SELF-SIMILAR MOTION OF A LOW-DENSITY PLASMA. II

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The general qualitative features of various self-similar motions of a quasineutral plasma are discussed. A detailed investigation is made of the expansion of a dense plasma into a less dense plasma and the expansion of a nonisothermal plasma into a vacuum.

1. CLASSIFICATION OF POSSIBLE CASES

In an earlier work [1] we have investigated the self-similar motion of a quasineutral low-density plasma—the flow of a plasma into a vacuum. In the present work we consider the question from a more general point of view, treating the qualitative features of various cases of self-similar motion.

The general formulation of the self-similar motion we consider is as follows. At an initial time $t = 0$ the plasma consists of two regions. For $x < 0$ (region 1) the distribution functions for the ions and the electrons are respectively

$$f_{1i}(v), \quad f_{1e}(v), \tag{1a}$$

and for $x > 0$ (region 2)

$$f_{2i}(v), \quad f_{2e}(v). \tag{1b}$$

At time $t = 0$ both regions start to intermix and the interface between them is smeared out. In the absence of collisions the equations for f_i and f_e become

$$\frac{\partial f_i}{\partial t}+v\frac{\partial f_i}{\partial x}-\frac{e}{M}\frac{\partial\varphi}{\partial x}\frac{\partial f_i}{\partial v}=0,$$

$$\frac{\partial f_e}{\partial t}+v\frac{\partial f_e}{\partial x}+\frac{e}{m}\frac{\partial\varphi}{\partial x}\frac{\partial f_e}{\partial v}=0, \tag{2}$$

where M is the ion mass, m is the electron mass and φ satisfies the Poisson equation

$$\Delta\varphi=-4\pi e\left(\int f_i dv-\int f_e dv\right). \tag{3}$$

In the course of time the effective dimensions of the interface between the two regions of plasma becomes larger than the Debye radius. The plasma then may be regarded as quasineutral so that Eq. (3) is replaced by the condition

$$\int f_i dv=\int f_e dv. \tag{4}$$

After this point the charge e disappears from the equations, as is evident from the possibility of the substitution

$$e\varphi=\psi. \tag{5}$$

This means that we have eliminated from the problem the single parameter with the dimensions of length-the Debye radius; hence the subsequent motion becomes self-similar, that is to say, f_i and f_e are functions only of the ratio $y = x/t$:

$$f_i=f_i(y,v), \quad f_e=f_e(y,v), \quad \psi=\psi(y). \tag{6}$$

As a result we obtain the following system of equations for f_i, f_e and ψ:

$$\frac{\partial f_i}{\partial y}(v-y)-\frac{1}{M}\frac{\partial\psi}{\partial y}\frac{\partial f_i}{\partial v}=0, \quad \frac{\partial f_e}{\partial y}(v-y)+\frac{1}{m}\frac{\partial\psi}{\partial y}\frac{\partial f_e}{\partial v}=0, \tag{7}$$

$$\int f_i dv=\int f_e dv.$$

The boundary conditions assume the form:

$$\begin{gathered} f_i\to f_{1i}, \quad f_e\to f_{1e} \text{ as } y\to-\infty, \\ f_i\to f_{2i}, \quad f_e\to f_{2e} \text{ as } y\to+\infty. \end{gathered} \tag{8}$$

We note that the equations in (7) are only a particular case of self-similar kinetic equations. A more general relation can be obtained by writing

$$f_i=t^{\alpha}h_i(y,v),$$

(where α is a parameter), and similarly for f_e. As a result we obtain

$$\frac{\partial h_i}{\partial y}(v-y)-\frac{1}{M}\frac{\partial\psi}{\partial y}\frac{\partial h_i}{\partial v}+\alpha h_i=0. \tag{9}$$

These equations describe self-similar motion that arises as a result of propagation of some initial perturbation which is concentrated in an infinitesimally thin layer close to the plane $x = 0$; the power α depends on the nature of the initial perturbation. Similar equations can be obtained for the two-dimensional and three-dimensional cases. Such equations would describe the propagation of perturbations which are initially concentrated close to a line (2-dimensional case) or a point (3-dimensional case).

One of the difficulties in the use of the general equations (9) lies in the formulation of the boundary conditions. It is evident that the layer can only be assumed to be infinitesimally thin at distances that are large compared with the thickness. Strictly speaking, the boundary conditions are determined by the motion in regions comparable with the thickness of the layer. However, it is evident that the motion is not self-similar in such regions. This difficulty does not arise in Eq. (7).

The most convenient choice for the boundary conditions (8) is to take f_{1i} and f_{2i} as Maxwellian distribution functions with some characteristic density, temperature and velocity. The plasma in region 1 can be assumed to be at rest:

$$\begin{gathered} f_{1i}=N_1\left(\frac{M}{2\pi T_{i1}}\right)^{1/2}\exp\left(-\frac{Mv^2}{2T_{i1}}\right), \\ f_{2i}=N_2\left(\frac{M}{2\pi T_{i2}}\right)^{1/2}\exp\left[-\frac{M(v-V_0)^2}{2T_{i2}}\right]. \end{gathered} \tag{10}$$

For values such that $y \ll \sqrt{T_i/m}$, in considering the electrons in Eq. (7) we can neglect y compared with v. Under these conditions the electron distribution is stationary. In particular, with the appropriate boundary conditions it can be a Boltzmann distribution. In any case it is necessary that the electrons at $x > 0$ and $x < 0$ have the same temperature:

$$f_{1e} = N_1\left(\frac{m}{2\pi T_e}\right)^{1/2} \exp\left(-\frac{mv^2}{2T_e}\right), \quad f_{2e} = N_2\left(\frac{m}{2\pi T_e}\right)^{1/2} \exp\left(-\frac{mv^2}{2T_e}\right). \tag{11}$$

Then the potential ψ is given by the explicit expression

$$\psi = T_e \ln \int f_i dv. \tag{12}$$

Introducing the variables

$$\tau = \left(\frac{M}{2T_e}\right)^{1/2} y, \quad g = \left(\frac{2\pi T_e}{M}\right)^{1/2} f_i, \quad u = v\left(\frac{M}{2T_e}\right)^{1/2},$$

we obtain an equation that has been obtained earlier:[1,2]

$$(u-\tau)\frac{\partial g}{\partial \tau} - \frac{1}{2}\frac{\partial g}{\partial u}\frac{d}{d\tau}\left(\ln \int_{-\infty}^{\infty} g du\right) = 0, \tag{13}$$

with the boundary conditions

$$g \to N_1\left(\frac{T_e}{T_{i1}}\right)^{1/2} \exp\left(-u^2\frac{T_e}{T_{i1}}\right), \quad \tau \to -\infty,$$
$$g \to N_2\left(\frac{T_e}{T_{i2}}\right)^{1/2} \exp\left(-u^2\frac{T_e}{T_{i2}}\right), \quad \tau \to +\infty. \tag{14}$$

Dividing (13) by $u - \tau$ and integrating with respect to du from $-\infty$ to $+\infty$ we obtain an interesting identity which can be imposed on the solution of (13):

$$\int g du = \frac{1}{2}\int \frac{\partial g}{\partial u}\frac{du}{u-\tau}.$$

In solutions of actual cases with boundary conditions (14), importance attaches to the general nature of the function $N(\tau) = \int g du$, in particular the existence of maxima and minima.

In order to obtain qualitative conclusions as to the nature of the function $N(y)$ we first consider the problem of self-similar motion of a neutral gas with mass M. In this case one easily finds

$$N(y) = \int_{-\infty}^{y} f_1(v) dv + \int_{y}^{\infty} f_2(v) dv. \tag{15}$$

The extrema of the function $N(y)$ are determined by the equation

$$dN/dy = f_1(y) - f_2(y) = 0. \tag{16}$$

We now consider some particular cases.

A. First assume $T_1 > T_2$ (Fig. 1-a). Then, when $y \to \pm\infty$

$$f_1(y) \sim e^{-y^2/2T_1} \gg f_2(y) \sim e^{-y^2/2T_2},$$
$$dN/dy > 0, \quad y \to \pm\infty. \tag{17}$$

This means that the curve $N(y)$ is either monotonic or that it has a maximum. It is easy to determine the conditions under which one or the other cases obtain. If N_2 and V_0 are small, then for $f_2(y) < f_1(y)$ for all y and $N(y)$ is monotonic. If, however, N_2 and V_0 are large then $f_2(y)$ intersects the curve $f_1(y)$ and the curve $N(y)$ will exhibit a maximum and a minimum.

B. Now let $T_1 = T_2 = T$ (Fig. 1-b). Then when $y \to \pm\infty$

$$f_1 \sim e^{-y^2/2T}, \quad f_2 \sim e^{-y^2/2T} e^{yV_0/T}. \tag{18}$$

Assume to be definite that $V_0 > 0$; then for $y \to -\infty$

$$f_1 \gg f_2, \quad dN/dy > 0.$$

For $y \to +\infty$ on the other hand,

$$f_1 \ll f_2, \quad dN/dy < 0.$$

In this case the curve $N(y)$ must exhibit an extremum—a maximum when $V_0 < 0$ and a minimum when $V_0 > 0$.

C. Now let $T_1 = T_2 = T$, $V_0 = 0$ and $N_1 > N_2$ (Fig. 1-c). In this case, for all y,

$$f_1 > f_2, \quad dN/dy > 0.$$

The curve $N(y)$ is monotonic.

We have carried out an analysis for the motion of neutral particles. It can be asserted, however, that the qualitative conclusions as to the nature of $N(y)$ also hold in the general motion of ions. The point is that for asymptotic behavior of the function dN/dy for $y \to \pm\infty$ only the particles with high velocities $v \sim y$ are important. In all cases aside from that considered in,[1] the special case of expansion of a plasma into vacuum, the jump in the electric potential $\varphi_{x\to-\infty} - \varphi_{x\to+\infty}$ is found to be finite. The distribution of these fast particles will not make a small modification of the electric field. Hence, the asymptotic expressions (17) and (18) also hold for ions. Consequently the qualitative characteristics of $N(y)$ hold for the various cases. A somewhat special situation arises in the case denoted by C. Here, the electric field can change the coefficient of $e^{-y^2/2T}$ in the asymptote. Physically, however, it is evident that there is no cause for a deviation from monotonic behavior of $N(y)$ in this case.

The relation in (15) also has general value for another reason. The results of the numerical calculations show that if $T_e \lesssim T_i$ the effect of the electric field on the ion motion is generally not important. Hence, Eq. (15) can give a good picture of the process if high accuracy is not required.

In a nonisothermal plasma with $T_e \gg T_i$ the effect of the field on the ion motion increases in proportion to the temperature ratio and can become decisive for large values of T_e/T_i. This will be seen below in Sec. 3, where we consider the flow of a nonisothermal plasma into a vacuum. The effect of the field is also

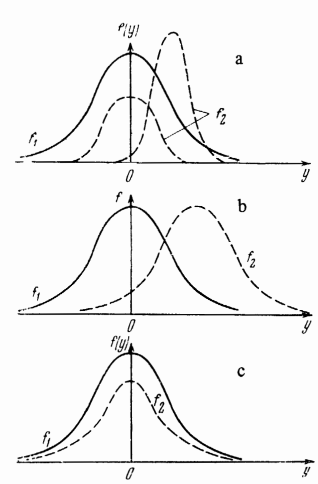

FIG. 1.

important in the case in which there is a nonmonotonic variation of potential. It has been shown above that the density distribution, and consequently the field potential for self-similar motion, can exhibit a nonmonotonic character. If the potential exhibits a maximum there arises a region of trapped electron motion in the electric field. The boundary conditions (1) and (11) determine the stationary electron distribution function only in the region of trapped motion. In the absence of collisions the distribution of electrons trapped by the field is sensitive to transient processess.

The distribution function for the trapped particles is not a Maxwellian distribution. This leads to a change in Eq. (12) for the field potential. The potential is increased. Consequently, the electric field is magnified by a large factor in Eq. (13). This question has been treated in[3] in which a kinetic equation has been used for self-similar motion taking account of trapped electrons.

We wish to comment on several qualitative features of the behavior of the distribution function as effected by the electric field in the plasma. First of all, the ion distribution function vanishes for some finite value of the velocity at the separatrix. This is associated with the energy effect of the field on the ions in the initial period of motion. Furthermore, close to the separatrix there also arise narrow peaks in the distribution function which actually represent plasma flux in the plasma.

The electric field can also give rise to discontinuities in the self-similar motion in the plasma. The structure of a weak discontinuity will be analyzed in Sec. 4.

2. EXPANSION OF A PLASMA INTO A PLASMA

As an example we now consider the expansion of a plasma into a plasma. Let $N_2 = \alpha N_1$. In order to be definite we assume that $\alpha < 1$. In the course of time the plasma must flow from the left half-space into the plasma in the right half-space. This process is self-similar and is described by Eq. (13). If the electron and ion temperatures are the same, the boundary conditions for this equation assume the form $T_{1i} = T_{2i} = T_e = T$

$$g/N_1 \to e^{-u^2}, \qquad \tau \to -\infty, \tag{19a}$$

$$g/N_1 \to \alpha e^{-u^2}, \qquad \tau \to +\infty. \tag{19b}$$

We shall first investigate the behavior of the characteristics of Eq. (18), that is to say, the curves in the u, τ plane in which the function g has a constant value. The equation for the characteristics is

$$\frac{du}{d\tau} = \frac{1}{2}\frac{F(\tau)}{u-\tau}, \tag{20}$$

where F is a dimensionless force that acts on the ions:

$$F(\tau) = -\frac{d}{d\tau}\left(\ln \int_{-\infty}^{\infty} g du\right) = -\frac{d \ln N}{d\tau}, \tag{21}$$

while $N(\tau)$ is the ion density (or electron density). When $\tau \to -\infty$, the function $N(\tau) \to N_1$ and when $\tau \to +\infty$ we have; $N(\tau) \to \alpha N_1$, $\alpha < 1$, i.e., $\alpha N_1 < N_1$.

In accordance with the results of Sec. 1 we assume that $N(\tau)$ falls of monotonically as τ changes from $-\infty$ to $+\infty$. This assumption is verified by the further calculations. In this case the force $F(\tau)$ is always positive $F(\tau) \geq 0$.

It has been shown in[1] that for this condition the characteristics in the u, τ plane that lie above the line $u = \tau$ do not intersect this line. Starting with $\tau \to -\infty$ in the region of finite values of u they extend to infinity. When $\tau \to \infty$ the characteristics crowd toward the line $u = \tau$ (by virtue of the fact that $dN/d\tau$ and, consequently $F(\tau)$, approach zero when $\tau \to \infty$). This behavior is evident, in particular, in Fig. 2 in which we show the characteristics for the solution of Eq. (13) for the case $\alpha = 1/2$.

The behavior of the characteristics that lie below the line $u = \tau$ is entirely different. In this region, as is evident from Eq. (20), $u(\tau)$ increases as τ diminishes. At the point at which $u(\tau)$ is comparable with τ, i.e., on the line $u = \tau$, the derivative $du/d\tau$ becomes infinite and undergoes a discontinuity, changing from $-\infty$ at $u = \tau - 0$ to $+\infty$ at $u = \tau + 0$. This means that all the characteristics that come from the right below the line $u = \tau$ intersect this line; they are reversed in direction, go to infinity, and crowd close to the line $u = \tau$, as shown in Fig. 2.

It should be noted that the characteristics do not intersect. This means, in particular, that the characteristics from the right and from the left occupy their own regions of the u, τ plane. These regions are separated by a curve, the separatrix. It is evident that close to the separatrix the characteristics crowd together. The separatrix itself lies above the line $u = \tau$ and approaches this line as $\tau \to \pm\infty$. The distribution function vanishes on the separatrix. The presence of a separatrix is a basic feature of the topology of the characteristics in the u, τ plane.

These properties of the characteristics of Eq. (13) have been used in numerical integration on an electronic computer. As an example we consider the case $\alpha = 1/2$, that is to say, we assume that a plasma with density N_1 flows into a plasma with density $N_1/2$. The solution is found by integration of the equations for 600 characteristics. For 300 characteristics the initial conditions are specified for $\tau = -3$ for values $u_{11} = -2.98$, $u_{12} = -2.96 \ldots u_{1300} = +3.00$. For the 300 other characteristics the initial conditions are given $\tau = +3$ for values of u from 2.98 to -3.00. In accordance with the boundary conditions (19a) the function g on the characteristics which come from the left is given by

$$g_1 = e^{-u^2},$$

while, in accordance with Eq. (19b), those coming from the right are given by

$$g_2 = {}^1/_2 e^{-u^2}.$$

These initial values of g are conserved on each characteristic.

The solution of Eq. (20) for the characteristics is found by steps in τ. The behavior of the characteristics is shown in Fig. 2. The value of the particle density is determined from the expression

$$N(\tau) = N_1(\tau) + N_2(\tau) + \Delta N_2(\tau). \tag{22}$$

Here, $N_1(\tau)$ is the density of particles coming from the left

$$N_1(\tau) = \sum_{k=1}^{300} \frac{g(u_{1k}) + g(u_{1k+1})}{2}[u_{1k+1}(\tau) - u_{1k}(\tau)]. \tag{23}$$

Correspondingly, $N_2(\tau)$ is the density of particles

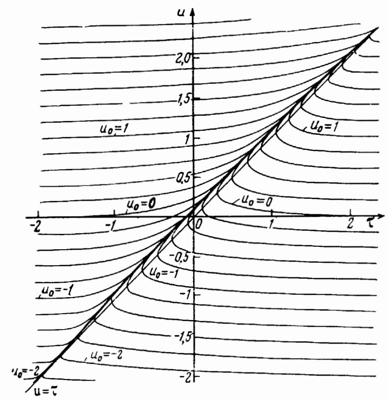

FIG. 2. Characteristics in the u, τ plane. The quantity u_0 is the value of u and $\tau \to \pm\infty$.

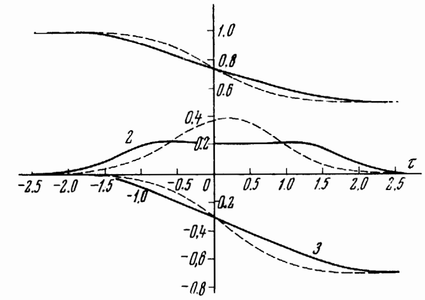

FIG. 3. The density distribution $N(\tau)N_1$ (curve 1), the pontential $e\varphi(T)/T_e$ (curve 3) and the dimensionless force $F(\tau)$ (curve 2). The dashed curves are the same quantities for the neutral gas (26).

coming from the right and lying below the line $u = \tau$ (that is to say, particles whose trajectories does not undergo reversal, Fig. 2):

$$N_2(\tau) = \sum_{k=k_0}^{300} \frac{g(u_{2k}) + g(u_{2k+1})}{2} [u_{2k+1}(\tau) - u_{2k}(\tau)]. \quad (24)$$

(here $k_0 = k_0(\tau)$ is the number of trajectories which do not reach the turning point $u = \tau$). Finally, ΔN_2 is the density of particles coming from the right that lie below the line $u = \tau$:

$$\Delta N_2(\tau) = \sum_{k=1}^{k_0} \frac{g(u_{2k}) + g(u_{2k+1})}{2} [u_{2k+1}(\tau) - u_{2k}(\tau)]. \quad (25)$$

The function $F(\tau)$ that appears in the equation for the characteristics (20) is related to $N(\tau)$ by (21).

The solution is obtained by an iteration method. We start with some initial function $F_0(\tau)$ and move in τ from left to right from the point $\tau = -3$, thus finding the characteristics that come from the left, and the density $N_{11}(\tau)$ [from Eq. (7)]. Then, moving in τ from the right to the left from the point $\tau = +3$ we find the characteristics coming from the right and compute $N_{21}(\tau)$. The turning point of the trajectory is retained. Finally, again moving from left to right we determine the trajectory after the turning point and compute the density ΔN_{21}. Summing N_{11}, N_{21}, and ΔN_{21} we determine the first iteration $N_1(\tau)$ and $F_1(\tau) = d \ln N_1/d\tau$. Then the entire procedure repeated, allowing us to determine the higher iterations and so on.

It is natural to take the zeroth iteration to be the solution for the collisionless flow of a neutral gas with density N_1 into a gas with density $N_1/2$:

$$N(\tau) = \frac{N_1}{4}\left(3 - \frac{2}{\sqrt{\pi}} \int_0^\tau e^{-x^2} dx\right). \quad (26)$$

It turns out, however, that the iteration does not converge in this case: the deviation of the zeroth iteration (26) from the final solution is found to be extremely significant. Hence it is necessary to choose a zeroth approximation that is closer to the final solution. More precisely, we take the zeroth approximation by trying a number of functions from which we choose those for which the successive iterations converge most rapidly. The answer obviously does not depend on the exact form of the zeroth approximation.

The final result of the calculation of $N(\tau)$ (with an accuracy of 0.5%) is shown in Fig. 3 (curve 1). It is evident that the function $N(\tau)$ falls off monotonically as τ increases, in accordance with the assumption made above. The dotted curve in the figure shows the distribution of neutral gas (26). It is evident that under the effect of the electric field the region of transition from the density N_1 to the density $N_1/2$ is extended to some extent, that is to say, the flow of plasma into plasma is accelerated. In addition we note that the deviation of the dotted curve from the solid curve is generally small: The difference in densities is never greater than 10–15%. The neutral gas approximation is thus a good approximation for the description of the variation in plasma density. This result is in agreement with the results of the solution of the problem of plasma expansion into a vacuum for $T_e = T_i$.[1] In Fig. 3 we show the potential $e\varphi/T_e$ (curve 3) and the force $F(\tau)$ (curve 2). The dashed curves are the same quantities for the distribution (26).

In Fig. 4 we show the ion velocity distribution in u for various values of τ. The deviation of this function from the distribution functions for the neutral particles is most pronounced in the vicinity of the separatrix (Fig. 2). On the separatrix the distribution function vanishes so that its form is highly distorted near the separatrix. This feature is especially pronounced for positive values of τ i.e., in the region in which the initial plasma density was smaller; near the separatrix two narrow peaks appear in the distribution function. The origin of these features is the effect of the electric field on the plasma. The most pronounced effect of the field appears in the initial phase of the motion, when the field strength is large $E \sim 1/t$. In this time period, however, the field is concentrated in a narrow region close to the initial discontinuity. The distribution of particles that enter this region is highly distorted by the effect of the field. For any values of x, and t this distortion is retained in the form of features near the separatrix. The separatrix itself corresponds to particles that lie on the discontinuity at the initial time. Here, the field intensity is infinite for $t = 0$. Hence, all of the particles acquire an infinite velocity; for finite values of τ these particles do not appear whether τ is positive or negative. For this reason the distribution function vanishes on the separatrix.

Since the field only accelerates particles in the direction of positive x or τ, for negative values of τ there is a dip in the distribution function near the separatrix; for positive values of τ there are two peaks, one above the separatrix and one below it. The peak for the faster particles corresponds to particles that are accelerated by the field from the region close to the initial discontinuity into the dense plasma N_1

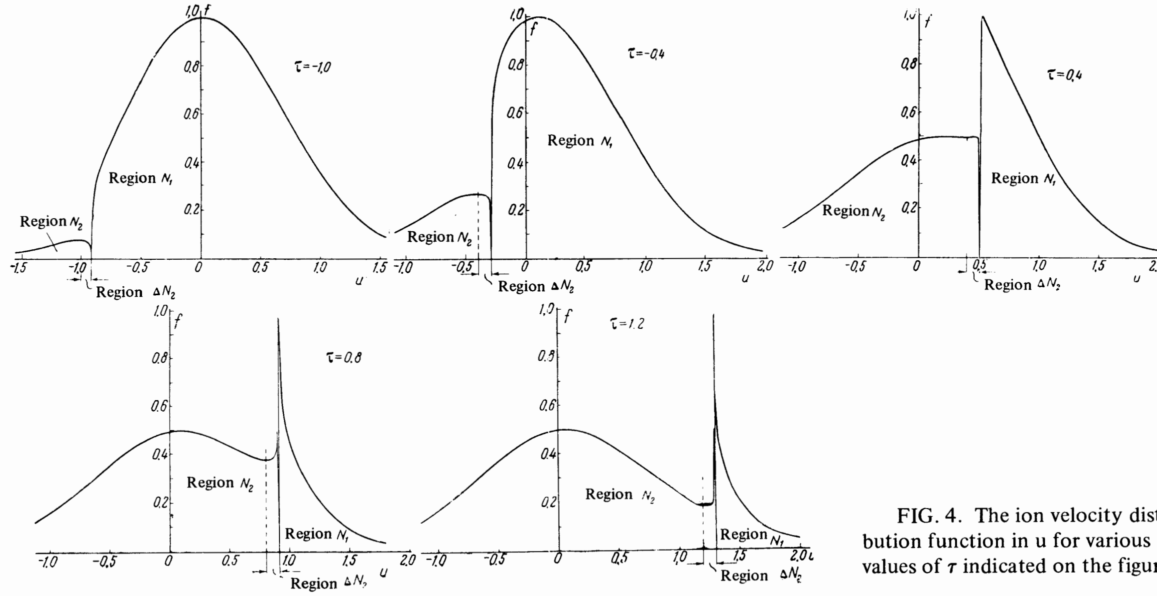

FIG. 4. The ion velocity distribution function in u for various values of τ indicated on the figure.

while the slow peak corresponds to those accelerated from the less dense plasma $N_1/2$[1].

If the plasma exhibits streams it is possible for a two-stream electrostatic instability to arise. The instability criterion is always satisfied for sufficiently large values of τ. Ion waves are excited, these waves moving with the beam velocity. The wavelength is of the order of the Debye radius; the growth rates are not large.

In addition to these waves, taking account of the possibility of magnetic-field oscillations can lead to non-electrostatic instabilities with wavelengths of order $\lambda_0 \sim cD/v_e$ (D is the Debye radius).[4] Under actual experimental conditions these instabilities can be neglected if the system length is restricted to a value L that satisfies the easily achieved condition $\lambda_0 \gg L \gg D$.

3. EXPANSION OF A NONISOTHERMAL PLASMA IN A VACUUM

The expansion into vacuum of a plasma with identical initial temperatures for the ions and electrons has been studied in[1]. In the present case we wish to investigate unequal temperatures. As before, the process is described by Eq. (13). The boundary conditions (14) now assume the form

$$g \to N_1 \sqrt{\frac{T_e}{T_i}} \exp\left(-u^2 \frac{T_e}{T_i}\right) \text{ as } \tau \to -\infty,$$

$$g \to 0 \text{ as } \tau \to +\infty. \tag{27}$$

As in the single-temperature plasma[4] all the characteristics go above the separatrix. As before they are determined by Eq. (20). A numerical integration of this equation has been carried out for 300 characteristics. The behavior of the quantity $N(\tau)$ obtained by numerical calculation for various values of the temperature ratio is shown in Fig. 5. It is evident from this figure that the difference in the behavior of the $N(\tau)$ curves as a function of T_e/T_i is in general small when $T_e/T_i \gtrsim 1$. This means that the effect of the initial thermal spread of the ion velocities is not very important and that their motion is determined primarily by the electric field. As in[1] it is found that the ion velocity distribution function is sharply compressed with increasing τ. In this region it is permissible to use the hydrodynamic equations, which apply to a plasma in which $T_e \gg T_i$. When the self-similar features are introduced these equations assume the form

$$(V-\tau)\frac{dN}{d\tau} + N\frac{dV}{d\tau} = 0, \quad (V-\tau)\frac{dV}{d\tau} + \frac{1}{2N}\frac{dN}{d\tau} = 0, \tag{28}$$

where V is the dimensionless hydrodynamic ion

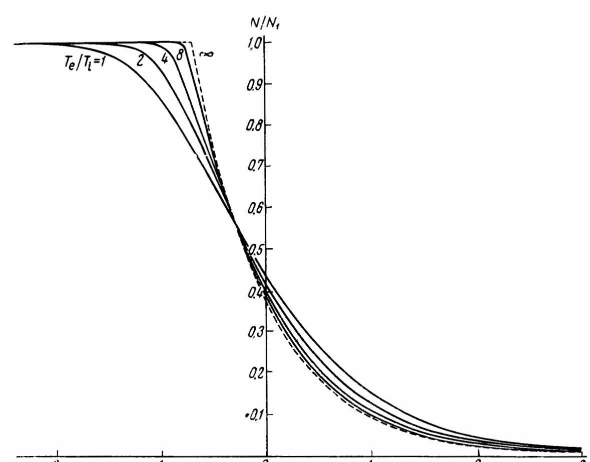

FIG. 5. The density distribution $N(\tau)/N_1$ for various values of T_e/T_i indicated in the figure. The dashed line shows the solution of the hydrodynamic equations.

[1]This can be shown easily by integrating the equation of motion of the particles. Knowing the equation for the characteristics $u = u(\tau, u_0)$ and assuming that $u = v_T^{-1}\, dx/dt$, $\tau = x/tv_T$ we find

$$\ln t = \int^{x/tv_T} \frac{d\tau}{u(\tau, u_0) - \tau}.$$

It is evident that even a small smearing of the initial discontinuity will have an important effect on the form of the distribution function in the region close to the separatrix.

velocity. The solution

$$N = C\exp(-\sqrt{2}\tau), \qquad V = \tau + 1/\sqrt{2} \tag{29}$$

describes the asymptotic behavior of the density for $\tau \gg 1$. The constant C is determined by joining the numerical solution with the asymptotic solution (29). When $T_e = T_i$ the constant $C \approx 0.70\ N_1$. This quantity diminishes as the ratio T_e/T_i increases.

When the temperature ratio $T_e/T_i \to \infty$ the hydrodynamic equations (28) and their solution (29) hold for any τ.

In this case, taking account of the boundary condition $N = N_1$, $V = 0$ when $\tau \to -\infty$ we have

$$V = \begin{cases} 0, & \tau \leqslant -1/\sqrt{2} \\ \tau + 1/\sqrt{2}, & \tau \geqslant -1/\sqrt{2} \end{cases}, \tag{30}$$

$$N = \begin{cases} N_1, & \tau \leqslant -1/\sqrt{2} \\ N_1 \exp(-\sqrt{2}\tau - 1), & \tau \geqslant -1/\sqrt{2} \end{cases}.$$

The density distribution obtained from (30) is shown by the dashed line in Fig. 5. The behavior of the derivative $dN/d\tau$ for various values of the ratio T_e/T_i is shown in Fig. 6. It is evident from this figure that as the ratio T_e/T_i increases the gradient of the function $dN/d\tau$ increases near the point $\tau = -1/\sqrt{2}$. The dashed curve in Figure 6 shows the behavior of $dN/d\tau$ for the hydrodynamic solution (30). At the point $\tau = -1/\sqrt{2}$ this solution exhibits a discontinuity in the derivative (weak discontinuity). For finite values of the ratio T_e/T_i the discontinuity is smeared out because of the thermal motion of the ions.

4. WEAK JUMPS

An important property of self-similar motion in conventional hydrodynamics is the fact that this motion allows discontinuous solutions--weak and strong and strong jumps (shock waves).

In particular, the decay of the initial discontinuities in hydrodynamics is always accompanied by the appearance of discontinuities in the appropriate self-similar solutions ([5] Sec. 93). We have seen this in the example given in Sec. 3. In collisionless kinetics of a neutral gas, on the other hand, the decay of an arbitrary initial discontinuity only leads to a smooth self-similar solution (cf. Sec. 1). This same situation has been found in [1] and in the present work in the consideration of examples of decay of initial discontinuities in a collisionless plasma.

However, as in the case of hydrodynamics, the solutions of the self-similar kinetic equations in a plasma (13) allow discontinuities. This can be easily seen if one analyzes the solution of Eq. (13) concerning the problem of plasma expansion into a vacuum.

Let us consider an arbitrary point τ_0. We assume that the solution undergoes a weak jump at this point. In other words, we assume that the density $N(\tau)$ is continuous at τ_0 and that the derivative exhibits the discontinuities:

$$dN/d\tau|_{\tau_0+0} \neq dN/d\tau|_{\tau_0-0}. \tag{31}$$

The force $F(\tau)$ also exhibits a discontinuity (21). It is clear from (20) that all the characteristics $u(\tau)$ exhibit a break at the point τ_0. The solution of Eq. (13) exists in the presence of discontinuity since the solution of the equation for the characteristics exists (20). The magnitude of the weak discontinuity is arbitrary. The single limitation is that the derivative (from the right beyond the discontinuity) must be negative:

$$dN/d\tau|_{\tau_0+0} < 0. \tag{32}$$

Thus, Eq. (13) with the boundary conditions in (27) allows an arbitrary weak discontinuity (subject only to the condition in (32))) in an arbitrary point τ_0. These jumps obviously have an effect on the solution of Eq. (13). For this reason the solution of Eq. (13) with the boundary conditions in (27) is not determined uniquely. It is only unique to within a class of continuous functions $g(u, \tau)$ with continous derivatives $\partial g/\partial \tau$.

Physically the possibility of jumps indicates that there can be charge sheets that move with a constant velocity $\tau_0 v_T$. We now show that such charge sheets can actually be described by means of the self-similar solution. For this purpose it is necessary to consider dimensions of the order of the Debye radius. We assume that the electron density obeys a Boltzmann dis-

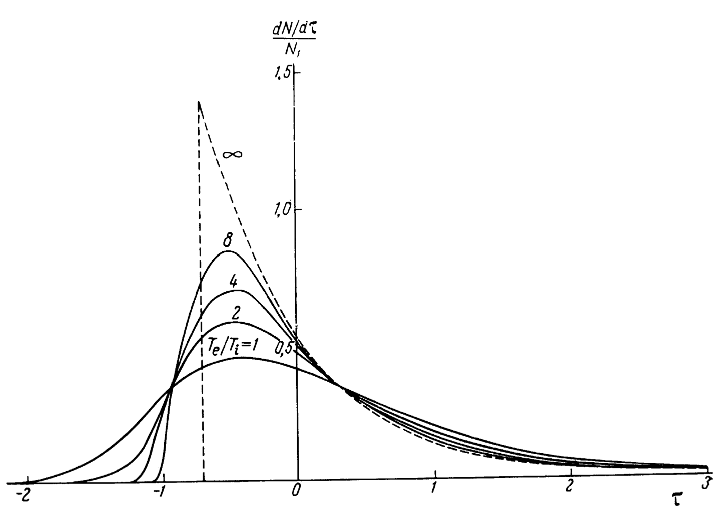

FIG. 6. The quantity $(dN/d\tau)/N_1$ as a function of τ for various values of T_e/T_i; the dashed line corresponds to the hydrodynamic equations.

tribution. Poisson's equation for the electric field equation potential is then

$$d^2\varphi / dx^2 = -4e[N_i(\tau) - N_1 e^{e\varphi/T_e}]. \quad (33)$$

Let τ_0 be the point of the weak discontinuity in $N_i(\tau_0)$ while

$$\varphi(\tau_0) = \frac{T_e}{e} \ln \frac{N_i(\tau_0)}{N_1}$$

is the value of the ion density and the field potential at the point $\tau_0 = x/tv_T$. In treating the region around the point τ_0 we can linearize Eq. (33) in terms of $\varphi - \varphi(\tau_0)$ and $\tau - \tau_0$. We also introduce the dimensionless variables

$$\varphi^* = \frac{e(\varphi - \varphi(\tau_0))}{T_e}, \quad \xi = \frac{x - tv_T\tau_0}{D}, \quad (34)$$
$$\frac{d^2\varphi^*}{d\xi^2} = \varphi^* - \begin{cases} \alpha_- D\xi/tv_T, & \xi < 0 \\ \alpha_+ D\xi/tv_T, & \xi > 0 \end{cases},$$

where (D is the Debye radius at the point τ_0)

$$D = \left[\frac{T_e}{4\pi e^2 N_i(\tau_0)}\right]^{1/2}, \quad \alpha_- = \frac{1}{N_i(\tau_0)}\left(\frac{dN_i}{d\tau}\right)_{\tau_0 - 0},$$
$$\alpha_+ = \frac{1}{N_i(\tau_0)}\left(\frac{dN_i}{d\tau}\right)_{\tau_0 + 0}$$

The solution of Eq. (34) with the boundary conditions

$$\varphi^*_{\xi\to-\infty} = \alpha_- \frac{D}{tv_T}\xi, \quad \varphi^*_{\xi\to+\infty} = \alpha_+ \frac{D}{tv_T}\xi \quad (35)$$

is

$$\varphi^* = \begin{cases} \alpha_- \dfrac{D}{tv_T}\xi - \dfrac{\alpha_- - \alpha_+}{2}\dfrac{D}{tv_T}e^{\xi}, & \xi < 0 \\ \alpha_+ \dfrac{D}{tv_T}\xi - \dfrac{\alpha_- - \alpha_+}{2}\dfrac{D}{tv_T}e^{-\xi}, & \xi > 0. \end{cases} \quad (36)$$

Converting to the usual variables we now rewrite (36) in the form

$$\varphi = \frac{T_e}{e}\left\{\ln\frac{N_i(\tau)}{N_1} - \frac{1}{2N_i(\tau_0)}\left[\left(\frac{dN_i}{d\tau}\right)_{\tau_0-0} - \left(\frac{dN_i}{d\tau}\right)_{\tau_0+0}\right]\right.$$
$$\left.\times \frac{D}{tv_T}\begin{bmatrix} \exp\left(-\dfrac{x - \tau_0 tv_T}{D}\right), & x > \tau_0 tv_T \\ \exp\left(\dfrac{x - \tau_0 tv_T}{D}\right), & x < \tau_0 tv_T \end{bmatrix}\right\}.$$

Thus, in addition to the usual expression for the potential $\varphi = e^{-1}T_e \ln(N_i/N_1)$ there is in a charge sheet with a charge distribution

$$N_i - N_e = \frac{1}{2}\left[\left(\frac{dN_i}{d\tau}\right)_{\tau_0-0} - \left(\frac{dN_i}{d\tau}\right)_{\tau_0+0}\right]\frac{D}{tv_T}$$
$$\times \begin{Bmatrix} \exp\left[-\dfrac{x - \tau_0 tv_T}{D}\right], & x > \tau_0 tv_T \\ \exp\left[\dfrac{x - \tau_0 tv_T}{D}\right], & x < \tau_0 tv_T \end{Bmatrix}.$$

This charge sheet moves with a velocity $\tau_0 v_T$. It provides the jump in the self-similar solution. It is important to note that the difference in the electron and ion densities and the field potential produced by the sheet is reduced within the course of time (going as $1/t$).

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