2(1 + 2\omega_{pe}^2/k^2v_0^2)^{-1},$$

$$\gamma = -\pi\frac{M}{m}\frac{\omega^3}{\omega_{He}^2}\frac{\omega}{|k_z|v_0}J_0(y)J_1(y),$$

$$y = \frac{k_\perp}{\omega_{He}}(v_0^2 - \omega^2/k_z^2)^{1/2}. \tag{9}$$

The growth rate is a maximum when $k_\perp v_0/\omega_{He} \sim 3$, $w/k_z \sim v_0$:

$$\gamma_{max} \approx \begin{cases} (\omega_{Hi}\omega_{He})^{1/2}, & \omega_{pe} > \omega_{He} \\ \omega_{pi}(\omega_{He}/\omega_{pe})^2, & \omega_{pe} < \omega_{He} \end{cases}. \tag{10}$$

This instability leads to a broadening of the electron distribution function toward lower energies in a time given approximately by $\gamma_{max}^{-1}\ln(nmv_0^2/2w_0)$ where w_0 is the energy density of the initial perturbations.

The instabilities considered in Secs. 1 and 2 obviously do not exhaust all the possibilities in a plasma with an isotropic distribution function which is weakly smeared out in energy. We have seen that in such a plasma when $k_\perp v_0^\alpha/\omega_{H\alpha} > 2.4$ the conditions required for an instability $\partial g(v_z)/\partial v_z > 0$ can always be satisfied. Hence we may advance the suggestion, which as yet is not contradicted by experiment, that in a magnetic field the particle energy distribution functions cannot have sharp peaks.

[1] L. V. Korablev, Zh. Eksp. Teor. Fiz. 53, 1600 (1967) [Sov. Phys.-JETP 26, 000 (1968)].
[2] V. D. Shafranov, Reviews of Plasma Physics, Consultants Bureau, New York, 1967, Vol. 3. pp. 84, 93, 94.

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