ALLOWANCE FOR THE COULOMB INTERACTION IN MULTIPHOTON IONIZATION

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Account is taken, within the framework of the quasiclassical approximation, of the Coulomb interaction between the electron and the atomic core upon ionization of an atom by a strong light wave. The obtained formulas pertain to the case $\sigma < \sigma_{cr}(\gamma)$, where σ is the Coulomb parameter, defined in (26), $\gamma = \omega/\omega_t = \omega\kappa/F$, and $\sigma_{cr}(\gamma)$ is the bifurcation point for Eq. (25). The values of $\sigma_{cr}(\gamma)$ are obtained by numerical calculation.

1. FORMULATION OF PROBLEM

THE study of multiphoton ionization of atoms and ions in the field of a strong light wave has been subject of a large number of recent experimental[1-8] and theoretical[9-16] papers. The problem of the theory is, primarily, to calculate the total ionization probability w and its dependence on the intensity of the incident light (i.e., the exponent k in the formula $w(F) \sim F^{2k}$, where F is the electric field intensity). Such calculations were performed both on the basis of the wave equation[11-13], and by the simpler quasiclassical method[13,14]. The final results coincide, inasmuch as the accurate solution of the Schrödinger equation calls for the calculation of rapidly oscillating integrals by the saddle-point method, corresponding to a transition to the quasiclassical approach in the final state of the calculation[1)].

A comparison of the formulas obtained in[11,12] with the experimental data on ionization of neutral atoms by laser light is made difficult by the fact that the Coulomb interaction of the emitted electron with the atomic remainder is neglected in these calculations. Yet the Coulomb correction to the ionization probability is not small. Thus, in the case of a constant field

$$w(F)\,/\,w_{sr}(F) = (2F_0/F)^{2\lambda}, \tag{1}$$

where $w_{sr}(F)$ is the probability of ionization of the level connected with the short-range forces, $F_0 = \kappa^3$ is the field inside the atom, and λ is the Coulomb parameter:

$$\lambda = \varkappa_c/\varkappa = Z(I/I_0)^{-1/2}. \tag{2}$$

Here Z is the charge of the atomic remainder (Z = 1 in the case of ionization of neutral atoms), and I and I_0 = 13.6 eV are the ionization potentials of the atom under consideration and of the hydrogen atoms[2)]. It follows from experiment that $2F_0/F \sim 10^3$, and allowance for the Coulomb interaction introduces in w(F) a factor $\sim 10^5$. The cause of its appearance lies in the fact that, owing to the Coulomb interaction, a pre-exponential factor $(\kappa r)^\lambda$, which takes into account the density of the electron cloud far from the nucleus, appears in the asymptotic form of the wave function of the bound electron. Putting $r \sim r_0 = \kappa^2/2F$ (r_0 is the width of the barrier in a constant field), we arrive at (1).

We shall consider below the influence of the Coulomb interaction the sub-barrier trajectory in a field F(t) = F cos ωt (the parameter $\gamma = \omega\kappa/F$ is arbitrary). When this influence is small, the Coulomb interaction is taken into account by perturbation theory (see Sec. 2). In the general case it is necessary to solve Eq. (25) for the sub-barrier motion, and this can be done only by numerical methods. The results obtained by such a calculation are described in Sec. 3. Section 5 is devoted to a discussion of the experimental data on the ionization.

2. ACCOUNT OF COULOMB INTERACTION BY PERTURBATION THEORY

In the quasiclassical approximation, the penetrability of the oscillating barrier is given by the function

$$\widetilde{W} = \int_{t_0}^{0}\left\{\frac{1}{2}\dot{\mathbf{r}}^2 - U(\mathbf{r}, t) + E_0\right\}dt - (\dot{\mathbf{r}}\mathbf{r})_{t=0}, \tag{3}$$

which is calculated along the sub-barrier trajectory $\mathbf{r} = \mathbf{r}_0(t)$ with the minimum value of Im W. Such a trajectory is determined by the classical equations of motion and by the boundary conditions[14]

$$\dot{\mathbf{r}}(t_0) = 0, \quad H(t_0) = E_0 = -\varkappa^2/2, \quad \operatorname{Im}\mathbf{r}(0) = 0. \tag{4}$$

Let now

$$U(\mathbf{r}, t) = V + \delta V, \quad \delta V \sim \mu V, \tag{5}$$

where μ is a small parameter. We obtain, accurate to μ^2, the correction to the barrier penetrability necessitated by the perturbation δV. Varying expression (3) with respect to μ, we obtain the following sum calculations[15]:

$$\delta\widetilde{W} = \delta\widetilde{W}_1 + \delta\widetilde{W}_2 + O(\mu^3); \tag{6}$$

$$\delta\widetilde{W}_1 = -\int_{t_0}^{0}\delta V(\mathbf{r}_0(t))\,dt,$$

$$\delta\widetilde{W}_2 = \frac{1}{2}\left[\delta V(t_0)\,\delta t_0 - \int_{t_0}^{0}\frac{\partial\delta V}{\partial\mathbf{r}_0}\mathbf{r}_1 dt\right]. \tag{7}$$

[1)]We note in this connection that the remark made in [12] concerning the discrepancy, by a factor of 2, between the results of [11] and [12], is based on a misunderstanding and is connected with the method used in [11] to normalize the wave function $\psi_i(x)$. We are grateful to V. I. Ritus for calling our attention to this circumstance.

[2)]For the hydrogen atom in a state with principal quantum number n, we get λ = n; for atoms of most transparent gases in the ground state λ ≈ 1 (see Table II below).

The first-order correction $\delta\widetilde{W}_1$ suffices for an account of the Coulomb interaction; the expression for $\delta\widetilde{W}_2$ will be needed in Sec. 4. We note that these formulas can be used if the perturbation δV is small along the entire sub-barrier trajectory. In our case $\delta V(r)$ is the atomic potential, which is singular at $r = 0$. In addition, the quasiclassical approach is still not valid in the region $\kappa r \lesssim 1$. It is therefore necessary to make the result continuous with the wave function of the free atom[16], after which we get

$$\delta\widetilde{W}_1 = -i\left[\lambda \ln \varkappa r_1 + \varkappa_c \int_{t_1}^{0} \frac{dt}{r(t)}\right], \quad r_1 = r(t_1), \tag{8}$$

where $r(t)$ is the trajectory unperturbed by the Coulomb interaction, r_1 is the continuity point ($1 \ll \kappa r_1 \ll (2I/\omega)\tanh(\tau_0/2)$, $\tau_0 = \sinh^{-1}\gamma$), and account is taken of the fact that $\delta V(r) \approx -\kappa_c/r$ when $\kappa r \gg 1$.

Changing over to the variables $\tau = i\omega t$ and $\xi = \omega x/\kappa$, we get the following expression for the sub-barrier trajectory with momentum p at the emergence point:

$$\xi_x = \frac{1}{\gamma}(\operatorname{ch}\tau_0' - \operatorname{ch}\tau) - i\frac{p_x}{\varkappa}(\tau_0' - \tau), \quad \xi_\perp = -i\frac{\mathbf{p}_\perp}{\varkappa}(\tau_0' - \tau), \tag{9}$$

$$\tau_0' = \operatorname{Arsh}\left\{\gamma\left(\sqrt{1 + \frac{p_\perp^2}{\varkappa^2}} + i\frac{p_x}{\varkappa}\right)\right\} \tag{10}$$

(in the case of linear polarization of the light). Substituting (9) in (8) we get

$$\delta\widetilde{W}_1 = -i\lambda\left[\ln\frac{2F_0}{F} + \Phi(\gamma, \mathbf{p})\right], \quad \lambda = \frac{\varkappa_c}{\varkappa}, \tag{11}$$

$$\Phi(\gamma, \mathbf{p}) = \ln\frac{\tau_0'}{2\gamma} + \int_0^{\tau_0'} d\tau\left[\frac{1}{\xi(\tau)} - \frac{1}{\tau_0' + \tau}\right], \quad \xi(\tau) = \sqrt{\xi_x^2 + \xi_\perp^2}. \tag{12}$$

The first term in (11) is the contribution from the extremal trajectory with $p = 0$. Accordingly, $\Phi(\gamma, p = 0) = 0$. It is interesting to note that although the form of the sub-barrier trajectory changes strongly when $\gamma \gg 1$ (for example, the point of emergence is given by $x(0) \sim \kappa^2/F\gamma$), the main Coulomb correction to the penetrability is given by formula (1) and does not depend on γ at all.

In the quasiclassical case, only a narrow beam of classical trajectories close to the extremal one takes part in the tunneling through the barrier. Expanding $\Phi(\gamma, p)$ at small values of p, we arrive at the following formula for the probability of ionization of the atom by the field of a light wave with linear polarization:

$$w = \sum_{n=k_0}^{\infty} w_n, \quad k_0 = [\nu + 1], \quad \nu = \frac{I}{\omega}\left(1 + \frac{1}{2\gamma^2}\right), \tag{13}$$

$$w_n = \omega\frac{|C_{\varkappa l}|^2}{\pi}\left(\frac{\omega}{2I}\frac{\operatorname{th}\tau_0}{\rho}\right)^{1/2} R_n\left(\frac{2F_0}{F}\right)^{2\lambda}\exp\left\{-\frac{2I}{\omega}f(\gamma)\right\}. \tag{14}$$

Here $[\nu]$ is the integer part of ν; $k_0 = [\nu + 1]$ is the photoionization threshold. The pre-exponential factor is given by

$$R_n = e^{-\alpha(n-\nu)}R(\beta(n-\nu)), \tag{15}$$

where

$$R(x) = w(\sqrt{x}) = e^{-x}\int_0^{\sqrt{x}} e^{y^2}dy \text{ for } x > 0,$$

$$R(x) = \frac{\sqrt{\pi}}{2}\operatorname{erf}(\sqrt{-x}) = \int_0^{\sqrt{-x}} e^{-y^2}dy \quad \text{for } x < 0 \tag{16}$$

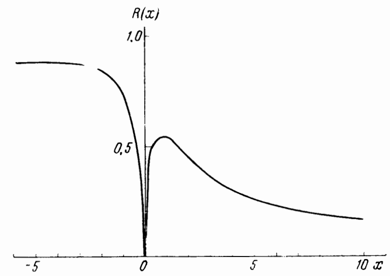

FIG. 1. Plot of the function R(x).

(a plot of the function $R(x)$ is shown in Fig. 1).

The quantities α, β, and ρ, which enter in these formulas, are functions of γ. The exact expressions for them are quite cumbersome and are given in the Appendix. In the limiting cases of large and small γ these formulas simplify to:

$$\alpha = \begin{cases} ^2/_3\gamma^3 & \text{for } \gamma \ll 1 \\ 2\tau_0(1 - \tau_0^2/3\gamma_k) & \text{for } \gamma \gg 1 \end{cases}, \tag{17}$$

$$\beta = \begin{cases} 2\gamma & \text{for } \gamma \ll 1 \\ 2(1 - \tau_0^3/\gamma_k) & \text{for } \gamma \gg 1 \end{cases}, \tag{18}$$

$$\rho = \begin{cases} 1 & \text{for } \gamma \ll 1 \\ |1 - \tau_0^3/\gamma_k| = {}^1/_2|\beta| & \text{for } \gamma \gg 1 \end{cases}. \tag{19}$$

Here $\gamma_k = 2I/\lambda\omega$, $\tau_0 = \ln 2\gamma$ at $\gamma \gg 1$. The dependence on the Coulomb interaction enters in (14) only via the preexponential factor.

Let us formulate now the condition for the applicability of the obtained formulas. Allowance for the Coulomb interaction in accordance with (8) is valid when the interaction does not distort noticeably the sub-barrier trajectory of the electron. This is the situation in the case of a constant field and in the adiabatic region $\gamma \ll 1$. However, as soon as γ becomes larger than unity, the width of the barrier $x(0)$ begins to decrease rapidly and the trajectory shifts towards the smaller values of r. When $\gamma \sim \gamma_0$, a qualitative change of the form of the sub-barrier trajectory[16] takes place, and therefore the sought-for condition takes the form[3)]

$$\gamma \ll \gamma_k \quad (\gamma_k = 2I/\lambda\omega). \tag{20}$$

Inasmuch as $\omega \ll I$, the parameter $\gamma_k \gg 1$. In principle, (20) can be satisfied also when $\gamma \gg 1$, provided the number of quanta involved in the process is large. At the present time, however, in experiments on ionization, $I/\omega \sim 10$, and the region $1 \ll \gamma \ll \gamma_k$ is practically nonexistent.

As $\lambda \to 0$ (and for all values of γ), we have $\alpha = 2(\tau_0 - \tanh\tau_0)$, $\beta = 2\tanh\tau_0$, and $\rho = 1$, and (14) goes over into a formula for the ionization probability of a level bound by short-range forces (see formula (51) of[14]).

Let us discuss the role of the Coulomb correction in two limiting cases.

3)We emphasize that condition (20) is only qualitative. As shown in the next section, the exact condition for the validity of formula (8) is of the form $\sigma \ll \sigma_{cr}(\gamma)$, where σ is defined in (26). From Fig. 4 (see below) we see that at not too large γ these two conditions can differ by a factor of several times.

1) $\gamma \ll 1$, adiabatic region. In this case the frequency of light ω is negligibly small compared with the two frequencies ω_0 and ω_t that are characteristic of these problems[4]: $\omega \ll \omega_t \ll \omega_0$. Therefore the quantum character of the absorption of light does not become manifest, and we can go over in (13) from summation over n to integration:

$$w = \omega_0 \left(\frac{6}{\pi}\right)^{1/2} |C_{\varkappa l}|^2 \left(\frac{2F_0}{F}\right)^{2\lambda - 3/2} \exp\left\{-\frac{2}{3}\frac{F_0}{F}\left(1 - \frac{1}{10}\gamma^2\right)\right\}. \quad (21)$$

The Coulomb correction has the same form as in a constant field.

2) Region $1 \ll \gamma \ll \gamma_k$. Here $w_{n+1}/w_n \sim (4\gamma^2)^{-1}$, making it possible to confine oneself in the first approximation to the term in (13) with $n = k_0$. This yields

$$w = A\omega \left(\frac{2I}{\omega}\right)^{2\lambda} e^{I/\omega} \left(\frac{1}{4\gamma^2}\right)^k. \quad (22)$$

In (22) A is a constant on the order of unity

$$k = [\nu + 1] - \lambda - \left(\frac{\nu}{1 + 2\gamma^2} + \frac{\tau_0^2 \delta}{3\gamma_k}\right),$$
$$\delta = [\nu + 1] - \nu, \quad 0 < \delta < 1. \quad (23)$$

Under the additional condition $\gamma \gg (I/\omega)^{1/2}$, the expression for k simplifies to

$$k = [\nu_0 + 1] - \lambda, \quad \nu_0 = I/\omega. \quad (24)$$

Inasmuch as $\omega_t \ll \omega \ll \omega_0$, the light absorption has here a quantum character. Expression (22) has the same form as the formula of the k-th order perturbation theory (process with absorption of k photons)[5]. The Coulomb interaction decreases the "multiquantum degree" of the process by an amount $\lambda \sim 1$, which can likewise be treated as an effective lowering of the end point of the continuous spectrum (see (41)). Formerly, this lowering of k is the result of the factor $(2F_0/F)^{2\lambda}$, which is the Coulomb correction to the exponential. When $\tau_0 \sim (I/\lambda\omega)^{1/3}$, a significant change in the pre-exponential factor takes place, namely, the coefficient β reverses sign (see (18)). Leading to a change in character of the dependence of w_n on the number n (see Fig. 1).

At the existing experimental accuracy, a verification of the preexponential factor in (14) is hardly possible. We shall henceforth confine ourselves to a study of the principal (exponential) factor in w, for which purpose it is sufficient to consider only the extremal trajectory.

3. INFLUENCE OF COULOMB INTERACTION ON THE SUB-BARRIER TRAJECTORY

The exact equation of the sub-barrier of motion is

$$\ddot{\xi} = \frac{1}{\gamma}\left(\frac{\sigma}{\xi^2} - \operatorname{ch}\tau\right), \quad -\tau_0 \leqslant \tau \leqslant 0 \quad (25)$$

(in the case of linear polarization of the incident light). Here

$$\sigma = \lambda\left(\frac{\omega}{2I}\right)^2 \frac{F_0}{F} = \frac{\gamma}{\gamma_k}, \quad \gamma_k = \frac{2I}{\lambda\omega}, \quad \tau_0 = \tau_0(\gamma, \sigma). \quad (26)$$

the solution of (25) satisfying the boundary conditions $\xi(0) = \xi_0$ and $\dot{\xi}(0) = 0$ is of the form

$$\xi(\tau) = \xi_0 - \frac{1}{\gamma}\left[\operatorname{ch}\tau - 1 - \frac{\sigma\tau^2}{2\xi_0^2} f(\tau)\right], \quad f(\tau) = 1 + \sum_{n=1}^{\infty} a_n \tau^{2n}, \quad (27)$$

where

$$a_1 = \frac{1}{6\gamma\xi_0}\left(1 - \frac{\sigma}{\xi_0^2}\right), \quad a_2 = \frac{1}{180\gamma\xi_0}\left[1 + \frac{1}{\gamma\xi_0}\left(1 - \frac{\sigma}{\xi_0^2}\right)\left(9 - 11\frac{\sigma}{\xi_0^2}\right)\right] \quad (28)$$

etc. When $\sigma = 0$ (short-range potential) we have for the point of emergence ξ_0

$$\xi_0 = \operatorname{th}\frac{\tau_0}{2} = 1 - \frac{1}{\gamma} + \ldots \text{ for } \gamma \gg 1 \quad (29)$$

(here $\tau_0 = \sinh^{-1}\gamma$). If $\xi_0 \sim 1$ also when $\gamma \gg 1$, but $\sigma \neq 0$ (as is verified by calculation), then $|a_n| \ll 1$ and we can put $f(\tau) \approx 1$. Then

$$\xi(\tau) = \xi_0 - \frac{1}{\gamma}\left(\operatorname{ch}\tau - 1 - \frac{\sigma}{2\xi_0^2}\tau^2\right). \quad (30)$$

Actually this approximation reduces to a replacement of the Coulomb force σ/ξ^2 by its value at the emergence point $\xi = \xi_0$. Inasmuch as the action S in a rapidly alternating field is accumulated essentially at the end of the sub-barrier trajectory[14], such an approximation seems reasonable when $\gamma \gg 1$. From (30) and the initial conditions $\xi(-\tau_0) = 0$ and $\dot{\xi}(-\tau_0) = 1$ we get the following equations for the determination of ξ_0 and τ_0:

$$\xi_0 = \frac{1}{\gamma}\left(\operatorname{ch}\tau_0 - 1 - \frac{\sigma}{2\xi_0^2}\tau_0^2\right), \qquad \operatorname{sh}\tau_0 - \frac{\sigma}{\xi_0^2}\tau_0 = \gamma. \quad (31)$$

Eliminating σ/ξ_0^2 from the second equation, we transform these equations into

$$\xi_0 = \varphi_1(\tau_0) = \varphi_2(\tau_0), \quad (32)$$

where

$$\varphi_1(\tau) = \frac{1}{\gamma}\left[\operatorname{ch}\tau - 1 - \frac{\tau}{2}(\operatorname{sh}\tau - \gamma)\right], \quad \varphi_2(\tau) = \left(\frac{\sigma\tau}{\operatorname{sh}\tau - \gamma}\right)^{1/2}. \quad (33)$$

The investigation of these equations is facilitated by the fact that the parameter σ enters only in $\varphi_2(\tau)$. It is clear from Fig. 2 that when $\sigma < \sigma_{cr}(\gamma)$ there are two solutions, $\xi'(\tau)$ and $\xi''(\tau)$. When $\sigma \to 0$, we get $\tau_0 \to \sinh^{-1}\gamma$, and the trajectory $\xi'(\tau)$ goes over continuously into the corresponding trajectory for $\sigma = 0$. This is the ordinary solution obtained by perturbation theory in the case of small σ. With increasing σ, both solutions come closer together, and finally, when $\sigma = \sigma_{cr}(\gamma)$, they coincide. When $\gamma \gg 1$ we can obtain the asymptotic formulas

$$\sigma_{cr}(\gamma) \approx \frac{8}{27}\frac{\gamma}{(\ln\gamma)^2}, \quad \xi_{cr}(0) \approx \frac{2}{3}, \quad \tau_0^{cr} \approx \ln 2\gamma + \frac{2}{3\ln 2\gamma}. \quad (34)$$

When $\sigma \sim \sigma_{cr}(\gamma)$, the influence of the Coulomb interaction on the sub-barrier trajectory becomes appreci-

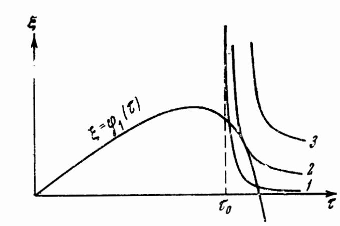

FIG. 2. Graphical solution of Eqs. (32). Curves 1–3 correspond to the function $\varphi_2(\tau)$: 1–at $\sigma < \sigma_{cr}$, 2–at $\sigma = \sigma_{cr}$, 3–at $\sigma > \sigma_{cr}$. Here $\tau_0 = \sinh^{-1}\gamma$.

[4]It is easily seen that $\omega_t/\omega_0 = 2F/F_0 \ll 1$.

[5]We note that $1/4\gamma^2 = (F/F_{eff})^2$, where $F_{eff} = \omega F_0/I$.

able. For example, comparison of (29) with (34) shows that the Coulomb interaction causes the width of the barrier at $\sigma = \sigma_{cr}$ to decrease by approximately 1/3. Therefore the analysis presented in Sec. 2 is valid only if

$$\sigma \ll \sigma_{cr}(\gamma). \tag{35}$$

These conclusions are confirmed by a numerical solution of the exact equation (25)[6].

The imaginary part of the action S accumulated by the particle along the extremal trajectory will be represented in the form

$$\operatorname{Im} S = \frac{I}{\omega} f(\gamma, \sigma) - \frac{\sigma}{\gamma} \ln \frac{2F_0}{F}, \tag{36}$$

where

$$f(\gamma, \sigma) = \int_0^{\tau_0} \left(1 + \dot{\xi}^2 - \frac{2\xi \operatorname{ch} \tau}{\gamma} - \frac{2\sigma}{\gamma\xi}\right) d\tau + \frac{2\sigma}{\gamma} \ln 2\gamma. \tag{37}$$

Accurate up to the pre-exponential factor, we have

$$w \sim \exp(-2 \operatorname{Im} S) = \text{const} \cdot \omega \left(\frac{2F_0}{F}\right)^{2\lambda} \exp\left\{-\frac{2I}{\omega} f(\gamma, \sigma)\right\}. \tag{38}$$

As seen from Fig. 3, the contribution of the second extremal trajectory to the barrier penetrability can be neglected (when $\omega \ll I$). For the first solution $\xi'(\tau)$, the dependence of $f(\gamma, \sigma)$ on σ is almost linear

$$f(\gamma, \sigma) \approx f(\gamma) - \frac{a(\gamma)}{\gamma} \sigma \quad (\text{if} \quad \sigma \ll \sigma_{cr}), \tag{39}$$

and the coefficient $a(\gamma)$ depends little on γ. When $\gamma > 10$, we have $a(\gamma) \approx 9.2 + 0.4 \ln 4\gamma^2$, which leads to a further decrease of the exponent k in formula (24):

$$k = [\nu_0 + 1] - 1.4\lambda. \tag{40}$$

Such a lowering of the photoionization threshold agrees with the effective decrease of the ionization potential I by an amount

$$\Delta I = -V_0(x_0) = \frac{\varkappa_c}{x_0} = \frac{\lambda}{\xi_0} \hbar\omega \sim (1.0 - 1.5)\hbar\omega. \tag{41}$$

When $\sigma > \sigma_{cr}(\gamma)$, the curves of Fig. 2 did not intersect, and the roots of (31) go off to the complex plane. It is easy to show[15] that in this case $\operatorname{Im} \xi_0 \neq 0$, i.e., the corresponding classical trajectory is not extremal and must therefore be discarded in the quasi-classical case. Consequently, when $\sigma > \sigma_{cr}$ there are no trajectories that realize a "smooth" minimum of Im W (i.e., sub-barrier trajectories for which the variation of Im W, following a small change of the trajectory itself, begins with the quadratic terms). There remains the possibility of the type of a "minimum on the edge"[7]. It is clear from physical considerations that the only distinct point corresponds to a trajectory with p = 0 at infinity. For a short-range

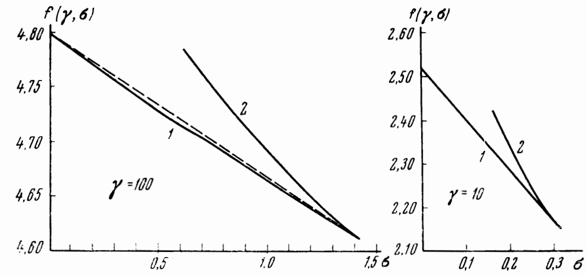

FIG. 3. The function f (γ,σ) from (38) from (38). Curves 1 and 2 correspond to the first and second solutions of the equations of the sub-barrier motion.

potential the average momentum of the particle at infinity coincides with its momentum at the point of emergence from under the barrier, but in the presence of a Coulomb "tail" $V_0(r)$ this is no longer the case.

When $\gamma \gg 1$ the form of trajectory after emerging from under the barrier can be obtained by the Kapitza method[18]. Substituting into the equation

$$\ddot{x} = -\frac{\varkappa_c}{x^2} + F \cos \omega t \quad (0 < t < +\infty) \tag{42}$$

$x(t)$ in the form $x(t) = X + \rho(t)$, where $\rho(t) = F\omega^{-2}(1 - \cos\omega t)$, and $X = \langle x(t) \rangle$ is the "center of gravity" of the orbit, and averaging over the period, we get

$$\ddot{X} = -\left\langle \frac{\varkappa_c}{(X + \rho(t))^2} \right\rangle = -\frac{\varkappa_c}{X^2}. \tag{43}$$

The discarded terms are of the order of $\rho/X \sim \gamma^{-1}$ (for $X \lesssim \kappa^2/F\gamma$). In order for the particle to able to go to infinity, it is necessary to have

$$\tfrac{1}{2}\dot{x}^2 \geqslant \varkappa_c / x \text{ at } t = 0 \tag{44}$$

(we note that $\rho(0) = \dot{\rho}(0) = 0$). On going over to the imaginary "time" $\tau = i\omega t$, this condition signifies that the particle velocity $\dot{\xi} = d\xi/d\tau$ at the instant of emergence $\tau = 0$ is pure imaginary and the condition $\operatorname{Im} \xi(0) = 0$ for the extremal trajectory is replaced by

$$\dot{\xi}^2(0) = -\lambda\omega / I\xi(0). \tag{45}$$

To calculate the ionization probability w at $\gamma \gg 1$ and $\sigma > \sigma_{cr}(\gamma)$ it is necessary to solve Eq. (25) with boundary condition (45) and with the ordinary conditions (4) at the initial instant $\tau = -\tau_0$. With this, however, τ_0 becomes complex (compare with expression (10) for τ_0 in the case of trajectories with non-zero momentum p in the absence of a Coulomb interaction), thus greatly complicating the numerical calculations. The resultant situation (encounter of two extremals and appearance of a new solution) can be lucidly illustrated by using as an example the plateau problem from the theory of minimal surfaces[15].

The value $\sigma = \sigma_{cr}$ at which the encounter of the two solutions takes place is called in mathematics the bifurcation point[19]. The final results on σ_{cr} are best represented by changing over to the variable s:

$$s = \sigma / \gamma = \lambda\omega / 2I = 1 / \gamma_k \tag{46}$$

(s does not depend on the field F). The curve $s = s_{cr}(\gamma)$ is shown in Fig. 4. Allowance for the Coulomb interaction, considered in Secs. 2 and 3, pertains to the points (γ, s) lying below this curve.

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6) This problem differs from the standard Cauchy problem of the theory of differential equations in that the boundary conditions $\xi(-\tau_0) = 0$, $H(-\tau_0) = 1/2$, and $\xi(0) = 0$ are specified at different ends of the integration interval (here $H = \frac{1}{2}\dot{\xi}^2 + \gamma^{-1}(\xi \cosh \tau + \sigma/\xi)$).

7) Compare with the paper by E. M. Lifshitz [17] concerning the disintegration of the deuteron in the Coulomb field of the nucleus, where the cross section of the process is determined when $E_d > 1.72\xi$ (ϵ_1 – binding energy of the deuteron) by a sub-barrier trajectory of just this type. The determination of the Coulomb correction with the aid of a diagram technique is the subject of [20].

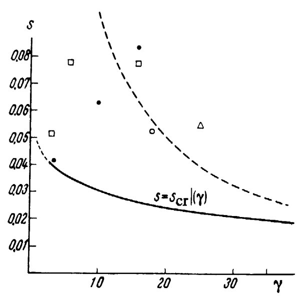

FIG. 4. Plot of $s = s_{cr}(\gamma)$ and values of the parameters γ and s for the ionization experiments performed on the following atoms: ● – Kr, □ – Xe, ○ – Ar, Δ – H_2 molecules (the literature references are given in Table II). The dashed curve represents the equation $s = 1/\gamma$ and corresponds to the equality $\gamma = \gamma_k$. Comparison of these curves shows that for small values of γ the condition (35) is more stringent than (20).

The results of the numerical calculations are listed in Table I.

4. CASE OF ELLIPTIC POLARIZATION

We consider now a more general case, when the light wave incident on the atom has elliptic polarization: $\mathbf{F}(t) = (F\cos\omega t, \epsilon F\sin\omega t, 0)$. Here ϵ is the ellipticity; the case $\epsilon = \pm 1$ corresponds to circular polarization of the light. In the equations for the subbarrier motion

$$\ddot{\xi}_x = \frac{1}{\gamma}\left(\frac{\sigma\xi_x}{\xi^3} - \operatorname{ch}\tau\right), \quad \ddot{\xi}_y = \frac{1}{\gamma}\left(\frac{\sigma\xi_y}{\xi^3} - i\varepsilon\operatorname{sh}\tau\right) \tag{47}$$

we replace the Coulomb force by its value at the point of emergence $\xi_x = \xi_0$) $\xi_y = 0$. In this approximation, the trajectory can be determined in explicit form:

$$\xi_x = \frac{1}{\gamma}\left[\operatorname{ch}\tau_0 - \operatorname{ch}\tau - \frac{\sigma}{2\xi_0^2}(\tau_0^2 - \tau^2)\right],$$
$$\xi_y = \frac{i\varepsilon}{\gamma}\left(\operatorname{sh}\tau_0\frac{\tau}{\tau_0} - \operatorname{sh}\tau\right). \tag{48}$$

The initial conditions $\xi(-\tau_0) = 0$ and $\dot{\xi}^2(-\tau_0) = 1$ give the pair of equations needed for the determination of ξ_0 and τ_0; these equations can be transformed to a form similar to (31) and (32) For details see [15]). When ϵ is not too close to ± 1, the $\varphi_1(\tau)$ and $\varphi_2(\tau)$ curves have qualitatively the same form as in Fig. 2. This leads to the existence of two extremal solutions, to their coming together with increasing σ, and to their encounter at $\sigma = \sigma_{cr}(\gamma, \epsilon)$. It is important that the value of σ_{cr} increases together with $|\epsilon|$. Therefore the calculation of the Coulomb correction by perturbation theory, given in [16], has at $\epsilon \neq 0$ a greater region of applicability than in the case of linear polarization ($\epsilon = 0$). When $\epsilon \to 1$ the character of the $\varphi_1(\tau)$

Table I

γ	$\sigma_{cr}(\gamma)$	$s_{cr}\cdot 100$	$f(\gamma,\sigma)$: $\sigma = 0$	$f(\gamma,\sigma)$: $\sigma = \sigma_{cr}$	$a(\gamma)$ from (39)	γ	$\sigma_{cr}(\gamma)$	$s_{cr}\cdot 100$	$f(\gamma,\sigma)$: $\sigma = 0$	$f(\gamma,\sigma)$: $\sigma = \sigma_{cr}$	$a(\gamma)$ from (39)
3	0.119	4.00	1.365	0.972	9.90	20	0,496	2.48	3.194	2,895	12,1
5	0.184	3,68	1.832	1.44	10,6	25	0.57	2.28	3.416	3.136	12,3
7	0.239	3.42	2.156	1.78	11.1	32,7	0.685	2,09	3.683	3,423	12,5
10	0,315	3,15	2.510	2,15	11,4	40	0.775	1,94	3.883	3,639	12.6
15	0.410	2.73	2.909	2.586	11.85	100	1,42	1,42	4.799	4,610	13,3

curve changes noticeably, and the question of the encounter of the solutions in the case of circular polarization calls for an additional study.

At small values of $|\epsilon|$, the ellipticity of the light can be accounted for by perturbation theory, representing the potential of the plane wave $U = -\mathbf{F}(t)\mathbf{x}$ in the form (5), where $V = -Fx\cos\omega t$, $\delta V = -\epsilon Fy\sin\omega t$. Since $y_0(t) \equiv 0$ and $\delta t_0 \sim \epsilon^2$, we find from (7) that

$$\delta W = \delta W_2 = \frac{\varepsilon F}{2}\int_{t_0}^{0} y_1(t)\sin\omega t\,dt$$
$$= -i\frac{I}{\omega}\left(\frac{\varepsilon}{\gamma}\right)^2\int_{-\tau_0}^{0} g(\tau)\operatorname{sh}\tau\,d\tau. \tag{49}$$

Here $y_1(t)$ is the correction to the extremal trajectory:

$$y_1(t) = \frac{\varepsilon\varkappa}{\omega\gamma}g(\tau)$$

The equation and the boundary conditions for $g(\tau)$ take the form

$$\ddot{g} - \frac{s}{[\xi_0(\tau)]^3}g = \operatorname{sh}\tau, \quad s = \frac{1}{\gamma_k}, \tag{50}$$

$$\dot{g}(0) = 0, \quad \begin{cases} g(-\tau_0) = 0 & \text{if} \quad \sigma \neq 0, \\ g(-\tau_0) = 0 & \text{if} \quad \sigma = 0. \end{cases} \tag{51}$$

In (50), $\xi_0(\tau)$ is the exact trajectory satisfying Eq. (25). We get therefore

$$f(\gamma,\sigma,\varepsilon) = f(\gamma,\sigma) + \varepsilon^2 f_1(\gamma,\sigma) + O(\varepsilon^4), \tag{52}$$

where

$$f_1(\gamma,\sigma) = -\frac{1}{\gamma^2}\int_{-\tau_0}^{0} g(\tau)\operatorname{sh}\tau\,d\tau. \tag{53}$$

The determination of the correction $f_1(\gamma,\sigma)$ requires, generally speaking, numerical calculations. Let us illustrate the obtained formula using as an example $\sigma = 0$ (short-range potential). From (50) and (51) we get

$$g(\tau) = \operatorname{sh}\tau - \frac{\operatorname{sh}\tau_0}{\tau_0}\tau.$$

Substitution in (53) yields

$$f_1(\gamma,0) = \frac{1}{2}\left(\frac{\tau_0}{\gamma^2} + \operatorname{cth}\tau_0\right) - \frac{1}{\tau_0} = \begin{cases} {}^1/_{45}\gamma^3 & \text{if} \quad \gamma \ll 1 \\ {}^1/_2 - \ln^{-1}2\gamma & \text{if} \quad \gamma \gg 1 \end{cases}. \tag{54}$$

the same result can be obtained by expanding directly the function $f(\gamma,\epsilon)$ from [13] at $\epsilon \to 1$.

5. COMPARISON WITH EXPERIMENT

In the experiments one usually measures not the field intensity F, but the photon flux J. The connection between them is given by the formula

$$F = \left[\frac{8\pi\hbar\omega}{(1+\varepsilon^2)c}J\right]^{1/2} \tag{55}$$

(here F is the maximum value of the field). It is easy to represent (55) in a form that is convenient for calculations:

$$F = a\left(\frac{J}{1+\varepsilon^2}\right)^{1/2}\cdot 10^{-8} \tag{56}$$

where J is measured in photons/cm²sec, F is measured in V/cm, and a is a numerical factor on the order of unity, equal to 1.46 for a ruby laser ($\hbar\omega = 1.785$ eV), 1.19 for a neodymium laser ($\hbar\omega$

Table II

Ionized atom	λ	I, eV	$\hbar\omega$, eV	$\nu_0 = \frac{I}{\omega}$	k_0	k exp	Δk	k from (40)	F	γ	Experiment
Kr	0,99	14.00	1.18	11,86	12	9,12±0.13	2.9	10.6	7,8	3,6	[5]
			1.785	7.85	8	6,31±0.11	1,7	6.6	7.53	10	[2]
			2.36	5.93	6	5,38±0,15	0.6	4,6	7.45	16	[8]
Xe	1.06	12.13	1.18	10,3	11	8,8 ±0.19	2,2	9.5	7.85	3,0	[5]
			1,785	6.81	7	6,23±0.14	0,8	5.5	7.29	16	[1]
			1,785	6.81	7	5,88±0,14	1,1	5.5	7.73	5.9	[8]
			2.36	5.14	6	4.4 ±0.21	1.6	4,5	7.45	15	[8]
$H_2 \to H_2^+ + e$	0,94	15.43	1.785	8,65	9	7,67±0,36	1.3	7.7	7.16	25	[3]
I^-	0	3,076	1.785	1.72	2	2,24±?		2.0	∼5.5	∼400	[6]

Note. $F = 10^n$ V/cm; the table lists the values of the exponent n.

= 1.18 eV), and 1.68 for the second harmonic of a neodymium laser ($\hbar\omega$ = 2.36 eV).

The adiabaticity parameter γ is best determined from the formula

$$\gamma = F_1/F = \sqrt{J_1/J}, \tag{57}$$

where

$$F_1 = \frac{1}{2}\left(\frac{I}{I_0}\right)^{1/2}\frac{\omega}{\omega_0}F_0, \quad J_1 = \frac{I\omega}{4I_0\omega_0}J_0, \tag{58}$$

I and $I_0 = \hbar\omega_0 = 13.6$ eV are the ionization potentials of the given atom and of the hydrogen atom, $F_0 = m^2e^5/\hbar^4 = 5.14 \times 10^9$ V/cm, and

$$J_0 = \frac{c}{4\pi}\left(\frac{me^2}{\hbar^2}\right)^3 = 1.61\cdot 10^{34}\ \frac{\text{photons}}{\text{cm}^2\,\text{sec}}.$$

The quantity measured most accurately in experiment is at present not the absolute ionization probability w, but the "multiquantum exponent" of the process k.

Table II shows a summary of the values of k obtained in different experiments. In all cases $k < k_0$, and in most cases even $k < k_0 - 1.4\lambda$. This does not contradict (40) directly, since, as can be seen from Fig. 4, all the experimental points lie in the region $\sigma > \sigma_{cr}(\gamma)$, when the ionization process is not described by the sub-barrier trajectory considered in Sec. 3. It is surprising, however, that the lowering of the threshold $\Delta k = k_0 - k$ is particularly large for those points which are closest to the limiting curve $\sigma = \sigma_{cr}(\gamma)$. This shows that the results of Sec. 3 cannot be simply extrapolated into the region $\sigma > \sigma_{cr}$. For comparison of the available experimental data with the quasiclassical theory of ionization, it is necessary to obtain a quasiclassical solution in the region beyond the encounter of their roots, i.e., to find the sub-barrier trajectory satisfying the boundary condition (45)[8)].

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APPENDIX

Expanding the function $\Phi(\gamma, p)$ from (12) as $p \to 0$, we get

$$\Phi(\gamma, \mathbf{p}) = ia_1\frac{p_x}{\varkappa} - a_2\frac{p_x^2}{\varkappa^2} + a_3\frac{p_\perp^2}{\varkappa^2} + \ldots, \tag{A.1}$$

where we obtain for the coefficients a_i the expressions

$$a_1 = \frac{1}{2} - \frac{\operatorname{th}\tau_0}{\tau_0} + \int_0^{\tau_0}\left[\frac{1}{(\tau_0-\tau)^2} - (\tau_0 - \operatorname{th}\tau_0 - \tau)X^2\right]d\tau, \tag{A.2}$$

$$a_2 = \frac{1}{8} + \frac{1}{6}\operatorname{th}^2\tau_0 - \frac{1}{2}(1 + \tau_0\operatorname{th}\tau_0)\left(\frac{\operatorname{th}\tau_0}{\tau_0}\right)^2$$

$$- \int_0^{\tau_0}\left\{\frac{\operatorname{th}^2\tau_0}{(\tau_0-\tau)^3} + \frac{1}{2}\frac{\operatorname{th}^3\tau_0}{(\tau_0-\tau)^2} - (\tau_0 - \operatorname{th}\tau_0 - \tau)^2X^3\right.$$

$$\left.- \frac{1}{2}\operatorname{th}\tau_0(1 + \operatorname{th}^2\tau_0)X^2\right\}d\tau, \tag{A.3}$$

$$a_3 = \frac{1}{4} + \frac{1}{2}\int_0^{\tau_0}\left\{\operatorname{th}\tau_0\left[\frac{1}{(\tau_0-\tau)^2} - X^2\right] + (\tau_0-\tau)^2X^3\right\}d\tau. \tag{A.4}$$

Here

$$\tau_0 = \operatorname{Arsh}\gamma, \quad \operatorname{th}\tau_0 = \frac{\gamma}{\sqrt{1+\gamma^2}}, \quad X(\tau) = \frac{\operatorname{sh}\tau_0}{\operatorname{ch}\tau_0 - \operatorname{ch}\tau}. \tag{A.5}$$

All the integrals written out above converge. Nonetheless, the singularities contained in the individual terms of the integrands make it very difficult to calculate the coefficients a_i explicitly. To overcome this difficulty, we used the following identity:

$$\int_0^{\tau_0} d\tau\left[\frac{1}{\tau_0-\tau} - X(\tau)\right] = \ln\frac{\tau_0}{2\operatorname{sh}\tau_0}, \tag{A.6}$$

and also two others obtained from (A.6) by successively differentiating with respect to the parameter τ_0. With the aid of these equalities we can transform the initial expressions for a_i to the more convenient form

$$a_1 = 1 + \int_0^{\tau_0}\{(\tau_0-\tau)X^2 - X\}\,d\tau, \tag{A.7}$$

$$a_2 = 2\operatorname{th}\tau_0\int_0^{\tau_0}\left\{X^2 - \left[(\tau_0-\tau) - \frac{(\tau_0-\tau)^2}{2\operatorname{th}\tau_0}\right]X^3\right\}d\tau, \tag{A.8}$$

$$a_3 = \frac{1}{2}\left\{1 + \int_0^{\tau_0}[(\tau_0-\tau)^2X^3 - X]\,d\tau\right\}. \tag{A.9}$$

From this we can readily obtain asymptotic formulas for a_i in two limiting cases:

$$a_2 = a_2(\gamma) = \begin{cases} {}^5/_6\gamma^2 & \text{if } \gamma \ll 1 \\ {}^1/_3\tau_0^3(1 - 3/\tau_0 + 6/\tau_0^2 + \ldots) & \text{if } \gamma \gg 1 \end{cases}, \tag{A.10}$$

$$a_3 = a_3(\gamma) = \begin{cases} {}^7/_4 & \text{if } \gamma \ll 1 \\ {}^1/_6\tau_0^3(1 - 3/\tau_0^2 + \ldots) & \text{if } \gamma \gg 1 \end{cases}. \tag{A.11}$$

The determination of a_i in the region $\gamma \sim 1$ calls for numerical calculations in accordance with formulas (A.7)–(A.9).

8)The Coulomb correction is considered by a diagram technique in [20].

With the aid of formula (53) of [12] we get the momentum spectrum of the emitted electrons:

$$|F(\mathbf{p})|^2 = \frac{\varkappa}{(2\pi)^3}|C_{\varkappa l}|^2 \frac{\omega}{2I} \operatorname{th}\tau_0\left(\frac{2F_0}{F}\right)^{2\lambda} \times \exp\left\{-\frac{2I}{\omega}\left[f(\gamma) + a\frac{p_\perp^2}{\varkappa^2} + b\frac{p_x^2}{\varkappa^2}\right]\right\}. \quad (A.12)$$

here

$$f(\gamma) = \left(1 + \frac{1}{2\gamma^2}\right)\tau_0 - \frac{1}{2}\operatorname{cth}\tau_0, \quad \tau_0 = \operatorname{Arsh}\gamma,$$
$$a = \tau_0 - \frac{\lambda\omega}{I}a_3, \quad b = \tau_0 - \operatorname{th}\tau_0 + \frac{\lambda\omega}{I}a_2. \quad (A.13)$$

Owing to the factor $I/\omega \gg 1$, we get $p \ll \kappa$. This justifies the expansion presented in (A.1). Taking into account the connection between the probability of n-quantum ionization w_n and the function $|F(p)|^2$ (see formula (14) of [13]), and integrating with respect to the momentum, we arrive at formulas (13) and (14) of the present article, in which

$$\alpha = 2\min(a, b), \quad \beta = 2(a-b) = 2\left[\operatorname{th}\tau_0 - \frac{\lambda\omega}{I}(a_2 + a_3)\right],$$
$$\rho = \left|\frac{\beta}{2\operatorname{th}\tau_0}\right| = \left|1 - \frac{\lambda\omega(a_2+a_3)}{I\operatorname{th}\tau_0}\right| \quad (A.14)$$

When $\lambda \to 0$ we get $a = \tau_0$, $b = \tau_0 - \tanh\tau_0$, and therefore

$$\alpha = 2(\tau_0 - \operatorname{th}\tau_0), \quad \beta = 2\operatorname{th}\tau_0, \quad \rho = 1. \quad (A.15)$$

Using for a_i the asymptotic expansions (A.10) and (A.11), we obtain formulas (17)–(19).

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