# *ELASTO-TEMPERATURE WAVES IN SOLIDS*

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The operator is found for the interaction of a classically described sound wave with the quantized field of thermal oscillations (phonons). An expression is obtained for the complex elastic modulus tensor with account of the space-time dispersion brought about by the thermal oscillations. A detailed analysis is given of the interaction of elastic waves (first sound) and temperature waves (second sound) in solids. The resulting renormalization of the velocities of first and second sound and their attenuation are calculated. The temperature dependence of these characteristics is analyzed. The parameters of the suggested theory are the harmonic and anharmonic constants of the continuous medium.

## 1. INTRODUCTION

ALL problems associated with the propagation and absorption of electromagnetic waves in matter are usually solved by means of Maxwell's equations. As the fundamental characteristic of the electromagnetic properties of the medium we have the complex dielectric tensor $\epsilon_{\alpha\beta}(\Omega, q)$ ($\Omega$ is the frequency of the wave and $q$ its wave vector).

In complete analogy with this, the problems of the propagation and absorption of sufficiently longwave sound waves ($q \ll a^{-1}$, $a$ being the lattice constant) in unbounded solids should be solved by means of the equations of elasticity theory. Here the fundamental characteristic of all the elastic properties of the medium is the complex elastic modulus tensor $\Lambda_{\alpha\beta\gamma\delta}(\Omega, q)$. Precisely this problem was posed in the research of Silin,[1] which was devoted to the study of the absorption of ultrasound by the electrons of a metal at high frequencies, when their collisions can be neglected. Silin found, the complex contribution to the elastic modulus tensor. This contribution determines both the absorption and the dispersion of the sound. In the work of Kontorovich,[2] the scattering of conduction electrons was taken into account in the relaxation time approximation, and also the difference of the properties of the quasimomentum of the conduction electrons from the momentum of free electrons. The ideas underlying these researches can be used in the solution of other problems. In[1,2] the sound was absorbed by a gas of quasiparticles—electrons. In order to take into account the ultrasonic absorption in a dielectric brought about, for example, by the anharmonism or by magnetostriction, by analogy with[1,2], it suffices to take into account the interaction of the sound with the quasi-particles—the thermal Debye phonons or magnons.

In our previous work,[3] the method of Silin[1] was used for the study of the absorption in dielectric of high-frequency sound with frequency $\Omega \gg \tau_N^{-1}$, $\tau_U^{-1}$ ($\tau_N^{-1}$ and $\tau_U^{-1}$ are relaxation frequencies corresponding to the normal collisions of thermal phonons and collisions with momentum loss, for example, in Umklapp processes or in scattering from impurities). In the present work, the results of[1-3] are used for the study of the absorption and dispersion of sound in its interaction with thermal phonons both in the region of low frequencies and in the region of high frequencies. The formulas found by us in the relaxation time approximation are formally valid for any values of $\Omega\tau$. It can be shown that they are asymptotically exact only in two limiting cases: a) $\Omega\tau_N \gg 1$, $\Omega\tau_U \gg 1$ (see[3]) and b) $\Omega\tau_N \ll 1$ (see the work of Gurzhi[4]), i.e., as $\tau \to \infty$ and $\tau \to 0$. In these two limiting cases, our formulas take into exact account the symmetry of the elastic properties of the medium. In all intermediate cases introduced into consideration, the variational parameters $\vartheta(r, t)$ and $v(r, t)$ (see Eq. (3.10)), which permit us to satisfy the laws of conservation of energy and momentum, and also the approximation of the collision integrals by relaxation times, make it possible to obtain only interpolation formulas, which can give only qualitatively correct results.

The region of oscillation frequencies $\tau_U^{-1} \ll \Omega \ll \tau_N^{-1}$ is of interest. Under these conditions, a new branch of oscillations arises—temperature waves or second sound. Actually, for $\Omega < \tau_N^{-1}$, it is possible to establish a local equilibrium in the field of the propagating wave in a system of quasiparticles with a temperature dependent on the coordinates and on the time. Here, if we can neglect the loss of the collective macroscopic momentum of the quasiparticles over the period of oscillation $1/\Omega$ ($\Omega \gg \tau_U^{-1}$), then collective oscillations appear in the gas of quasiparticles (second sound). Such oscillations in dielectrics differ from elastic waves, since they are accompanied by weakly damped oscillations of the local macroscopic velocity of the quasiparticles.

A detailed study of the conditions for the appearance of such temperature waves in a system of quasiparticles, and also the damping brought about by the collisions of the quasiparticles, have been set forth in the paper of Gurzhi.[4]

In the present work, the interaction is considered between ordinary sound and the temperature waves, leading to coupled elasto-temperature waves. In contrast with the work of Kwok and Martin[5], where only longitudinal elasto-temperature waves were considered, we have obtained an expression for the complex elastic modulus tensor with account of space-time dispersion, which is asymptotically exact for $\Omega\tau_U \gg 1$, $\Omega\tau_N \gg 1$

and $\tau_U^{-1} \ll \Omega \ll \tau_N^{-1}$. The case of the propagation in an isotropic medium of both longitudinal and transverse waves is studied in detail; a specific dependence of the characteristics of these waves on all the elastic and anharmonic constants and the temperature has been found.[1)]

## 2. ENERGY OF INTERACTION OF SOUND WITH THERMAL PHONONS AND THE EQUATION OF MOTION OF THE LATTICE

The interaction of sound propagating in a medium with thermal phonons is brought about by the anharmonism of the oscillations. In the study of dispersion and absorption of sound waves whose wavelength is long in comparison with the lattice constant, one can restrict oneself to the approximation of a continuous medium. Starting out from the specific properties of the symmetry of a continuous medium, one can write down an expression for the energy with account of the anharmonism of third order. In this approximation, there arise both sound absorption and a dependence of the sound velocity on the temperature. The amplitude of the displacement of the lattice can be represented as the sum of the displacement brought about by the propagating sound and the displacement brought about by the thermal motion. Substituting such an expansion in the expression for the energy, with account of anharmonism of third order, one can separate out the part that is linear in the sound amplitude and quadratic in the amplitude of temperature oscillations. If the amplitude of thermal oscillations is quantized according to the correspondence principle, then, in accord with[3] for the case of an isotropic medium, we find the following formula which describes the interaction of sound with phonons:

$$\hat{H}_{\text{int}} = {}^1/_2\hbar \sum_{\nu\nu'} M_{\alpha\beta}(\nu, \nu') \{u_{\alpha\beta}(\mathbf{k}-\mathbf{k}') b_{\nu'}^+ b_\nu + u_{\alpha\beta}(-\mathbf{k}+\mathbf{k}') b_\nu b_{\nu'}^+ - u_{\alpha\beta}(-\mathbf{k}+\mathbf{k}') b_{\nu'}^+ b_\nu^+ - u_{\alpha\beta}(\mathbf{k}+\mathbf{k}') b_\nu b_{\nu'}\}, \quad (2.1)$$

where $u_{\alpha\beta} = \partial u_\alpha / \partial x_\beta$; $u(r, t)$ is the displacement vector, due to the sound wave;

$$u_{\alpha\beta}(\mathbf{k}) = V^{-1} \int d\mathbf{r}\, u_{\alpha\beta} e^{i\mathbf{k}\mathbf{r}};$$

$b_\nu^+$ and $b_\nu$ are the creation and annihilation operators of thermal phonons; $\nu \equiv \mathbf{k}, \lambda$ is the set of quantum numbers characterizing the state of the phonon; $\mathbf{k}$ is the propagation vector, $\lambda$ the index of polarization.

The form of the tensor $M_{\alpha\beta}(\nu, \nu')$ is determined by the symmetry of the medium. For the special case of an isotropic medium, the method of construction of such a tensor is given in detail in[2]. It is obvious that this method also remains in force for a medium with arbitrary symmetry.

By taking into account the given formula, we represent that part of the energy operator which depends on u in the following form:

$$\hat{H}_{\text{ac}} = \hat{H}_0 + \hat{H}_{\text{int}}; \quad \hat{H}_0 = {}^1/_2 \int d\mathbf{r}\, \{\Lambda^{(0)}_{\alpha\beta\gamma\delta} u_{\alpha\beta} u_{\gamma\delta} + \rho \dot{\mathbf{u}}^2\}, \quad (2.2)$$

$\Lambda^{(0)}_{\alpha\beta\gamma\delta}$ is the elastic modulus tensor in the absence of thermal motion, the mass density of the lattice. Averaging (2.2) over the Gibbs canonical ensemble with the energy operator of free phonons

$$\hat{H}_{p^0} = \sum_\nu \varepsilon_\nu b_\nu^+ b_\nu, \quad (2.3)$$

and then applying the variational principle of Hamilton, we find

$$\rho \ddot{u}_\alpha - (\Lambda^{(0)}_{\alpha\beta\gamma\delta}) \frac{\partial^2 u_\delta}{\partial x_\beta \partial x_\gamma} = F_\alpha, \quad (2.4)$$

where

$$F_\alpha = \frac{\partial}{\partial x_\beta} \sum_{\nu,\nu'} \hbar M_{\alpha\beta}(\nu, \nu') \{ e^{i(\mathbf{k}-\mathbf{k}')\mathbf{r}} N_{\nu\nu'} + e^{-i(\mathbf{k}-\mathbf{k}')\mathbf{r}} (N_{\nu\nu'} + \delta_{\nu\nu'}) - e^{-i(\mathbf{k}+\mathbf{k}')\mathbf{r}} L^+_{\nu\nu'} - e^{i(\mathbf{k}+\mathbf{k}')\mathbf{r}} L_{\nu\nu'}\}, \quad (2.5)$$

$$N_{\nu\nu'} = \langle b_\nu^+ b_{\nu'} \rangle, \quad L^+_{\nu\nu'} = \langle b_\nu^+ b_{\nu'}^+ \rangle, \quad L_{\nu\nu'} = \langle b_\nu b_{\nu'} \rangle,$$
$$\langle \ldots \rangle = \mathrm{Sp}\,(\ldots, e^{-\hat{H}_{p^0}/T}) / \mathrm{Sp}\,(e^{-\hat{H}_{p^0}/T}). \quad (2.6)$$

We note here that the equation of motion of the lattice (2.4) is not exact if there exist processes of scattering of the thermal phonons with loss of momentum (scattering from impurities and defects, Umklapp processes). In this case, the total momentum of the phonons and the lattice is conserved. Therefore, on the right hand side of (2.4), along with $F_\alpha$ there should be a force $(F_{st})_\alpha$, brought about by the loss of momentum of the thermal phonons in collisions. In order of magnitude, this force will be small and it need not be considered if $\Omega\tau_U \gg 1$ ($\tau_U^{-1}$ is the characteristic frequency of relaxation of thermal phonons with loss of momentum). It is assumed everywhere below that this inequality is satisfied.

## 3. DENSITY MATRIX OF THERMAL PHONONS

We shall describe the thermal phonons by the Hamiltonian

$$\hat{H}_p = \hat{H}_{p^0} + \hat{H}_{\text{int}} + \hat{H}_{pp} + \hat{H}_{imp}, \quad (3.1)$$

where $\hat{H}_p^0$ and $\hat{H}_{\text{int}}$ are determined in (2.3) and (2.1), $H_{pp}$ describes the interaction between the thermal phonons, $H_{imp}$ is the scattering of phonons by impurities and defects.

Using the equation of motion of the operators, we find with the aid of (3.1), and after averaging over the canonical ensemble, the following equation for $N_{\nu\nu'}$:

$$\left( i\hbar \frac{\partial}{\partial t} + \varepsilon_{\nu'} - \varepsilon_\nu \right) N_{\nu\nu'} - \frac{1}{2} \sum_{\varkappa\varkappa'} \hbar M_{\alpha\beta}(\varkappa, \varkappa') [u_{\alpha\beta}(k_\varkappa - k_{\varkappa'}) + u_{\alpha\beta}(-k_\varkappa + k_{\varkappa'})] [\delta_{\varkappa\nu} N_{\varkappa'\nu'} - \delta_{\nu'\varkappa'} N_{\nu\varkappa}] = \langle [(\hat{H}_{pp} + \hat{H}_{imp}),\ b_\nu^+ b_{\nu'}]_- \rangle. \quad (3.2)$$

In this equation, terms containing $\langle b_\kappa^+ b_\kappa^+ \rangle$ and $\langle b_\kappa b_{\kappa'} \rangle$ are neglected. Such terms in (3.2) are quantities of second order in $u_{\alpha\beta}$.

On the right side of (3.2) are the averages of the product of three operators, i.e., Eq. (3.2) is not closed. For these averages of the product of three operators one can write their own equation of motion, and so on. We proceed as is usually done in the method of

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1)After this paper had been written and gone to press, the researches [6-8] were published, which also were devoted to the consideration of elasto-temperature waves. The results of these researches only partially overlap our own.

Bogolyubov, i.e., we form an infinite chain and express the right side in (3.2) in terms of the matrix $N_{\nu\nu'}$. Completion of such a program leads to the following kinetic equation for $N_{\nu\nu'}$:

$$\left(i\hbar\frac{\partial}{\partial t}+\varepsilon_{\nu'}-\varepsilon_{\nu}\right)N_{\nu\nu'}-\frac{1}{2}\sum_{\varkappa,\varkappa'}\hbar M_{\alpha\beta}(\varkappa,\varkappa')[u_{\alpha\beta}(k_\varkappa-k_{\varkappa'})$$
$$-u_{\alpha\beta}(-k_\varkappa+k_{\varkappa'})][\delta_{\varkappa\nu}N_{\varkappa'\nu'}-\delta_{\nu'\varkappa'}N_{\nu\varkappa}]=I_N[N_{\nu\nu'}]+I_U[N_{\nu\nu'}], \quad (3.3)$$

where $I_N[N]$ is an integral describing the normal collisions between phonons (with conservation of energy and momentum), while $I_U[N]$ is an integral describing the collisions with loss of momentum (scattering from impurities and defects and from boundaries, and Umklapp processes). We shall not write out here the explicit form of the collision integrals, inasmuch as in what follows we shall approximate them by a relaxation time.

We note that the nondiagonal matrices $N_{\nu\nu'}$ describe the spatially inhomogeneous distributions. It is not difficult to establish this fact if we transform to a displaced representation of the density matrix (the Wigner representation) according to the formula

$$N_{\lambda,\lambda'}(\mathbf{k},\mathbf{r})=\sum_{\mathbf{k}'}e^{-i(\mathbf{k}-\mathbf{k}')\mathbf{r}}N_{\nu\nu'}. \quad (3.4)$$

We shall seek a solution of Eq. (3.3) in the form

$$N_{\nu\nu'}=N^0_{\nu\nu'}+f_{\nu\nu'}. \quad (3.5)$$

In the simultaneous account of spatial inhomogeneities of the phonon distribution it is not possible to choose $N_{\nu\nu'}$ in (3.5) in the form of a thermodynamic-equilibrium Planck function i.e., $N^0_{\nu\nu'}=N_0(\epsilon_\nu/T)$ ($N_0(\epsilon_\nu/T)$ is the Planck function). Such a choice of $N^0_{\nu\nu'}$ in (3.5) leads to the result that the solution will not satisfy the local laws of energy and momentum conservation, and for systems with a constant number of particles—the equation of continuity. In order to satisfy the local conservation laws, it is necessary to put $N^0_{\nu\nu'}$ in a more general form, i.e., to find the more general solution of the equation

$$I_N[N^0_{\nu\nu'}]+I_U[N^0_{\nu\nu'}]=0. \quad (3.6)$$

Two limiting cases are possible, namely:[2)]

$$I_N[N]\gg I_U[N], \quad (3.7)$$
$$I_U[N]\gg I_N[N]. \quad (3.8)$$

In the first of these cases, we can find $N^{0\,\nu\nu'}$ approximately from the equation

$$I_N[N^0]\approx 0. \quad (3.9)$$

Inasmuch as for normal collisions, momentum and energy of the phonons are conserved, the most general solution of Eq. (3.9) is the operator $\hat{N}^0$ which corresponds to local equilibrium with local temperature $T(\mathbf{r},t)=T(1+\vartheta(\mathbf{r},t))$ and the local macroscopic velocity of phonons $\mathbf{v}(\mathbf{r},t)$. This operator (assuming $\vartheta(\mathbf{r},t)$ and $\mathbf{v}(\mathbf{r},t)$ to be small) can be represented in the form

$$N^0=N^0(H^*/T), \quad (3.10)$$

where $N^0(x)$ is the Planck function and the operator

$$H^*\equiv H_p+H_{\rm int}-\{\hat{\mathbf{p}},\mathbf{v}(\mathbf{r},t)\}_+-\{H_p{}^0,\vartheta\}_+ \quad (3.11)$$

can be assumed to be the effective Hamiltonian of the phonon, defining local equilibrium; $\hat{\mathbf{p}}$ is the momentum operator and $\{A,B\}_+=(AB+BA)/2$.

We note that the parameters $\mathbf{v}$ and $\vartheta$ have the direct physical meaning of the local velocity and temperature only in the case in which the characteristic frequency of the macroscopic process $\Omega\ll\tau_N^{-1}$. In the case of intermediate frequencies ($\Omega\sim\tau_N^{-1}$) $\mathbf{v}$ and $\vartheta$ are auxiliary variational parameters, which allow us to satisfy the local conservation laws for energy and momentum; finally, in the case of high frequencies, $\Omega\gg\tau_N^{-1}$, these parameters fall out of the solution, together with $\tau_N^{-1}\to 0$. The values of the parameters $\mathbf{v}(\mathbf{r},t)$ and $\vartheta(\mathbf{r},t)$ are determined from the equations of conservation of momentum and energy. To obtain such equations, we multiply (3.3) successively by the matrix elements of the momentum density operator $\mathbf{P}_{\nu\nu'}$ and the energy density operator $H_{\nu\nu'}$, and then take the trace. This gives

$$\sum_{\nu\nu'}\left\{\left(i\hbar\frac{\partial}{\partial t}+\varepsilon_{\nu'}-\varepsilon_{\nu}\right)N_{\nu\nu'}-\frac{1}{2}\sum_{\varkappa\varkappa'}\hbar M_{\alpha\beta}(\varkappa,\varkappa')[u_{\alpha\beta}(k_\varkappa-k_{\varkappa'})\right.$$
$$\left.+u_{\alpha\beta}(-k_\varkappa+k_{\varkappa'})](\delta_{\varkappa\nu}N_{\varkappa'\nu'}-\delta_{\varkappa'\nu'}N_{\nu\varkappa})-I_U[N_{\nu\nu'}]\right\}\mathbf{P}_{\nu'\nu}=0,$$
$$\sum_{\nu,\nu'}\left\{\left(i\hbar\frac{\partial}{\partial t}+\varepsilon_{\nu'}-\varepsilon_{\nu}\right)N_{\nu\nu'}-\frac{1}{2}\sum_{\varkappa\varkappa'}\hbar M_{\alpha\beta}(\varkappa,\varkappa')[u_{\alpha\beta}(k_\varkappa-k_{\varkappa'})\right.$$
$$\left.+u_{\alpha\beta}(-k_\varkappa+k_{\varkappa'})](\delta_{\varkappa\nu}N_{\varkappa'\nu'}+\delta_{\varkappa'\nu'}N_{\nu\varkappa})\right\}H_{\nu\nu'}=0. \quad (3.12)$$

To obtain these equations, it is taken into account that

$$\sum_{\nu\nu'}\{I_N[N_{\nu\nu'}]+I_U[N_{\nu\nu'}]\}H_{\nu'\nu}=0,$$
$$\sum_{\nu\nu'}\{I_N[N_{\nu\nu'}]\}\mathbf{P}_{\nu\nu'}=0. \quad (3.13)$$

In (3.11), (3.12), it is necessary to substitute the solution of Eq. (3.3), which is linear in $u_{\alpha\beta}$, $\mathbf{v}$ and $\vartheta$; we then get a set of equations with the help of which we express $\mathbf{v}$ and $\vartheta$ in terms of $u_{\alpha\beta}$. Equations (3.12) are the condition of solvability of the kinetic equation (3.3) for the nonequilibrium contribution to the density matrix $f_{\nu\nu'}$. These equations, upon substitution in them of $N_{\nu\nu'}=N^0_{\nu\nu'}$ from (3.14), are identical with the corresponding equations of the work of Gurzhi,[4] if we neglect the interaction between the first and second sound, which is described by the tensor $M_{\alpha\beta}$ which was not considered in[4].

We now write down the matrix element of the operator $\hat{N}^0$ entering into (3.5). From (3.10) we find in the linear approximation in $\vartheta$, $u_{\alpha\beta}$, and $\mathbf{v}$

$$N^0_{\nu\nu'}=N_\nu{}^0\delta_{\nu\nu'}+\frac{N_{\nu'}{}^0-N_\nu{}^0}{\varepsilon_{\nu'}-\varepsilon_\nu}[(\hat{H}_{\rm int})_{\nu\nu'}-(\{\hat{\mathbf{p}},\mathbf{v}\}_+)_{\nu\nu'}-(\{\hat{H}_p{}^0,\vartheta\}_+)_{\nu\nu'}], \quad (3.14)$$

where $N^0_\nu=N_0(\epsilon_\nu/T)$. We further approximate the collision integrals by relaxation times. For normal collisions, in accord with (3.9), (3.14) and (3.5), we set

$$I_N[N^0_{\nu\nu'}+f_{\nu\nu'}]=-i\hbar f_{\nu\nu'}/(\tau_N)_{\nu\nu'}. \quad (3.15)$$

The collision integral with momentum losses in the

2) A separate communication will be devoted to consideration of the case (3.8).

matrixes $N^0_{\nu\nu'}$ from (3.14) does not vanish for $v \neq 0$ (see, for example, [9]); therefore we get

$$I_U[N^0_{\nu\nu'} + f_{\nu\nu'}] = i\hbar\left[\frac{N_{\nu'}^0 - N_\nu^0}{\varepsilon_{\nu'} - \varepsilon_\nu}(\{\hat{\mathbf{p}}, \mathbf{v}\}_+)_{\nu\nu'} - f_{\nu\nu'}\right] / (\tau_U)_{\nu\nu'}. \quad (3.16)$$

In the case of a steady-state wave process, one can set $u_{\alpha\beta} \sim \mathbf{v} \sim \vartheta \sim e^{-i\Omega t + i\mathbf{q}\cdot\mathbf{r}}$. Taking this into account and Eqs. (3.5), (3.14)–(3.16) from (3.3) in the linear approximation in $u_{\alpha\beta}$, $\vartheta$, and $\mathbf{v}$

$$N_{\nu\nu'} = N_\nu^0\delta_{\nu\nu'} + \frac{N_{\nu'}^0 - N_\nu^0}{\varepsilon_{\nu'} - \varepsilon_\nu}\Big\{W^{(1)}_{\nu\nu'}(\mathbf{q}) - \frac{\Omega W^{(1)}_{\nu\nu'}(q) + i(\mathbf{P}_{\nu\nu'}\mathbf{v})/(\tau_N)_{\nu\nu'} + iH^0_{\nu\nu'}\vartheta/\tau_{\nu\nu'}}{\omega_{\nu'} - \omega_\nu + \Omega + i\tau^{-1}_{\nu\nu'}}\Big\}. \quad (3.17)$$

Here

$$W^{(1)}_{\nu\nu'}(q) = \hbar M_{\alpha\beta}(\nu\nu')u_{\alpha\beta}(\mathbf{q})\delta_{\mathbf{k}'-\mathbf{k}+\mathbf{q}},$$

$$\mathbf{P}_{\nu\nu'} = \langle\nu|\{e^{i\mathbf{q}\mathbf{r}}, \hat{\mathbf{p}}\}_+|\nu'\rangle = \frac{\hbar}{2}(\mathbf{k}+\mathbf{k}')\delta_{\mathbf{k}'-\mathbf{k}+\mathbf{q}},$$

$$H^0_{\nu\nu'} = \langle\nu|\{e^{i\mathbf{q}\mathbf{r}}, \hat{H}^0\}_+|\nu'\rangle = {}^1/_2(\varepsilon_{\nu'} + \varepsilon_\nu)\delta_{\mathbf{k}'-\mathbf{k}+\mathbf{q}},$$

$$\omega_\nu = \frac{\varepsilon_\nu}{\hbar}, \quad \tau^{-1}_{\nu\nu'} = \frac{1}{(\tau_N)_{\nu\nu'}} + \frac{1}{(\tau_U)_{\nu\nu'}}. \quad (3.18)$$

The matrices $\overset{+}{L}_{\nu\nu''}$ and $L_{\nu\nu'}$ are found in similar fashion. Omitting the calculations, we write out only the result here:

$$\overset{+}{L}_{\nu\nu'} = \left(-1 + \frac{\hbar\Omega}{\varepsilon_{\nu'} + \varepsilon_\nu - \hbar\Omega - i\hbar/\tau_{\nu\nu'}}\right)\frac{1 + N_{\nu'}^0 + N_\nu^0}{\varepsilon_\nu + \varepsilon_{\nu'}} \cdot \hbar M_{\alpha\beta}(\nu\nu')u_{\alpha\beta}(\mathbf{q})\delta(\mathbf{k}+\mathbf{k}'-\mathbf{q}), \quad (3.19)$$

$$L_{\nu\nu'} = \left(-1 - \frac{\hbar\Omega}{\varepsilon_{\nu'} + \varepsilon_\nu + \hbar\Omega + i\hbar/\tau_{\nu\nu'}}\right)\frac{1 + N_{\nu'}^0 + N_\nu^0}{\varepsilon_{\nu'} + \varepsilon_\nu} \cdot \hbar M_{\alpha\beta}(\nu\nu')u_{\alpha\beta}(\mathbf{q})\delta(\mathbf{k}+\mathbf{k}'+\mathbf{q}), \quad (3.20)$$

Substituting (3.17) in (3.11) and (3.12), we find

$$A_{\alpha\beta}u_\beta + B_\alpha\vartheta + C_{\alpha\beta}v_\beta = 0, \quad D_\beta u_\beta + G\vartheta + B_\beta v_\beta = 0, \quad (3.21)$$

where

$$A_{\alpha\beta} = \hbar\Omega q_\gamma \sum_{\nu\nu'} \varphi_{\nu\nu'}M_{\gamma\beta}(\nu\nu')(P_\alpha^*)_{\nu\nu'},$$

$$B_\alpha = i\hbar^{-1}\sum_{\nu\nu'}\varphi_{\nu\nu'}(\varepsilon_{\nu'} - \varepsilon_\nu + \hbar\Omega)H_{\nu\nu'}(P_\alpha^*)_{\nu'\nu},$$

$$C_{\alpha\beta} = \hbar\Omega q_\gamma\sum_{\nu\nu'}\varphi_{\nu\nu'}(\varepsilon_{\nu'} - \varepsilon_\nu + \hbar\Omega + i\hbar/(\tau_U)_{\nu\nu'})(P_\beta)_{\nu\nu'}(P_\alpha^*)_{\nu'\nu},$$

$$D_\alpha = \hbar\Omega q_\gamma\sum_{\nu\nu'}\varphi_{\nu\nu'}((\tau_N)_{\nu\nu'}/\tau_{\nu\nu'})M_{\gamma\alpha}(\nu\nu')H^*_{\nu'\nu},$$

$$G = i\hbar^{-1}\sum_{\nu\nu'}((\tau_N)_{\nu\nu'}/\tau_{\nu\nu'})(\varepsilon_{\nu'} - \varepsilon_\nu + \hbar\Omega)H_{\nu'\nu}H^*_{\nu\nu'},$$

$$\varphi_{\nu\nu'} = \frac{N_{\nu'}^0 - N_\nu^0}{\varepsilon_{\nu'} - \varepsilon_\nu}\,\frac{-i\hbar/(\tau_N)_{\nu\nu'}}{\varepsilon_{\nu'} - \varepsilon_\nu + \hbar\Omega + i\hbar/\tau_{\nu\nu'}}. \quad (3.22)$$

Equations (3.21) agrees in accuracy with the corresponding equations from the work of Gurzhi[4], if we set $M_{\alpha\beta} = 0$ here (not taking into account the interaction between first and second sound), which also means $A_{\alpha\beta} = D_\beta = 0$, and setting $\tau = 0$ (not taking into account departures from local equilibrium i.e., $f_{\nu\nu'} = 0$.

Using (3.17), (3.19), and (3.20), we can find an expression for the forces (2.5) exerted by the phonons on the lattice. If we substitute this force in (2.4), then we get for the Fourier component $u_\alpha$ of the lattice vibrations the equation

$$(-\rho\Omega^2\delta_{\alpha\beta} + \overset{(0)}{\Lambda}_{\alpha\gamma\delta\beta}q_\gamma q_\delta + \overset{(1)}{\Lambda}_{\alpha\gamma\delta\beta}q_\gamma q_\delta)u_\beta = i\Omega^{-1}(D_\alpha\vartheta + A_{\alpha\beta}v_\beta), \quad (3.23)$$

where

$$\overset{(1)}{\Lambda}_{\alpha\gamma\delta\beta} = \hbar^2\sum_{\nu\nu'}M_{\alpha\gamma}(\nu\nu')M_{\delta\beta}(\nu\nu')\Big\{\Big(1 - \frac{\hbar\Omega}{\varepsilon_{\nu'} - \varepsilon_\nu + \hbar\Omega + i\hbar\tau^{-1}_{\nu\nu'}}\Big) \times \frac{N_{\nu'}^0 - N_\nu^0}{\varepsilon_{\nu'} - \varepsilon_\nu}\delta_{\mathbf{k}'-\mathbf{k}+\mathbf{q}} - \frac{1}{2}\Big(2 - \frac{\hbar\Omega}{\varepsilon_{\nu'} + \varepsilon_\nu + \hbar\Omega + i\hbar\tau^{-1}_{\nu\nu'}} + \frac{\hbar\Omega}{\varepsilon_{\nu'} + \varepsilon_\nu - \hbar\Omega - i\hbar\tau^{-1}_{\nu\nu'}}\Big)\frac{1 + N_{\nu'}^0 + N_\nu^0}{\varepsilon_{\nu'} + \varepsilon_\nu}\delta_{\mathbf{k}'+\mathbf{k}-\mathbf{q}}\Big\} \quad (3.24)$$

are contributions to the elastic modulus tensor, due to interaction of the sound with the thermal phonons.

We eliminate $\vartheta$ and $\mathbf{v}$ from (3.23). With the help of (3.21) we find

$$(-\rho\Omega^2\delta_{\alpha\beta} + \Lambda_{\alpha\gamma\delta\beta}q_\gamma q_\delta)u_\beta = 0, \quad (3.25)$$

where

$$\Lambda_{\alpha\beta\gamma\delta} = \overset{(0)}{\Lambda}_{\alpha\beta\gamma\delta} + \overset{(1)}{\Lambda}_{\alpha\beta\gamma\delta} + \overset{(2)}{\Lambda}_{\alpha\beta\gamma\delta},$$

$$\overset{(2)}{\Lambda}_{\alpha\gamma\delta\beta}q_\gamma q_\delta = i\Omega^{-1}(G - B_{\beta'}B'_{\beta'})^{-1} \times\{(D_\alpha - A_{\alpha\beta'}B_{\beta'}')(D_\beta - B_{\beta'}A'_{\beta'\beta}) + A_{\alpha\beta'}A'_{\beta'\beta}(G - B_{\beta'}B_{\beta'}')\}, \quad (3.26)$$

$$B_\alpha' = C_{\alpha\beta}{}^{-1}B_\beta, \quad A_{\alpha\beta}' = C_{\alpha\gamma}{}^{-1}A_{\gamma\beta}, \quad C_{\alpha\gamma}{}^{-1}C_{\gamma\beta} = \delta_{\alpha\beta}. \quad (3.27)$$

Additional renormalization of the elastic modulus tensor is connected with the interaction of sound oscillations with the ordered motion of the phonons.

The equation for the eigenfrequencies follows from (3.25):

$$\det(-\rho\Omega^2\delta_{\alpha\beta} + \operatorname{Re}\Lambda_{\alpha\gamma\delta\beta}q_\gamma q_\delta) = 0, \quad (3.28)$$

and the formula for the damping decrement is

$$\Gamma = \frac{1}{2\rho\Omega}(\operatorname{Im}\Lambda_{\alpha\gamma\delta\beta}q_\gamma q_\delta\bar{u}_\alpha\bar{u}_\beta), \quad (3.29)$$

where $\bar{u}_\alpha$ and $\bar{u}_\beta$ are unit polarization vectors of the propagating sound wave. In the next section, we apply these formulas to the case of an isotropic medium.

## 4. RENORMALIZATION OF THE VELOCITY AND THE SOUND DAMPING DECREMENT IN AN ISOTROPIC MEDIUM

We now turn to the solution of the dispersion equation (3.28) and the calculation of the damping decrement (3.29) in the case of the sound propagation in an isotropic medium. The form of the tensor $M_{\alpha\beta}(\nu\nu')$ for such a medium is given in[2]. As a final result, we express all the quantities in terms of the elastic moduli K, $\mu$, three anharmonic constants A, B, C,[10] and the characteristics of thermal phonons. In what follows, we shall assume the relaxation time of thermal phonons to be independent of the wave numbers, but different for different polarizations. We also recall the range of frequencies $\tau^{-1}_{U\lambda} < \Omega < \tau^{-1}_{N\lambda}$ is considered.

Longitudinal waves. We direct the $x_3$ axis along the direction of propagation of the wave ($\mathbf{q} \parallel x_3$). Using the explicit form of the tensor $M_{\alpha\beta}(\nu\nu')$, we can show that, of all the coefficients that differ from zero, only $A_{33}$, $B_3$, $C_{33}$, $D_3$ and G remain. The dispersion equation here is materially simplified:

$$-\rho\Omega^2 + \overset{0}{\Lambda}_{3333}q^2 + \overset{(1)}{\Lambda}_{3333}q^2 - \frac{i\hbar^2\Omega^{-1}}{C_{33}G - B_3^2}(2A_{33}B_3D_3 - A_{33}^2G - D_3^2C_{33}) = 0. \quad (4.1)$$

First of all, we turn our attention to the fact that the last term in (4.1) has a singularity for $C_{33}G - B_3^2 = 0$. This pole, in the case considered by us, $\tau_{U\lambda}^{-1} < \Omega < \tau_{N\lambda}^{-1}$ gives a new branch of oscillations—second sound. In the case $\Omega < \tau_N^{-1} < \tau_U^{-1}$, the pole is shifted along the imaginary axis and the additional solutions are associated with the ordinary thermal conductivity.

We rewrite Eq. (4.1) in a different form:

$$(\Omega^2 - s_1^2 q^2 + 2i\Gamma_1\Omega)(\Omega^2 - s_2^2 q^2 - 2i\Gamma_2\Omega) - s_{12}^2 q^2(\Omega^2 + 2i\Gamma D) = 0. \tag{4.2}$$

Here $s_1$, $\Gamma_1$ and $s_2$, $\Gamma_2$ are the velocity and damping decrements of waves of first and second sound, respectively, equal to

$$s_1^2 = (K + {}^4/_3\mu - \Lambda_3')/\rho, \quad \Gamma_1 = \Lambda_3'' q^2/2\rho, \quad s_2^2 = \frac{C_V}{3\sum_\lambda C_{V\lambda}/s_\lambda^2}, \tag{4.3}$$

$$\Gamma_2 = \frac{3}{2C_V}\sum_\lambda C_{V\lambda}\left\{\frac{s_2^2}{\tau_{U\lambda}s_\lambda^2} + \tau_{N\lambda}\left(\frac{s_\lambda^2}{9s_2^2} + \frac{s_2^2}{s_\lambda^2} - \frac{2}{5}\right)\Omega^2\right\}, \tag{4.4}$$

where $s_\lambda$ is the phase velocity of thermal phonons with polarizations $\lambda$; $C_{V\lambda}$ is the specific heat of the phonons:

$$C_V = \sum_\lambda C_{V\lambda}, \quad C_{V\lambda} = \frac{T^3}{2\pi^2\hbar^3 s_\lambda^3}\int_0^{\Theta/T} e^z z^4 (e^z - 1)^{-2} dz,$$

and $\Lambda_3'$ and $\Omega\Lambda_3''$ are the real and imaginary contributions to the elastic modulus tensor $\Lambda_{3333}^{(1)}$. Equations for $s_2$ and $\Gamma_2$ are identical with the well-known expressions for the velocity and damping decrement of second sound.[4,5]

The complex contribution to the elastic modulus tensor can be expanded in a series in $\tau_{N\lambda}\Omega \ll 1$. Terms of zeroth order make a contribution only to the real part of the tensor. In this approximation, which does not take into account the dispersion of the tensor, both the scattering processes $(l + \lambda \to \lambda')$ and the decay processes $(l \to \lambda + \lambda')$ play an identical role. In the next approximation (linear in $\tau_{N\lambda}\Omega$), in addition to the contribution to the real part, there appears an imaginary contribution which corresponds to dissipation processes. In this approximation, the dispersion of the tensor is taken into account, and the role of the processes of scattering and decay is seen to be different. The most important processes for $\tau_N\Omega < 1$ are the scattering processes, since for $\tau\Omega > 1$ and $\hbar\Omega \gg T$, the principal contribution to the scattering of the longitudinal wave are made by the decay processes.[11]

It follows from what has been set forth that the dispersion of the elastic modulus tensor is more sensitive to the various processes of interaction of the sound with the thermal phonons. Therefore, for the explanation of the role of these or other interactions, it is necessary to study the dispersion of the elastic modulus tensor. Here, only the dispersion of the imaginary part is considered. The study of the dispersion of the real part, i.e., the dispersion of the sound velocity, is also of undoubted interest.

By carrying out the expansion in $\tau_{N\lambda}\Omega$ we get[3)]

$$\frac{\Lambda_3'}{2} = \sum_\lambda\left(\frac{Q_{l\lambda,\lambda}}{\rho s_\lambda^2}\right)^2 C_{V\lambda}T + \sum_{\lambda\neq\lambda'}\left(\frac{Q_{l\lambda,\lambda'}}{\rho s_\lambda s_{\lambda'}}\right)^2\frac{s_{\lambda'}\varepsilon_{T\lambda}}{s_{\lambda'} - s_\lambda} + \sum_{\lambda,\lambda'}\left(\frac{Q_{l\lambda,\lambda'}}{\rho s_\lambda s_{\lambda'}}\right)^2\frac{s_{\lambda'}\varepsilon_{T\lambda}}{s_{\lambda'} + s_\lambda}$$
$$+ \frac{\hbar}{8\pi^2}\sum_{\lambda,\lambda'}\left(\frac{Q_{l\lambda,\lambda'}}{\rho s_\lambda s_{\lambda'}}\right)^2\frac{s_\lambda s_{\lambda'}}{s_{\lambda'} + s_\lambda}k_{max}^4, \tag{4.5}$$

$$\frac{\Lambda_3''}{2} = \sum_\lambda\left(\frac{Q_{l\lambda,\lambda}}{\rho s_\lambda^2}\right)\tau_{N\lambda}C_{V\lambda}T. \tag{4.6}$$

Here $\epsilon_{T\lambda}$ is the energy of the thermal phonons:

$$\varepsilon_{T\lambda} = \frac{T^4}{2\pi^2\hbar^3 s_\lambda^3}\int_0^{\Theta/T} dz\, z^3(e^z - 1), \quad \varepsilon_T = \sum_\lambda \varepsilon_{T\lambda}.$$

The separate terms in (4.5) define the contribution to the renormalized elastic modulus tensor as the result of processes of scattering or decay, $Q_{l,\lambda\lambda'}$. Here,

$$Q_{lt,t}^2 = (K + B)^2 + {}^{32}/_{45}\mu^2 + {}^1/_{10}A^2 + {}^{16}/_{105}C^2 + {}^4/_{35}AC$$
$$+ (K + B)({}^2/_3\mu + A/2 + {}^4/_{15}C) + \mu({}^7/_{15}A + {}^{88}/_{315}C),$$
$$Q_{ll,l}^2 = {}^{47}/_{15}(K + 2B)^2 + {}^{416}/_{405}\mu^2 + {}^4/_5A^2 + {}^4/_9C^2$$
$$+ {}^2/_5(K + 2B)({}^{103}/_9\mu + {}^{22}/_3A + {}^{34}/_7C) + {}^4/_{15}\mu({}^{32}/_3A + 8C) + {}^8/_7AC,$$
$$Q_{lt,l}^2 = {}^4/_{15}(K + 2B + {}^7/_3\mu + A)^2 + {}^8/_{35}(K + 2B + {}^7/_3\mu + A)C + {}^{16}/_{63}C^2,$$
$$Q_{l\lambda,\lambda'} = Q_{l\lambda',\lambda} = Q_{l,\lambda\lambda'} = Q_{l,\lambda'\lambda}; \quad Q_{lt,t'} = 0. \tag{4.7}$$

Substituting (4.5) in (4.3), we get an expression for the renormalized velocity of longitudinal sound:

$$s_1^2 = s_l^2 + \frac{\bar{Q}_1^2}{\rho^3 s^4}C_V T - \frac{\bar{Q}_2^2}{\rho^3 s^4}\varepsilon_T, \tag{4.8}$$

which gives, as it ought, a decrease in the elasticity with rise in temperature. Equations (4.5) and (4.7) determine the value of the coefficients $Q_1^2/s^4$, $Q_2^2/s^4$ in terms of the elastic and anharmonic constants. For the damping decrement $\Gamma_1$, we obtain a formula which was first discovered by A. I. Akhiezer:[12]

$$\Gamma_1 \approx \tau\bar{Q}'^2 C_V T/\rho^3 s^4. \tag{4.9}$$

It follows from (4.8) and (4.9) that equal contributions are made to the renormalization of the sound velocity (neglecting dispersion) by the processes of scattering and decay, while the damping of the sound is determined only by the scattering processes (in linear approximation in $\tau\Omega$).

We now turn to the dispersion equation (4.2) and consider the second component in it. The values of $s_{12}^2$ and $\Gamma_D$ appearing in it are equal to

$$s_{12}^2 = \frac{1}{\rho C_V}\left\{\sum_\lambda\left(\frac{D_\lambda}{\rho s_\lambda^2}\right)C_{V\lambda}\right\}^2 T, \quad \Gamma_D \approx \Gamma_2, \tag{4.10}$$

where

$$D_t = K + B + {}^2/_3\mu + {}^1/_3A + {}^2/_{15}C,$$
$$D_l = {}^5/_3(K + 2B) + {}^8/_9\mu + {}^2/_3A + {}^2/_5C. \tag{4.11}$$

The term considered in the dispersion equation describes the interaction of first and second sounds. In other words, in a definite frequency range, the existence of coupled elasto-temperature waves is possible. Assuming the corresponding coupling coefficient to be small,[4)] we obtain the following expression for the frequencies of the coupled elasto-temperature waves:

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3)We note that account of anharmonism of fourth order leads to a change in the numerical values of the coefficient of $\epsilon_{T\lambda}$ only, and does not affect the imaginary part of the elastic modulus tensor in the approximation considered.

4)We note that the ratio $s_{12}^2/s^2 \sim \epsilon_T/\rho s^2 \sim (q_T a^3)\,T/Ms^2$ serves as the small parameter in our case, where a is the lattice constant, M the atomic mass and $q_T = T/hs$ is the mean momentum of the thermal phonon. For $T = 10°K$, this ratio is of the order of $10^{-4}$.

$$\Omega_1^2 = s_1^2\left(1 + \frac{s_{12}^2}{s_1^2 - s_2^2}\right) q^2 = \tilde{s}_1^2 q^2,$$

$$\Omega_2^2 = s_2^2\left(1 - \frac{s_{12}^2}{s_1^2 - s_2^2}\right) q^2 = \tilde{s}_2^2 q^2. \quad (4.12)$$

These relations were obtained in [5] in a somewhat different form. In contrast with that research, where the anharmonicity was taken into account by the introduction of a single averaged constant, here the specific dependence of the value of $s_{12}$ on the elastic and anharmonic constants is determined. This makes it possible to estimate the value of the contributions made to the longitudinal sound velocity from the various mechanisms of interaction of sound with thermal vibrations.

Using the expressions for $s_1$, $s_2$, and $s_{12}$, we find the approximate formulas for the renormalized velocities:

$$\tilde{s}_1^2 = s_l^2 - (\bar{Q}_1^2 C_V T + \bar{Q}_2^2 \varepsilon_T)/\rho^3 s^4 + \frac{\bar{D}^2}{\rho^3 s^4} C_V T,$$
$$\tilde{s}_2^2 = s_2^2 - {}^1/_2 (\bar{D}^2/\rho^3 s^4) C_V T. \quad (4.13)$$

It is seen from (4.13) that while the contribution to the velocity of longitudinal sound from processes of scattering and decay enters with a minus sign, the contribution arising from interaction with temperature waves is positive. Thus, on the one hand, the presence of anharmonism leads to a decrease in the elasticity with increase in T; on the other hand, the number of thermal phonons increases with T, i.e., the pressure of the phonon gas increases, and so does its elasticity. If, by making use of (4.5) and (4.10), we calculate the total effect of these two contributions, we then always obtain a decrease in the sound velocity for solids ($K \sim \mu$) with increase in the temperature $\sim T^4$ for $T < \Theta$.

We write down the expression for the damping decrement associated with elasto-temperature waves. It follows from (4.1) that

$$\tilde{\Gamma} = \frac{\Gamma_1(\Omega^2 - s_2^2 q^2) + \Gamma_2(\Omega^2 - s_1^2 q^2) - s_{12}^2 q^2 \Gamma_D}{\Omega^2 - s_2^2 q^2 + \Omega^2 - s_1^2 q^2 - s_{12}^2 q^2}. \quad (4.14)$$

Substituting the expression for the frequencies $\Omega_1$ and $\Omega_2$ and expanding in a series in $s_{12}^2/s^2$, we get

$$\tilde{\Gamma}_1 \approx \Gamma_1 + \frac{\varepsilon_{12}^2 s_2^2}{(s_1^2 - s_2^2)^2}(\Gamma_2 - \Gamma_1), \quad \tilde{\Gamma}_2 \approx \Gamma_2 - \frac{s_{12}^2 s_2^2}{(s_1^2 - s_2^2)^2}(\Gamma_2 - \Gamma_1). \quad (4.15)$$

Both terms in $\tilde{\Gamma}_1$ are of the same order and consequently, near the lower boundary of the existence of elasto-temperature waves $\Omega \sim \tau_U^{-1}$, the damping decrement of first sound is less than that of second sound. Here, there are terms in $\tilde{\Gamma}_1$, just as in $\tilde{\Gamma}_2$, that do not depend on the frequency.

The temperature and frequency dependence of the damping decrements of first and second sound have the form

$$\tilde{\Gamma}_1 \sim a_1 \tau_N \Omega^2 \varepsilon_T + a_2 \tau_U^{-1} \varepsilon_T,$$
$$\tilde{\Gamma}_2 \sim b_1 \tau \Omega^2 + b_2 \tau_U^{-1} + b_3 \tau_U^{-1} \varepsilon_T. \quad (4.16)$$

Transverse Waves. Let $\mathbf{q} \parallel \mathbf{x}_3$ and $\mathbf{u} \parallel \mathbf{x}_1$; then Eq. (3.28) takes the form

$$-\rho\Omega^2 + \overset{0}{\Lambda}_{1331} q^2 + \Lambda_{1331}^{(1)} q^2 + i\Omega^{-1} A_{11}^2 / C_{11} = 0. \quad (4.17)$$

By estimating the last term in (4.17), we can show that it will make a contribution to the frequency and damping decrement of higher order in $\Omega\tau$. Omitting it, we get

$$\Omega^2 - s_t^2 q^2 + 2i\Omega\Gamma_t = 0, \quad (4.18)$$

where

$$s_t^2 = \mu/\rho - \Lambda_1'/\rho, \quad \Gamma_t = \Lambda_1'' q^2 / 2\rho. \quad (4.19)$$

The quantities $\Lambda_1'$ and $\Lambda_1''$ are determined by Eqs. (4.5) and (4.6) and by replacement of $l$ by t, we have

$$Q_{tl,l} = {}^4/_{15}(K + 2B + {}^7/_3\mu + A)^2,$$
$$Q_{tt,t} = {}^{11}/_{15}(\mu + {}^1/_4 A)^2,$$
$$Q_{tt,l} = {}^{13}/_{30}(K + B + {}^1/_3\mu + {}^1/_2 A)(2\mu + {}^1/_2 A + B). \quad (4.20)$$

The difference of the dispersion equation (4.18) from the corresponding equation for longitudinal waves (4.1) is significant. First, as should be expected, additional branches of oscillations do not appear for transverse waves. Second, the specific dependence of $\tilde{s}_t$ and $\Gamma_t$ on the constants K, $\mu$, A, B, C differs appreciably from such a dependence for longitudinal waves. There is also a difference in the frequency and temperature dependence. All this can be used for the experimental determination of elastic and anharmonic constants.


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